ON \( L^p \) ESTIMATES FOR SQUARE ROOTS OF SECOND ORDER ELLIPTIC OPERATORS ON \( \mathbb{R}^n \)

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Abstract

We prove that the square root of a uniformly complex elliptic operator \( L = -\text{div}(A \nabla) \) with bounded measurable coefficients in \( \mathbb{R}^n \) satisfies the estimate \( \|L^{1/2}f\|_p \lesssim \|\nabla f\|_p \) for \( \sup(1, \frac{2n}{n+4} - \epsilon) < p < \frac{2n}{n+4} + \epsilon \), which is new for \( n \geq 5 \) and \( p < 2 \) or for \( n \geq 3 \) and \( p > \frac{2n}{n+4} \). One feature of our method is a Calderón-Zygmund decomposition for Sobolev functions. We make some further remarks on the topic of the converse \( L^p \) inequalities (i.e. Riesz transforms bounds), pushing the recent results of \([BK2]\) and \([HM]\) for \( \frac{2n}{n+2} < p < 2 \) when \( n \geq 3 \) to the range \( \sup(1, \frac{2n}{n+2} - \epsilon) < p < 2 + \epsilon' \).

In particular, we obtain that \( L^{1/2} \) extends to an isomorphism from \( \dot{W}^{1,p}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) for \( p \) in this range. We also generalize our method to higher order operators.

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\textit{Key words.} Calderón-Zygmund decomposition, elliptic operators, square roots, functional calculus.