GROUP RINGS IN WHICH EVERY
LEFT IDEAL IS A RIGHT IDEAL

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ABSTRACT. Let $K[G]$ denote the group ring of $G$ over the field $K$. In this note we characterize those group rings in which all left ideals are right ideals.

Let $R$ be a ring. We say that $R$ is l.i.r.i. if every left ideal is a right ideal. A ring is l.a.r.i. if every left annihilator is a right ideal. Our notation follows that of [2].

The main results are

THEOREM 1. Let $K$ be a field and let $G$ be a nonabelian locally finite group. Then if $K[G]$ is l.a.r.i., one of the following occurs

(i) $\text{Char } K = 0$ and $G$ is a Hamilton group such that for each odd exponent, $n$, of $G$ the quaternion algebra over the field $K(\z_n)$, where $\z_n$ is a primitive $n$-root of the unity, is a division ring.

(ii) $\text{Char } K = 2$ and $K$ does not contain any primitive 3-root of the unity. Moreover $G \cong Q\times A$, where $Q$ is the quaternion group of order 8 and $A$ is abelian in which each element has odd order and if $n$ is an exponent for $A$, then the least integer $m \geq 1$ satisfying $2^m \equiv 1 \pmod{n}$ is odd.
Conversely if $K[G]$ satisfies (i) or (ii), then $K[G]$ is l.i.r.i. and, in particular, it is l.a.r.i.

--- Observe that if char $K > 2$ and $G$ is locally finite, then $K[G]$ is l.a.r.i. if and only if $G$ is abelian.

THEOREM II. Let $K[G]$ denote the group ring over a nonabelian group. Then the following are equivalent

(i) $K[G]$ is l.i.r.i.

(ii) $G$ is locally finite and if $\alpha/\beta \in K[G]$ with $\alpha/\beta = 0$, then $\beta \alpha = 0$.

(iii) $G$ is locally finite and $K[G]$ is l.a.r.i.

If we combine the above theorems we get necessary and sufficient conditions for $K[G]$ to be l.i.r.i.

By using the antiautomorphism of $K[G]$ given by

$$\sum_{x \in G} a_x x \mapsto \sum_{x \in G} a_x x^{-1}$$

we see that $K[G]$ is l.i.r.i. (l.a.r.i) if and only if $K[G]$ is r.i.l.i. (r.a.l.i.).

LEMMA 1. (i) $K[G]$ is l.i.r.i. if and only if for every finitely generated subgroup $H \leq G$, $K[H]$ is l.i.r.i.

(ii) If $K[G]$ is l.i.r.i., then all subgroups of $G$ are normal.

(iii) Suppose that $G$ is locally finite. If $K[G]$ is l.a.r.i., then all subgroups of $G$ are normal.

PROOF. (i) First we suppose that for every finitely generated subgroup $H \leq G$, $K[H]$ is l.i.r.i. Let $I \leq K[G]$ a left ideal. Let $\alpha \in I$, $g \in G$. We set $H = \langle g, s \alpha \omega \rangle$. Then
I \cap K[H] is a left ideal of K[H] and hence I \cap K[H] is an ideal of K[H], since H is finitely generated. Now g \in H and 
\alpha \in I \cap K[H] so \omega g \in I \cap K[H] \subseteq I. Therefore we have shown that I g \subseteq I for any g \in G and so I is a right ideal. Conversely 
let H be a subgroup of G and suppose that I \subseteq K[H] is a left 
ideal of K[H]. Let \{x_i\} be a set of left coset representatives 
for H in G. Then K[G] is a free right K[H] - module with 
basis \{x_i\}. Thus we have K[G] = \sum x_i K[H]. Denote \sum x_i I 
by J. Clearly J is a left ideal of K[G]. If we suppose that 
K[G] is l.i.r.i., then we have that J is a right ideal of 
K[G]. Let h \in H. Then 

In \subseteq Jh \cap K[H] \subseteq J \cap K[H] = I 

and so I is a right ideal.

(ii) In order to prove that all subgroups of G are normal it 
suffices to see that all cyclic subgroups are normal. Let 
a, g \in G. Consider the left ideal I = K[G](1 - a). Then I is 
an ideal, since K[G] is l.i.r.i.. Thus \alpha^{-1}(1 - a)g \subseteq I and 
1 - g^{-1}ag = \alpha (1 - a) for a suitable element \alpha \in K[G]. Now 
we use the K[\langle a \rangle] - homomorphism \Theta : K[G] \rightarrow K[\langle a \rangle] in which 
\sum a_x x \mapsto \sum a_x x \quad \text{and we obtain} \quad 1 - \Theta(g^{-1}ag) = \Theta(\langle a \rangle)(1 - a). 

