The Fourier transform and Hausdorff dimension

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Hausdorff dimension

The s-dimensional *Hausdorff measure* $\mathcal{H}^s$, $s \geq 0$, is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

where, for $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_j d(E_j)^s : A \subset \bigcup_j E_j, d(E_j) < \delta\}.$$

Here $d(E)$ denotes the diameter of the set E.
The *Hausdorff dimension* of $A \subset \mathbb{R}^n$ is

$$\dim A = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

Frostman's lemma

For $A \subset \mathbb{R}^n$, let $\mathcal{M}(A)$ be the set of Borel measures μ such that $0 < \mu(A) < \infty$ and μ has compact support $\operatorname{spt}\mu \subset A$.

Theorem (Frostman's lemma)

Let $0 \leq s \leq n$. For a Borel set $A \subset \mathbb{R}^n, \mathcal{H}^s(A) > 0$ if and only there is $\mu \in \mathcal{M}(A)$ such that

$$\mu(B(x,r)) \leq r^s \quad \text{for all } x \in \mathbb{R}^n, r > 0. \tag{1}$$

In particular,

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that (1) holds}\}.$$

Riesz energies and Hausdorff dimension

The *s-energy*, $s > 0$, of a Borel measure μ is

$$I_s(\mu) = \iint |x-y|^{-s}\, d\mu x\, d\mu y = \int k_s * \mu\, d\mu,$$

where k_s is the *Riesz kernel*:

$$k_s(x) = |x|^{-s}, \quad x \in \mathbb{R}^n.$$

Theorem

For a closed set $A \subset \mathbb{R}^n$,

$$\dim A = \sup\{s : \textit{there is } \mu \in \mathcal{M}(A) \textit{ such that } I_s(\mu) < \infty\}.$$

The Fourier transform and Hausdorff dimension

The Fourier transform of $\mu \in \mathcal{M}(\mathbb{R}^n)$ is

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} \, d\mu x, \quad \xi \in \mathbb{R}^n.$$

The s-energy of $\mu \in \mathcal{M}(\mathbb{R}^n)$ can be written in terms of the Fourier transform:

$$I_s(\mu) = c(n,s) \int |\widehat{\mu}(x)|^2 |x|^{s-n} \, dx.$$

Thus we have

$$\dim A =$$

$$\sup\{s < n : \exists \mu \in \mathcal{M}(A) \text{ such that } \int |\widehat{\mu}(x)|^2 |x|^{s-n} \, dx < \infty\}.$$

The Fourier dimension

The Fourier dimension of a set $A \subset \mathbb{R}^n$ is

$$\dim_F A =$$
$$\sup\{s \leq n : \exists \mu \in \mathcal{M}(A) \text{ such that } |\widehat{\mu}(x)| \leq |x|^{-s/2}\ \forall x \in \mathbb{R}^n\}.$$

Then

$$\dim_F A \leq \dim A.$$

A is called a Salem set if $\dim_F A = \dim A$.
Examples of Salem sets are smooth planar curves with non-zero curvature and trajectories of Brownian motion. But line segments in $\mathbb{R}^n$, $n \geq 2$, have zero Fourier dimension.

The Fourier dimension of graphs

Fraser, Orponen and Sahlsten, IMRN 2014, proved that

Theorem

For any function $f : A \to \mathbb{R}^{n-m}, A \subset \mathbb{R}^m$, *we have for the graph* $G_f = \{(x, f(x)) : x \in A\}$,

$$\dim_F G_f \leq m.$$

The Hausdorff dimension of one-dimensional Brownian graphs is almost surely 3/2, so they are not Salem sets, in fact, they have almost surely Fourier dimension one due to Fraser and Sahlsten 2015.

Projections and dimension

How do the projections

$$p_\theta(x, y) = x \cos\theta + y \sin\theta, \quad (x, y) \in \mathbb{R}^2, \theta \in [0, \pi),$$

affect the Hausdorff dimension? Notice that p_θ is essentially the orthogonal projection onto the line making angle θ with the x-axis.

