The Fourier transform and Hausdorff dimension

Pertti Mattila

University of Helsinki

Sant Feliu de Guíxols
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The s-dimensional *Hausdorff measure* $\mathcal{H}^s$, $s \geq 0$, is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

where, for $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_j d(E_j)^s : A \subset \bigcup_j E_j, d(E_j) < \delta \right\}.$$

Here $d(E)$ denotes the diameter of the set $E$.

The *Hausdorff dimension* of $A \subset \mathbb{R}^n$ is

$$\dim A = \inf \{ s : \mathcal{H}^s(A) = 0 \} = \sup \{ s : \mathcal{H}^s(A) = \infty \}.$$
For $A \subset \mathbb{R}^n$, let $\mathcal{M}(A)$ be the set of Borel measures $\mu$ such that $0 < \mu(A) < \infty$ and $\mu$ has compact support $\text{spt} \mu \subset A$.

**Theorem (Frostman’s lemma)**

Let $0 \leq s \leq n$. For a Borel set $A \subset \mathbb{R}^n$, $\mathcal{H}^s(A) > 0$ if and only there is $\mu \in \mathcal{M}(A)$ such that

$$\mu(B(x, r)) \leq r^s \quad \text{for all } x \in \mathbb{R}^n, \ r > 0. \quad (1)$$

In particular,

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } (1) \text{ holds}\}.$$
Riesz energies and Hausdorff dimension

The $s$-energy, $s > 0$, of a Borel measure $\mu$ is

$$I_s(\mu) = \int\int |x - y|^{-s} \, d\mu x \, d\mu y = \int k_s * \mu \, d\mu,$$

where $k_s$ is the Riesz kernel:

$$k_s(x) = |x|^{-s}, \quad x \in \mathbb{R}^n.$$

**Theorem**

*For a closed set $A \subset \mathbb{R}^n$,*

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}.$$
The Fourier transform and Hausdorff dimension

The Fourier transform of \( \mu \in \mathcal{M}(\mathbb{R}^n) \) is

\[
\hat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} \, d\mu(x), \quad \xi \in \mathbb{R}^n.
\]

The \( s \)-energy of \( \mu \in \mathcal{M}(\mathbb{R}^n) \) can be written in terms of the Fourier transform:

\[
I_s(\mu) = c(n, s) \int |\hat{\mu}(x)|^2 |x|^{s-n} \, dx.
\]

Thus we have

\[
\dim A = \sup \{ s < n : \exists \mu \in \mathcal{M}(A) \text{ such that } \int |\hat{\mu}(x)|^2 |x|^{s-n} \, dx < \infty \}.
\]
The Fourier dimension

The Fourier dimension of a set $A \subset \mathbb{R}^n$ is

$$\dim_F A = \sup\{s \leq n : \exists \mu \in \mathcal{M}(A) \text{ such that } |\hat{\mu}(x)| \leq |x|^{-s/2} \forall x \in \mathbb{R}^n\}.$$

Then

$$\dim_F A \leq \dim A.$$

$A$ is called a Salem set if $\dim_F A = \dim A$.

Examples of Salem sets are smooth planar curves with non-zero curvature and trajectories of Brownian motion. But line segments in $\mathbb{R}^n$, $n \geq 2$, have zero Fourier dimension.
The Fourier dimension of graphs

Fraser, Orponen and Sahlsten, IMRN 2014, proved that

**Theorem**

*For any function* \( f : A \to \mathbb{R}^{n-m} \), \( A \subset \mathbb{R}^m \), *we have for the graph* \( G_f = \{(x, f(x)) : x \in A\} \),

\[
\dim_F G_f \leq m.
\]