Since 1 - a is not invertible we have that \Theta(g^{-1}ag) \neq 0. Hence 
g^{-1}ag \in \langle a \rangle.

(iii) Suppose that G is locally finite and K[G] is l.a.r.i. Let 
H be a finite subgroup of G. Then Lemma 1.2 [2, Chap. 3] yields
that \( \mathcal{L}(\hat{H}) = K[G] \omega(K[H]) \). In other hand we have that
\[
H = \{ x \in G : x - 1 \in K[G] \omega(K[H]) \}.
\]
By hypothesis \( \mathcal{L}(\hat{H}) \) is an ideal, then it is easy to see that \( H \) is normal in \( G \).

We recall that a nonabelian group \( G \) such that all subgroups are normal is a Hamilton group, that is [see 1, Th. 12.5.4]
\[
G \cong Q \times A \times B
\]
where \( Q \) is the quaternion group of 3 elements, \( A \) is an abelian group such that every element has odd order, and \( B \) is an abelian group of exponent 2. For the rest of this paper we fix this notation.

**Lemma 2.** Suppose that \( G \) is locally finite and \( K[G] \) is l.a.r.i. Let \( \alpha, \beta \in K[G] \) such that \( \alpha \beta = 0 \). Then \( \beta \alpha = 0 \).

**Proof.** If \( G \) is abelian the result is trivial. If \( G \) is not abelian, Lemma 1 (iii) yields that \( G \) is a Hamilton group. Put \( G = Q \times A \times B \). If \( Q \) is generated by \( a, b \) with the relations \( a^4 = 1, aba = b, a^2 = b^2 \), put \( H = \langle a^2 \rangle x A x B \). \( H \) is the center of \( G \). By using the map \( \theta : K[G] \rightarrow K[H] \) in which
\[
\sum a_x x \mapsto \sum a_x x
\]
we can write any element \( \alpha \in K[G] \) as
\[
(\ast) \quad \alpha = \theta(\alpha) + \theta(a^{-1} \alpha)a + \theta(b^{-1} \alpha)b + \theta(b^{-1}a^{-1} \alpha)ab.
\]
Suppose now that \( \alpha \beta = 0 \). A computation proves that \( \theta(\alpha \beta) = \theta(\beta \alpha) \)
Therefore \( \theta(\beta \alpha) = 0 \). Since \( \alpha \in \mathcal{L}(\beta) \) and, by hypothesis, \( \mathcal{L}(\beta) \) is an ideal we have \( \alpha x \beta = 0 \) for any \( x \in G \). Thus
$\Theta(x^\beta \alpha) = 0$. By considering $(\alpha)$ for $\beta \alpha$ we conclude that $\beta \alpha = 0$.

In characteristic 2 we need the following

**Lemma 4.** Let $K$ be a field of characteristic 2. Suppose that $K$ does not contain any primitive 3-root of the unity. Put $Q = \langle a, b \rangle$. Then if $\alpha = \sum a_x x \in K[\langle a \rangle]$ such that $|\alpha| = 1$ (where $|\alpha| = \sum a_x$) we have

$$1 + (\alpha b)^2 = (1 + a^2)u$$

where $u \in K[\langle a \rangle]$ is a unit.

**Proof.** Let $\alpha = a_1 + a_2 a_3 + a_4 a_5 a_6 a_7 a_8 a_9 a_{10} \in K[\langle a \rangle]$ with $\sum a_i = 1$.

Then a calculation proves that

$$1 + (\alpha b)^2 = (1 + a^2)(1 + (a_1 + a_3)(a_2 + a_4)a).$$

Since $Q$ is a 2-group and $\text{char} \ K = 2$ we know that $K[Q]$ is a local ring whose maximal ideal is $\{\alpha \in K[Q] : |\alpha| = 0\}$. Suppose by way of contradiction that $1 + (a_1 + a_3)(a_2 + a_4)a$ is not a unit. Then $(a_1 + a_3)(a_2 + a_4) = 1$, and since $\sum a_i = 1$ we see that $a_1 + a_3$ is a primitive 3-root of the unity. Since $X$ does not contain any primitive 3-root of the unity we have a contradiction.