Projections and dimension

The following theorem was proved by John Marstrand in 1954 ($\mathcal{L}^m$ denotes the Lebesgue measure in $\mathbb{R}^m$), a Fourier-analytic proof was given by Kaufman 1968:

Theorem

Let $A \subset \mathbb{R}^2$ be a Borel set. If $\dim A \leq 1$, then

$$\dim p_\theta(A) = \dim A \quad \textit{for almost all } \theta \in [0, \pi). \tag{2}$$

If $\dim A > 1$, then

$$\mathcal{L}^1(p_\theta(A)) > 0 \quad \textit{for almost all } \theta \in [0, \pi). \tag{3}$$

Projections and dimension

Generalized projections: Peres and Schlag 2000
Discrete versions: Katz and Tao 2001, Bourgain 2010, Orponen 2015
Self-similar and related sets: Peres and Shmerkin 2009, Hochman and Shmerkin 2012, Shmerkin and Suomala 2014, Falconer and Jin 2014, Simon and Rams 2014
Restricted families of projections: E. and M. Järvenpää and Keleti 2014, Fässler and Orponen 2014, D.M. and R. Oberlin 2013
Heisenberg groups: Balogh, Durand Cartegena, Fässler, Mattila and Tyson 2013

Distance sets and dimension

The *distance set* of $A \subset \mathbb{R}^n$ is

$$D(A) = \{|x-y| : x, y \in A\} \subset [0, \infty).$$

The following Falconer's conjecture seems plausible:

Conjecture

If $n \geq 2$ and $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n/2$, then $\mathcal{L}^1(D(A)) > 0$, or even $\mathrm{Int}(D(A)) \neq \emptyset$.

Falconer proved in 1985 that $\dim A > (n+1)/2$ implies $\mathcal{L}^1(D(A)) > 0$, and we also have then $\mathrm{Int}(D(A)) \neq \emptyset$ by Sjölin and myself 1999.

Distance sets and dimension

The best known result is due to Wolff 1999 for $n = 2$ and to Erdogan 2005 for $n \geq 3$:

Theorem

If $n \geq 2$ and $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n/2 + 1/3$, then $\mathcal{L}^1(D(A)) > 0$.

The proof uses restriction and Kakeya methods and results. In particular, the case $n \geq 3$ relies on Tao's bilinear restriction theorem.

Distance sets and dimension

A group-theoretic approach: Greenleaf, Iosevich, Liu and Palsson 2013
Erdös problem on finite sets: Guth and Katz 2015
Distance sets in finite fields: Iosevich and Rudnev 2007 and others
Angles, directions and other configurations: Iosevich, Laba and others

Besicovitch sets

We say that a Borel set in $\mathbb{R}^n$, $n \geq 2$, is a *Besicovitch set*, or a Kakeya set, if it has zero Lebesgue measure and it contains a line segment of unit length in every direction. This means that for every $e \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ there is $b \in \mathbb{R}^n$ such that $\{te + b : 0 < t < 1\} \subset B$. It is not obvious that Besicovitch sets exist but they do in every $\mathbb{R}^n$, $n \geq 2$.

Conjecture (Kakeya conjecture)

All Besicovitch sets in $\mathbb{R}^n$ have Hausdorff dimension n.

Besicovitch sets

Theorem (Besicovitch 1919)

For any $n \geq 2$ there exists a Borel set $B \subset \mathbb{R}^n$ such that $\mathcal{L}^n(B) = 0$ and B contains a whole line in every direction. Moreover, there exist compact Besicovitch sets in $\mathbb{R}^n$.

The proof of Besicovitch from 1964 uses duality between points and lines.

Theorem (Davies 1971)

For every Besicovitch set $B \subset \mathbb{R}^n$, $\dim B \geq 2$. In particular, the Kakeya conjecture is true in the plane.

D. Oberlin proved in 2006 that even $\dim_F B \geq 2$.

Line segment conjecture

Conjecture (Keleti 2014)

If A is the union of a family of line segments in $\mathbb{R}^n$ and B is the union of the corresponding lines, then $\dim A = \dim B$.

This is true in the plane:

Theorem (Keleti 2014)

The conjecture is true in $\mathbb{R}^2$.

Line segment conjecture

Theorem (Keleti 2014)

(1) If the line segment conjecture is true for some n, then, for this n, every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension at least $n-1$.
(2) If the line segment conjecture is true for all n, then every Besicovitch set in $\mathbb{R}^n$ has packing and upper Minkowski dimension n for all n.

Subsets of hyperplanes

For $p = (a, b) \in \mathbb{R}^{n-1} \times \mathbb{R}$ let $L(p) = L(a, b)$ denote the hyperplane $\{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y = a \cdot x + b\}$. If $E \subset \mathbb{R}^n$ let $L(E) = \bigcup_{p \in E} L(p)$.