The Hausdorff dimension of one-dimensional Brownian graphs is almost surely 3/2, so they are not Salem sets, in fact, they have almost surely Fourier dimension one due to Fraser and Sahlsten 2015.
How do the projections

\[ p_\theta(x, y) = x \cos \theta + y \sin \theta, \quad (x, y) \in \mathbb{R}^2, \theta \in [0, \pi), \]

affect the Hausdorff dimension? Notice that \( p_\theta \) is essentially the orthogonal projection onto the line making angle \( \theta \) with the \( x \)-axis.
The following theorem was proved by John Marstrand in 1954 ($\mathcal{L}^m$ denotes the Lebesgue measure in $\mathbb{R}^m$), a Fourier-analytic proof was given by Kaufman 1968:

**Theorem**

Let $A \subset \mathbb{R}^2$ be a Borel set. If $\dim A \leq 1$, then

$$\dim p_\theta(A) = \dim A \quad \text{for almost all } \theta \in [0, \pi).$$

(2)

If $\dim A > 1$, then

$$\mathcal{L}^1(p_\theta(A)) > 0 \quad \text{for almost all } \theta \in [0, \pi).$$

(3)
Projections and dimension

Generalized projections: Peres and Schlag 2000
Heisenberg groups: Balogh, Durand Cartegena, Fässler, Mattila and Tyson 2013
Distance sets and dimension

The distance set of $A \subset \mathbb{R}^n$ is

$$D(A) = \{|x - y| : x, y \in A\} \subset [0, \infty).$$

The following Falconer’s conjecture seems plausible:

**Conjecture**

If $n \geq 2$ and $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n/2$, then $\mathcal{L}^1(D(A)) > 0$, or even $\text{Int}(D(A)) \neq \emptyset$.

Falconer proved in 1985 that $\dim A > (n + 1)/2$ implies $\mathcal{L}^1(D(A)) > 0$, and we also have then $\text{Int}(D(A)) \neq \emptyset$ by Sjölin and myself 1999.
Distance sets and dimension

The best known result is due to Wolff 1999 for $n = 2$ and to Erdogan 2005 for $n \geq 3$:

**Theorem**

If $n \geq 2$ and $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n/2 + 1/3$, then $\mathcal{L}^1(D(A)) > 0$.

The proof uses restriction and Kakeya methods and results. In particular, the case $n \geq 3$ relies on Tao’s bilinear restriction theorem.
A group-theoretic approach: Greenleaf, Iosevich, Liu and Palsson 2013
Erdös problem on finite sets: Guth and Katz 2015
Distance sets in finite fields: Iosevich and Rudnev 2007 and others
Angles, directions and other configurations: Iosevich, Laba and others
We say that a Borel set in $\mathbb{R}^n$, $n \geq 2$, is a *Besicovitch set*, or a Kakeya set, if it has zero Lebesgue measure and it contains a line segment of unit length in every direction. This means that for every $e \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ there is $b \in \mathbb{R}^n$ such that $\{te + b : 0 < t < 1\} \subset B$. It is not obvious that Besicovitch sets exist but they do in every $\mathbb{R}^n$, $n \geq 2$.

**Conjecture (Kakeya conjecture)**

*All Besicovitch sets in $\mathbb{R}^n$ have Hausdorff dimension $n$.***
Besicovitch sets

Theorem (Besicovitch 1919)

For any $n \geq 2$ there exists a Borel set $B \subset \mathbb{R}^n$ such that $\mathcal{L}^n(B) = 0$ and $B$ contains a whole line in every direction. Moreover, there exist compact Besicovitch sets in $\mathbb{R}^n$.

The proof of Besicovitch from 1964 uses duality between points and lines.

Theorem (Davies 1971)

For every Besicovitch set $B \subset \mathbb{R}^n$, $\dim B \geq 2$. In particular, the Kakeya conjecture is true in the plane.

D. Oberlin proved in 2006 that even $\dim_F B \geq 2$. 
Conjecture (Keleti 2014)

If $A$ is the union of a family of line segments in $\mathbb{R}^n$ and $B$ is the union of the corresponding lines, then $\dim A = \dim B$.

