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RINGS WHOSE MODULES HAVE MAXIMAL SUBMODULES

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Dedicated to Laci Fuchs on his 70th birthday

Abstract

A ring R is a **right max ring** if every right module $M \neq 0$ has at least one maximal submodule. It suffices to check for maximal submodules of a single module and its submodules in order to test for a max ring; namely, any cogenerating module E of mod-R; also it suffices to check the submodules of the injective hull $E(V)$ of each simple module V (Theorem 1). Another test is transfinite nilpotence of the radical of E in the sense that $\operatorname{rad}^{\alpha} E = 0$; equivalently, there is an ordinal α such that $\operatorname{rad}^{\alpha}(E(V)) = 0$ for each simple module V. This holds iff each $\operatorname{rad}^{\beta}(E(V))$ has a maximal submodule, or is zero (Theorem 2). If follows that R is right max iff every nonzero (subdirectly irreducible) quasi-injective right R-module has a maximal submodule (Theorem 3.3). We characterize a right max ring R via the endomorphism ring Λ of any injective cogenerator E of mod-R; namely, Λ/L has a minimal submodule for any left ideal $L = \operatorname{ann}_{\Lambda} M$ for a submodule (or subset) $M \neq 0$ of E (Theorem 8.8). Then Λ/L_0 has socle $\neq 0$ for: (1) any finitely generated left ideal $L_0 \neq \Lambda$; (2) each annihilator left ideal $L_0 \neq \Lambda$; and (3) each proper left ideal $L_0 = L + L'$, where $L = \operatorname{ann}_{\Lambda} M$ as above (e.g. as in (2)) and L' finitely generated (Corollary 8.9A).

[1]Hamsher modules are called max modules by Shock [**S**].