Theorem (Falconer and Mattila 2014)

Let $E \subset \mathbb{R}^n$ be a non-empty Borel set and let $A \subset \mathbb{R}^n$ be a Borel set such that $\mathcal{L}^{n-1}\big(L(p) \cap A\big) > 0$ for all $p \in E$. Then

$$\dim\big(L(E) \cap A\big) = \dim L(E) = \min\{\dim E + n - 1, n\}.$$

Moreover, if $\dim E > 1$, *then*

$$\mathcal{L}^n\big(L(E) \cap A\big) > 0.$$

Kakeya maximal function

For $a \in \mathbb{R}^n, e \in S^{n-1}$ and $\delta > 0$, define the tube $T_e^\delta(a)$ with center a, direction e, length 1 and radius δ:

$$T_e^\delta(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot e| \leq 1/2, |x-a-((x-a)\cdot e)e| \leq \delta\}.$$

Then $\mathcal{L}^n(T_e^\delta(a)) = \alpha(n-1)\delta^{n-1}$, where $\alpha(n-1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

Definition

The *Kakeya maximal function* with width δ of $f \in L^1_{loc}(\mathbb{R}^n)$ is

$$\mathcal{K}_\delta f : S^{n-1} \to [0, \infty],$$

$$\mathcal{K}_\delta f(e) = \sup_{a \in \mathbb{R}^n} \frac{1}{\mathcal{L}^n(T_e^\delta(a))} \int_{T_e^\delta(a)} |f| \, d\mathcal{L}^n.$$

Kakeya maximal conjecture

We have the trivial but sharp proposition:

Proposition

For all $0 < \delta < 1$ *and* $f \in L^1_{loc}(\mathbb{R}^n)$,

$$\|\mathcal{K}_\delta f\|_{L^\infty(S^{n-1})} \leq \|f\|_{L^\infty(\mathbb{R}^n)},$$

$$\|\mathcal{K}_\delta f\|_{L^\infty(S^{n-1})} \leq \alpha(n-1)^{1-n}\delta^{1-n}\|f\|_{L^1(\mathbb{R}^n)}.$$

Conjecture

$$\|\mathcal{K}_\delta f\|_{L^n(S^{n-1})} \leq C(n,\epsilon)\delta^{-\epsilon}\|f\|_{L^n(\mathbb{R}^n)}$$

for all $\epsilon > 0, 0 < \delta < 1, f \in L^n(\mathbb{R}^n)$.

Kakeya maximal conjecture

Theorem (Córdoba 1977)

$$\|\mathcal{K}_\delta f\|_{L^2(S^1)} \leq C\sqrt{\log(1/\delta)}\|f\|_{L^2(\mathbb{R}^2)}$$

for all $0 < \delta < 1, f \in L^2(\mathbb{R}^2)$. *In particular, the Kakeya maximal conjecture is true in the plane.*

The factor $\sqrt{\log(1/\delta)}$ is sharp due to Keich 1999.

Kakeya and Hausdorff dimension

Theorem (Bourgain 1991)

Suppose that $1 < p < \infty, \beta > 0$ *and* $n - \beta p > 0$. *If*

$$\|\mathcal{K}_\delta f\|_{L^p(S^{n-1})} \leq C(n,p,\beta)\delta^{-\beta}\|f\|_p \quad \text{for } 0 < \delta < 1, f \in L^p(\mathbb{R}^n),$$

then the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ *is at least* $n - \beta p$. *In particular, if for some* $p, 1 < p < \infty$,

$$\|\mathcal{K}_\delta f\|_{L^p(S^{n-1})} \leq C(n,p,\epsilon)\delta^{-\epsilon}\|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $\epsilon > 0, 0 < \delta < 1, f \in L^p(\mathbb{R}^n)$, *then the Hausdorff dimension of every Besicovitch set in* $\mathbb{R}^n$ *is* n. *Thus the Kakeya maximal conjecture implies the Kakeya conjecture.*

Discretized Kakeya

Proposition

Let $1 < p < \infty, q = \frac{p}{p-1}, 0 < \delta < 1$ *and* $0 < M < \infty$. *Suppose that*

$$\|\sum_{k=1}^{m} t_k \chi_{T_k}\|_{L^q(\mathbb{R}^n)} \leq M$$

whenever $T_1, \dots, T_m$ *are* δ*-separated (in directions)* δ*-tubes and* $t_1, \dots, t_m$ *are positive numbers with*

$$\delta^{n-1} \sum_{k=1}^{m} t_k^q \leq 1.$$

Then

$$\|\mathcal{K}_\delta f\|_{L^p(S^{n-1})} \leq C(n) M \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in L^p(\mathbb{R}^n).$$

Bourgain and dimension of Besicovitch sets

Theorem (Bourgain 1991)

For all Lebesgue measurable sets $E \subset \mathbb{R}^n$,

$$\sigma^{n-1}(\{e \in S^{n-1} : \mathcal{K}_\delta(\chi_E)(e) > \lambda\}) \leq C(n)\delta^{1-n}\lambda^{-n-1}\mathcal{L}^n(E)^2$$

for all $0 < \delta < 1$ and $\lambda > 0$. In particular, the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $(n+1)/2$.