This is true in the plane:

Theorem (Keleti 2014)

The conjecture is true in $\mathbb{R}^2$. 
Theorem (Keleti 2014)

(1) If the line segment conjecture is true for some $n$, then, for this $n$, every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension at least $n - 1$.

(2) If the line segment conjecture is true for all $n$, then every Besicovitch set in $\mathbb{R}^n$ has packing and upper Minkowski dimension $n$ for all $n$. 
Subsets of hyperplanes

For \( p = (a, b) \in \mathbb{R}^{n-1} \times \mathbb{R} \) let \( L(p) = L(a, b) \) denote the hyperplane \( \{ (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y = a \cdot x + b \} \). If \( E \subset \mathbb{R}^n \) let \( L(E) = \bigcup_{p \in E} L(p) \).

**Theorem (Falconer and Mattila 2014)**

Let \( E \subset \mathbb{R}^n \) be a non-empty Borel set and let \( A \subset \mathbb{R}^n \) be a Borel set such that \( \mathcal{L}^{n-1}(L(p) \cap A) > 0 \) for all \( p \in E \). Then

\[
\dim (L(E) \cap A) = \dim L(E) = \min \{ \dim E + n - 1, n \}.
\]

Moreover, if \( \dim E > 1 \), then

\[
\mathcal{L}^n(L(E) \cap A) > 0.
\]
Kakeya maximal function

For $a \in \mathbb{R}^n$, $e \in S^{n-1}$ and $\delta > 0$, define the tube $T_{e}^{\delta}(a)$ with center $a$, direction $e$, length 1 and radius $\delta$:

$$T_{e}^{\delta}(a) = \{ x \in \mathbb{R}^n : |(x-a) \cdot e| \leq 1/2, |x-a-((x-a) \cdot e)e| \leq \delta \}.$$

Then $\mathcal{L}^n(T_{e}^{\delta}(a)) = \alpha(n-1)\delta^{n-1}$, where $\alpha(n-1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

**Definition**

The **Kakeya maximal function** with width $\delta$ of $f \in L^1_{loc}(\mathbb{R}^n)$ is

$$K_{\delta}f : S^{n-1} \rightarrow [0, \infty],$$

$$K_{\delta}f(e) = \sup_{a\in\mathbb{R}^n} \frac{1}{\mathcal{L}^n(T_{e}^{\delta}(a))} \int_{T_{e}^{\delta}(a)} |f| \ d\mathcal{L}^n.$$
We have the trivial but sharp proposition:

**Proposition**

For all $0 < \delta < 1$ and $f \in L^1_{loc} (\mathbb{R}^n)$,

\[
\|K_\delta f\|_{L^\infty (S^{n-1})} \leq \|f\|_{L^\infty (\mathbb{R}^n)},
\]

\[
\|K_\delta f\|_{L^\infty (S^{n-1})} \leq \alpha (n - 1)^{1-n} \delta^{1-n} \|f\|_{L^1 (\mathbb{R}^n)}.
\]

**Conjecture**

\[
\|K_\delta f\|_{L^n (S^{n-1})} \leq C(n, \epsilon) \delta^{-\epsilon} \|f\|_{L^n (\mathbb{R}^n)}
\]

for all $\epsilon > 0$, $0 < \delta < 1$, $f \in L^n (\mathbb{R}^n)$. 
Kakeya maximal conjecture

Theorem (Córdoba 1977)

\[ \|K_\delta f\|_{L^2(S^1)} \leq C \sqrt{\log(1/\delta)} \|f\|_{L^2(\mathbb{R}^2)} \]

for all \(0 < \delta < 1\), \(f \in L^2(\mathbb{R}^2)\). In particular, the Kakeya maximal conjecture is true in the plane.