**The Proof of Theorem I.** Suppose that $G$ is a nonabelian locally finite group and $K[Q]$ is l.a.r.i. Then Lemma 1(iii) yields that $G = Q \times A \times B$. First we observe that the case $\text{char} \ K > 2$ is not possible. Since $K[G]$ is l.a.r.i. clearly $K[Q]$ so. But in char $> 2$ we have

$$K[Q] \cong K \times K \times K \times K \times M(2, K)$$
and this is a contradiction, since \( M(2, K) \) is not l.a.r.i.

Suppose \( \text{char} K = 0 \). Let \( n \) be an exponent for \( A \) and let \( x \in A \)
such that \( o(x) = n \). Then \( K[<x>] \) is a product of fields

\[
K[<x>] \cong K(\xi_n) \times L_1 \times \cdots \times L_m
\]

where \( o(\xi_n) = n \). In other hand we have

\[
K[Q] \cong K \times K \times K \times K \times \left( \frac{-1,-1}{K} \right)
\]

where the last factor is the quaternion algebra over \( K \). Since

\[
K[Q \times <x>] = K[Q] \otimes_K K[<x>]
\]

we get that \( \left( \frac{-1,-1}{K} \right) \otimes_K K(\xi_n) = \left( \frac{-1,-1}{K(\xi_n)} \right) \)

is a direct factor of \( K[Q \times <x>] \) and so \( \left( \frac{-1,-1}{K(\xi_n)} \right) \) is l.a.r.i.

Therefore the quaternion algebra over \( K(\xi_n) \) is a division ring.

Conversely suppose that \( K[G] \) satisfies (i). Then we will prove

that \( K[G] \) is l.i.r.i. It follows from Lemma 1(i) that it

suffices to consider \( G \) finite. Then

\[
G \cong \mathbb{Q} \times A \times (\mathbb{Z}/2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/2\mathbb{Z})^{2m}
\]

and we get

\[
K[G] = K[Q \times A] \times \cdots \times K[Q \times A]
\]

Clearly we can suppose that \( G = Q \times A \). Then it is easy to see

that

\[
K[G] = K[A] \times K[A] \times K[A] \times K[A] \times \prod_{i} \left( \frac{-1,-1}{K(\xi_i)} \right)
\]

where \( o(\xi_i) \) are exponents for \( A \). Hence we see that \( K[G] \) is
a product of l.i.r.i. rings. Therefore $K[G]$ is l.i.r.i.

Char $K = 2$. First we observe that if $K$ contains a primitive $3$-root of the unity, then $K[G]$ is not l.a.r.i. From Lemma 2 it suffices to exhibit elements $\alpha, \beta \in K[G]$ such that $\alpha \beta = 0$ but $\beta \alpha \not= 0$. If $\bar{3}$ is a primitive $3$-root of the unity we set

\[ \alpha = (1 + \bar{3})(1 + \bar{3}a)b \]
\[ \beta = (1 + \bar{3}(1 + \bar{3}a)b)(1 + a)b. \]

A calculation proves that $\alpha \beta = 0$ but $\beta \alpha \not= 0$. We now prove that $G = Q \times A$. If this is not the case there exists an element $x \in G - Q$ of order $2$ which centralizes $G$. Again there exist elements

\[ \alpha = 1 + (a + b + ab)x \]
\[ \beta = (a + b + ab)(1 + a) + (1 + a)x \]

such that $\alpha \beta = 0$ but $\beta \alpha \not= 0$ and so $K[G]$ is not l.a.r.i.

Let $n$ be an exponent for $A$ and $x \in A$ such that $o(x) = n$. Since char $K = 2$ we have that $K[<x>]$ is semisimple, and so

\[ K[<x>] = K(\xi) \times \ldots \times L_m \] where $o(\xi) = n$.

Then $K[Q] \otimes K(\xi) \cong K(\xi)[Q]$ is a direct factor of $K[Q \times <x>].$ By hypothesis $K(\xi)[Q]$ is l.a.r.i. By above $K(\xi)$ does not contain any primitive $3$-root of the unity. Therefore $2 \not| m$, where $m$ is the degree of the extension $(\mathbb{Z}/2\mathbb{Z}(\xi))/\mathbb{Z}/2\mathbb{Z}$). But $m$ is precisely the least integer satisfying $2^m \equiv 1 \pmod{n}$.