The above restricted weak type inequality is very close to,

$$\|\mathcal{K}_\delta f\|_{L^q(S^{n-1})} \leq C(n,p,\epsilon)\delta^{-(n/p-1+\epsilon)}\|f\|_p$$

for all $\epsilon > 0$ with $p = (n+1)/2, q = n+1$.

Wolff and dimension of Besicovitch sets

Bourgain's proof used bushes; many tubes containing some point. Wolff replaced this with hairbrushes; many tubes intersecting some tube.

Theorem (Wolff 1995)

Let $0 < \delta < 1$. Then for $f \in L^{\frac{n+2}{2}}(\mathbb{R}^n)$,

$$\|\mathcal{K}_\delta f\|_{L^{\frac{n+2}{2}}(S^{n-1})} \leq C(n,\epsilon)\delta^{\frac{2-n}{2+n}-\epsilon}\|f\|_{L^{\frac{n+2}{2}}(\mathbb{R}^n)} \tag{4}$$

for all $\epsilon > 0$. In particular, the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $(n+2)/2$.

A combinatorial theorem

Theorem (Bourgain 1999, Katz and Tao 1999)

Let $\epsilon_0 = 1/6$. Suppose that A and B are finite subsets of $\lambda\mathbb{Z}^m$ for some $m \in \mathbb{N}$ and $\lambda > 0$, $\#A \leq N$ and $\#B \leq N$. Suppose also that $G \subset A \times B$ and

$$\#\{x + y : (x, y) \in G\} \leq N. \tag{5}$$

Then

$$\#\{x - y : (x, y) \in G\} \leq N^{2-\epsilon_0}.$$

The best value of ϵ_0 is not known, but it cannot be taken bigger than $\log 6/\log 3 = 0.39907\ldots$.

Application to dimension of Besicovitch sets

Theorem (Bourgain 1999, Katz and Tao 1999)

For any Besicovitch set B in $\mathbb{R}^n$, $\dim B \geq 6n/11 + 5/11$.

Theorem (Katz and Tao 2002)

For any Besicovitch set B in $\mathbb{R}^n$, $\dim B \geq (2-\sqrt{2})(n-4)+3$.

The second theorem improves Wolff's $(n+2)/2$ bound for all $n \geq 5$. Wolff's estimate is still the best known for $n = 3, 4$.

Restriction problem

When does $\widehat{f}|S^{n-1}$ make sense? If $f \in L^1(\mathbb{R}^n)$ it obviously does, if $f \in L^2(\mathbb{R}^n)$ it obviously does not.
$\widehat{f}|S^{n-1}$ makes sense for $f \in L^p(\mathbb{R}^n)$ if we have for some $q < \infty$ an inequality

$$\|\widehat{f}\|_{L^q(S^{n-1})} \leq C(n,p,q)\|f\|_{L^p(\mathbb{R}^n)} \tag{6}$$

valid for all $f \in \mathcal{S}(\mathbb{R}^n)$.
The restriction problems ask for which p and q (6) holds.
By duality (6) is equivalent, with the same constant $C(n,p,q)$, to

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq C(n,p,q)\|f\|_{L^{q'}(S^{n-1})}. \tag{7}$$

Here p' and q' are conjugate exponents of p and q and $\widehat{f}$ means the Fourier transform of the measure $f\sigma^{n-1}$.

Stein-Tomas theorem

Theorem (Tomas 1975, Stein 1986)

We have for $f \in L^2(S^{n-1})$,

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^2(S^{n-1})}$$

for $q \geq 2(n+1)/(n-1)$. The lower bound $2(n+1)/(n-1)$ is the best possible.

The sharpness of the range of q follows using the Knapp example.

Restriction conjecture

Conjecture

$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^p(S^{n-1})}$ *for* $q > 2n/(n-1)$ *and* $q = \frac{n+1}{n-1}p'$.

This is equivalent to

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^\infty(S^{n-1})} \quad \text{for } q > 2n/(n-1),$$

and to

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^q(S^{n-1})} \quad \text{for } q > 2n/(n-1).$$

The range $q > 2n/(n-1)$ would be optimal. Stein-Tomas theorem implies that these inequalities are true when $q \geq 2(n+1)/(n-1)$.

Restriction conjecture in the plane

Fefferman 1970 and Zygmund 1974 proved in the plane

$$\|\widehat{f}\|_{L^q(\mathbb{R}^2)} \leq C(q)\|f\|_{L^p(S^1)} \text{ for } q > 4 \text{ and } q = 4p'$$

.