The factor \(\sqrt{\log(1/\delta)}\) is sharp due to Keich 1999.
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Theorem (Bourgain 1991)

Suppose that $1 < p < \infty$, $\beta > 0$ and $n - \beta p > 0$. If

$$\|K_{\delta} f\|_{L^p(S^{n-1})} \leq C(n, p, \beta)\delta^{-\beta}\|f\|_p \quad \text{for } 0 < \delta < 1, f \in L^p(\mathbb{R}^n),$$

then the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $n - \beta p$. In particular, if for some $p$, $1 < p < \infty$,

$$\|K_{\delta} f\|_{L^p(S^{n-1})} \leq C(n, p, \epsilon)\delta^{-\epsilon}\|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $\epsilon > 0$, $0 < \delta < 1$, $f \in L^p(\mathbb{R}^n)$, then the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is $n$. Thus the Kakeya maximal conjecture implies the Kakeya conjecture.
Proposition

Let $1 < p < \infty$, $q = \frac{p}{p-1}$, $0 < \delta < 1$ and $0 < M < \infty$. Suppose that

$$\left\| \sum_{k=1}^{m} t_k \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \leq M$$

whenever $T_1, \ldots, T_m$ are $\delta$-separated (in directions) $\delta$-tubes and $t_1, \ldots, t_m$ are positive numbers with

$$\delta^{n-1} \sum_{k=1}^{m} t_k^q \leq 1.$$

Then

$$\| \mathcal{K}_\delta f \|_{L^p(S^{n-1})} \leq C(n) M \| f \|_{L^p(\mathbb{R}^n)} \text{ for all } f \in L^p(\mathbb{R}^n).$$
Bourgain and dimension of Besicovitch sets

**Theorem (Bourgain 1991)**

For all Lebesgue measurable sets $E \subset \mathbb{R}^n$,

$$\sigma^{n-1}(\{ e \in S^{n-1} : K_\delta(\chi_E)(e) > \lambda \}) \leq C(n)\delta^{1-n}\lambda^{-n-1}\mathcal{L}^n(E)^2$$

for all $0 < \delta < 1$ and $\lambda > 0$. In particular, the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $(n + 1)/2$.

The above restricted weak type inequality is very close to,

$$\|K_\delta f\|_{L^q(S^{n-1})} \leq C(n, p, \epsilon)\delta^{-(n/p-1+\epsilon)}\|f\|_p$$

for all $\epsilon > 0$ with $p = (n + 1)/2$, $q = n + 1$. 
Bourgain’s proof used bushes; many tubes containing some point. Wolff replaced this with hairbrushes; many tubes intersecting some tube.

**Theorem (Wolff 1995)**

Let $0 < \delta < 1$. Then for $f \in L^{\frac{n+2}{2}}(\mathbb{R}^n)$,

$$\|\mathcal{K}_\delta f\|_{L^{\frac{n+2}{2}}(S^{n-1})} \leq C(n, \epsilon)\delta^{\frac{2-n}{2+n}} \|f\|_{L^{\frac{n+2}{2}}(\mathbb{R}^n)}$$

for all $\epsilon > 0$. In particular, the Hausdorff dimension of every Besicovitch set in $\mathbb{R}^n$ is at least $(n + 2)/2$. 
A combinatorial theorem

Theorem (Bourgain 1999, Katz and Tao 1999)

Let $\epsilon_0 = 1/6$. Suppose that $A$ and $B$ are finite subsets of $\lambda \mathbb{Z}^m$ for some $m \in \mathbb{N}$ and $\lambda > 0$, $\#A \leq N$ and $\#B \leq N$. Suppose also that $G \subset A \times B$ and

$$\#\{x + y : (x, y) \in G\} \leq N. \quad (5)$$

Then

$$\#\{x - y : (x, y) \in G\} \leq N^{2-\epsilon_0}.$$ 

The best value of $\epsilon_0$ is not known, but it cannot be taken bigger than $\log 6 / \log 3 = 0.39907 \ldots$. 
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Application to dimension of Besicovitch sets

**Theorem (Bourgain 1999, Katz and Tao 1999)**

For any Besicovitch set $B$ in $\mathbb{R}^n$, $\dim B \geq \frac{6n}{11} + \frac{5}{11}$.