Conversely suppose that $K[G]$ satisfies (ii). We will prove that
$K[Q]$ is l.i.r.i.. Again from Lemma 1(i) we can consider that $G$ is finite. Then

$$K[A] \cong K(\xi_1) \times \cdots \times K(\xi_n)$$

and so

$$K[Q \times A] \cong K(\xi_1)[Q] \times \cdots \times K(\xi_n)[Q].$$

By hypothesis the field $K(\xi_i)$ does not contain any primitive 3-root of the unity. Since a product of l.i.r.i. rings is a l.i.r.i., we have only to prove that if a field $K$ does not contain any primitive 3-root of the unity, then $K[Q]$ is l.i.r.i.

Let $I \subseteq K[Q]$ a left ideal. Suppose that $\alpha \in I$. We can write $\alpha$ in the form $\alpha = \alpha_1 + \alpha_2 b$, where $\alpha_1, \alpha_2 \in K[<a>]$. The first task is to show that $\alpha_1(1+a^2) \in I$. Note that if $\alpha_1(1+a^2) \in I$, then, since $1+a^2$ is central, $\alpha_2 b(1+a^2) \in I$. Again $\alpha_2(1+a^2)$ is central and therefore $b \alpha_2(1+a^2) \in I$. Since $I$ is a left ideal $\alpha_2(1+a^2) \in I$. Thus we need only to prove that $\alpha_1(1+a^2) \in I$. If $\alpha$ is a unit, then $I = K[Q]$. Thus we may suppose that $\alpha$ is not a unit. Then we have $|\alpha_1| + |\alpha_2| = 0$. Suppose that $\alpha_1$ is a unit. Then $1+\alpha_1^{-1} \alpha_2 b \in I$. Clearly $1+(\alpha_1^{-1} \alpha_2 b) \in I$, so Lemma 4 yields that $1+a^2 \in I$. Hence $\alpha_1(1+a^2) \in I$. If $\alpha_1$ is not a unit, then we have $|\alpha_1| = 0$ and hence $|\alpha_2| = 0$. Therefore $\alpha_1 = \beta_1(1+a)$ and $\alpha_2 = \beta_2(1+a)$ for suitable elements $\beta_1 \in K[<a>]$. Thus $\alpha = (\beta_1 + \beta_2 ab)(1+a)$. If $\beta_1 + \beta_2 ab$ is a unit we obtain that $1+a \in I$ and so $\alpha_1(1+a^2) = \alpha_1(1+a)^2 \in I$. Hence we may consider that $|\beta_1| + |\beta_2| = 0$. If $\beta_1$ is a unit, then $(1+\beta_1^{-1} \beta_2 ab)(1+a) \in I$. Again we use Lemma 4 and we get that $(1+a^2)(1+a) \in I$. Thus
\[ \alpha_1(1+a^2) = \beta_1(1+a)(1+a^2) \in I. \] Finally if \( \beta_1 \) is not a unit we have \( \beta_1 = \gamma_1(1+a) \) for certain \( \gamma_1 \in K[<a>] \). Therefore

\[ \alpha_1(1+a^2) = \gamma_1(1+a^2)(1+a^2) = 0 \quad \text{and, certainly, } \alpha_1(1+a^2) \in I. \]

Now we will prove that \( \alpha x \in I \) for any \( x \in Q \). Since \( Q = \langle a, b \rangle \) it suffices to see that \( \alpha a, \alpha b \in I \). By using the automorphism of \( Q \) given by \( a \mapsto b, b \mapsto a \) we see that we have only to prove that \( \alpha a \in I \). But

\[ \alpha a = \alpha_1 a + \alpha_2 ba = a \alpha + ab \alpha_2(1+a^2). \]

Since \( a \in I \) and by above \( \alpha_2(1+a^2) \in I \), the result follows.

THE PROOF OF THEOREM II. (i) \( \implies \) (ii). It follows from Lemma 1 (ii) that all subgroups of \( G \) are normal. Since \( G \) is not abelian, it is a Hamilton group and, clearly, locally finite. If a ring is l.i.r.i., then it is l.a.r.i. Lemma 2 completes the proof. Trivially (ii) implies (iii). It follows from Th. I that (iii) implies (i). The result follows.

REFERENCES

ADDENDUM

In the paper entitled "Group rings in which every left ideal is a right ideal" by P. Menal, which appeared in Pub. Mat. U.A.B. 6 (1977) 97-105, the following has been omitted on p. 97: "Preprint. To appear in Proc. AMS".