Thus the restriction conjecture is true in the plane.

Restriction implies Kakeya

Theorem (Bourgain 1991)

Suppose that $2n/(n-1) < q < \infty$ and

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^q(S^{n-1})} \quad \text{for } f \in L^q(S^{n-1}). \tag{8}$$

Then with $p = q/(q-2)$,

$$\|\mathcal{K}_\delta f\|_{L^p(S^{n-1})} \lesssim_{n,q} \delta^{4n/q-2(n-1)} \|f\|_p$$

for all $0 < \delta < 1, f \in L^p(\mathbb{R}^n)$. In particular, the restriction conjecture implies the Kakeya maximal conjecture.

The proof uses Khintchine's inequalities and the Knapp example.

Bilinear restriction

Theorem (Tao 2003)

Let $c > 0$ and let $S_j \subset \{x \in S^{n-1} : x_n > c\}, j = 1, 2$, with $d(S_1, S_2) \geq c > 0$. Then

$$\|\widehat{f_1}\widehat{f_2}\|_{L^q(\mathbb{R}^n)} \leq C(n, q, c)\|f_1\|_{L^2(S_1)}\|f_2\|_{L^2(S_2)}$$

for $q > (n+2)/n$ and for all $f_j \in L^2(S_j)$ with spt $f_j \subset S_j, j = 1, 2$.

The lower bound $(n+2)/n$ is the best possible due to the Knapp example.

Progress on restriction conjecture

Conjecture: $\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(S^{n-1})}$ for $q > 2n/(n-1)$,
$q > 3$ for $n = 3$.
Tomas 1975: $q > (2n+2)/(n-1)$, $q > 4$ for $n = 3$.
Stein 1986: $q = (2n+2)/(n-1)$, $q = 4$ for $n = 3$.
Bourgain 1991: $q > (2n+2)/(n-1) - \epsilon_n$,
$q > 31/8 = 4 - 1/8$ for $n = 3$.
Tao, Vargas and Vega 1998, Tao 2003 by bilinear restriction:
$q > (2n+4)/n$,
$q > 10/3 = 31/8 - 13/24$ for $n = 3$.
Bennett, Carbery and Tao 2006, Bourgain and Guth 2011 by multilinear restriction: $q > 33/10 = 10/3 - 1/30$ for $n = 3$.
(Dvir 2009), Guth 2014 by polynomial method:
$q > 13/4 = 33/10 - 3/40 = 3 + 1/4$ for $n = 3$.

Distance sets and bilinear restriction

The proof of the Wolff-Erdogan distance set theorem is based on the following: Define quadratic spherical averages of $\mu \in \mathcal{M}(\mathbb{R}^n)$ for $r > 0$,

$$\sigma(\mu)(r) = \int_{S^{n-1}} |\widehat{\mu}(rv)|^2 \, d\sigma^{n-1} v.$$

If $s > n/2$, $I_s(\mu) < \infty$ and $\int_1^\infty \sigma(\mu)(r)^2 r^{n-1} \, dr < \infty$, then $\delta(\mu) \ll \mathcal{L}^1$.
This gives: If $t \leq s, s + t \geq n$, $I_s(\mu) < \infty$ and $\sigma(\mu)(r) \leq Cr^{-t}$ for all $r > 0$, then $\mathcal{L}^1(D(spt\mu)) > 0$.

Distance sets and bilinear restriction

Wolff proved that for all $0 < s < 2, \epsilon > 0$ and $\mu \in \mathcal{M}(\mathbb{R}^2)$ with $\mathrm{spt}\mu \subset B(0,1)$,

$$\sigma(\mu)(r) \leq C(s,\epsilon) r^{\epsilon - s/2} I_s(\mu) \quad \text{for } r > 1.$$

Erdogan extended this to higher dimensions: For all $(n-2)/2 < s < n, n \geq 2, \epsilon > 0$ and $\mu \in \mathcal{M}(\mathbb{R}^n)$ with $\mathrm{spt}\mu \subset B(0,1)$,

$$\sigma(\mu)(r) \leq C(n,s,\epsilon) r^{\epsilon - (n+2s-2)/4} I_s(\mu) \quad \text{for } r > 1.$$

Combining these leads to

$$\dim A > n/2 + 1/3 \quad \text{implies} \quad \mathcal{L}^1(D(A)) > 0.$$

Wolff's power $s/2$ is sharp, Erdogan's $(n+2s-2)/4$ probably is not sharp when $n > 2$. The proofs of these estimates are based on Kakeya and restriction type methods. Erdogan's estimate explicitly uses Tao's bilinear theorem.