**Theorem (Katz and Tao 2002)**

For any Besicovitch set $B$ in $\mathbb{R}^n$, $\dim B \geq (2 - \sqrt{2})(n - 4) + 3$.

The second theorem improves Wolff’s $(n + 2)/2$ bound for all $n \geq 5$. Wolff’s estimate is still the best known for $n = 3, 4$. 
When does $\hat{f}|S^{n-1}$ make sense? If $f \in L^1(\mathbb{R}^n)$ it obviously does, if $f \in L^2(\mathbb{R}^n)$ it obviously does not. $\hat{f}|S^{n-1}$ makes sense for $f \in L^p(\mathbb{R}^n)$ if we have for some $q < \infty$ an inequality

$$\|\hat{f}\|_{L^q(S^{n-1})} \leq C(n, p, q)\|f\|_{L^p(\mathbb{R}^n)}$$

valid for all $f \in \mathcal{S}(\mathbb{R}^n)$.

The restriction problems ask for which $p$ and $q$ (6) holds. By duality (6) is equivalent, with the same constant $C(n, p, q)$, to

$$\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq C(n, p, q)\|f\|_{L^{q'}(S^{n-1})}.$$  

(7)

Here $p'$ and $q'$ are conjugate exponents of $p$ and $q$ and $\hat{f}$ means the Fourier transform of the measure $f \sigma^{n-1}$. 
Theorem (Tomas 1975, Stein 1986)

We have for $f \in L^2(S^{n-1})$,

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n, q)\|f\|_{L^2(S^{n-1})}$$

for $q \geq 2(n + 1)/(n - 1)$. The lower bound $2(n + 1)/(n - 1)$ is the best possible.

The sharpness of the range of $q$ follows using the Knapp example.
Restriction conjecture

\[ \| \hat{f} \|_{L^q(\mathbb{R}^n)} \leq C(n, q) \| f \|_{L^p(S^{n-1})} \quad \text{for } q > 2n/(n-1) \text{ and } q = \frac{n+1}{n-1} p'. \]

This is equivalent to

\[ \| \hat{f} \|_{L^q(\mathbb{R}^n)} \leq C(n, q) \| f \|_{L^\infty(S^{n-1})} \quad \text{for } q > 2n/(n-1), \]

and to

\[ \| \hat{f} \|_{L^q(\mathbb{R}^n)} \leq C(n, q) \| f \|_{L^q(S^{n-1})} \quad \text{for } q > 2n/(n-1). \]

The range \( q > 2n/(n-1) \) would be optimal. Stein-Tomas theorem implies that these inequalities are true when \( q \geq 2(n+1)/(n-1) \).
Restriction conjecture in the plane

Fefferman 1970 and Zygmund 1974 proved in the plane

$$\|\hat{f}\|_{L^q(\mathbb{R}^2)} \leq C(q)\|f\|_{L^p(S^1)}$$ for $q > 4$ and $q = 4p'$.

Thus the restriction conjecture is true in the plane.
Restriction implies Kakeya

**Theorem (Bourgain 1991)**

Suppose that $2n/(n-1) < q < \infty$ and

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^q(S^{n-1})} \quad \text{for } f \in L^q(S^{n-1}). \quad (8)$$

Then with $p = q/(q-2)$,

$$\|\mathcal{K}_\delta f\|_{L^p(S^{n-1})} \lesssim_{n,q} \delta^{4n/q-2(n-1)} \|f\|_p$$

for all $0 < \delta < 1$, $f \in L^p(\mathbb{R}^n)$. In particular, the restriction conjecture implies the Kakeya maximal conjecture.

The proof uses Khintchine’s inequalities and the Knapp example.
Bilinear restriction

**Theorem (Tao 2003)**

Let \( c > 0 \) and let \( S_j \subset \{ x \in S^{n-1} : x_n > c \} \), \( j = 1, 2 \), with \( d(S_1, S_2) \geq c > 0 \). Then

\[
\| \hat{f}_1 \hat{f}_2 \|_{L^q(\mathbb{R}^n)} \leq C(n, q, c) \| f_1 \|_{L^2(S_1)} \| f_2 \|_{L^2(S_2)}
\]

for \( q > (n + 2)/n \) and for all \( f_j \in L^2(S_j) \) with \( \text{spt } f_j \subset S_j, j = 1, 2 \).

The lower bound \( (n + 2)/n \) is the best possible due to the Knapp example.
Conjecture: $\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(S^{n-1})}$ for $q > 2n/(n-1)$, $q > 3$ for $n = 3$.

Tomas 1975: $q > (2n + 2)/(n - 1)$, $q > 4$ for $n = 3$.
Stein 1986: $q = (2n + 2)/(n - 1)$, $q = 4$ for $n = 3$.
Bourgain 1991: $q > (2n + 2)/(n - 1) - \epsilon_n$,
$q > 31/8 = 4 - 1/8$ for $n = 3$.
Tao, Vargas and Vega 1998, Tao 2003 by bilinear restriction:
$q > (2n + 4)/n$,
$q > 10/3 = 31/8 - 13/24$ for $n = 3$.
Bennett, Carbery and Tao 2006, Bourgain and Guth 2011 by multilinear restriction: $q > 33/10 = 10/3 - 1/30$ for $n = 3$.
(Dvir 2009), Guth 2014 by polynomial method:
$q > 13/4 = 33/10 - 3/40 = 3 + 1/4$ for $n = 3$. 
The proof of the Wolff-Erdogan distance set theorem is based on the following: Define quadratic spherical averages of $\mu \in \mathcal{M}(\mathbb{R}^n)$ for $r > 0$,

$$\sigma(\mu)(r) = \int_{S^{n-1}} |\hat{\mu}(rv)|^2 d\sigma^{n-1}v.$$ 

If $s > n/2$, $l_s(\mu) < \infty$ and $\int_1^\infty \sigma(\mu)(r)^2 r^{n-1} dr < \infty$, then $\delta(\mu) \ll L^1$.

This gives: If $t \leq s$, $s + t \geq n$, $l_s(\mu) < \infty$ and $\sigma(\mu)(r) \leq Cr^{-t}$ for all $r > 0$, then $L^1(D(spt\mu)) > 0$. 
Wolff proved that for all $0 < s < 2$, $\epsilon > 0$ and $\mu \in \mathcal{M}(\mathbb{R}^2)$ with $\text{spt} \mu \subset B(0,1)$,

$$\sigma(\mu)(r) \leq C(s, \epsilon)r^{\epsilon-s/2}l_s(\mu) \quad \text{for } r > 1.$$  

Erdogan extended this to higher dimensions: For all $(n-2)/2 < s < n$, $n \geq 2$, $\epsilon > 0$ and $\mu \in \mathcal{M}(\mathbb{R}^n)$ with $\text{spt} \mu \subset B(0,1)$,

$$\sigma(\mu)(r) \leq C(n, s, \epsilon)r^{\epsilon-(n+2s-2)/4}l_s(\mu) \quad \text{for } r > 1.$$  

Combining these leads to

$$\dim A > n/2 + 1/3 \quad \text{implies} \quad \mathcal{L}^1(D(A)) > 0.$$  

Wolff’s power $s/2$ is sharp, Erdogan’s $(n + 2s - 2)/4$ probably is not sharp when $n > 2$. The proofs of these estimates are based on Kakeya and restriction type methods. Erdogan’s estimate explicitly uses Tao’s bilinear theorem.