THE NORMAL AND MACKEY TOPOLOGIES
ON CO-ECHELON SPACES

by

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Abstract. A necessary and sufficient condition for the coincidence of the normal and Mackey topologies on a co-echelon space of order one is studied.

Introduction. Let \( a_n^{(k)} \), \( n = 0, 1, \ldots \), \( k = 1, 2, \ldots \) be such that the following conditions are satisfied.

a) \( a_n^{(k)} > 0 \), for each \( k \) and \( n \).

b) \( a_n^{(1)} \leq a_n^{(2)} \leq a_n^{(3)} \) ... \( n = 0, 1, \ldots \)

Let \( E \) and \( E^X \) be the echelon and co-echelon spaces respectively, corresponding to the steps \( a_n^{(k)} \) [1, p.419].

In this paper, the following theorem is proved.

Theorem. The normal and Mackey topologies in \( E^X \) coincide if and only if \( E \) with the normal topology is nuclear.

It is known that the normal and Mackey topologies in \( E^X \) coincide if and only if for every sequence \( (x_n) \) satisfying \( \lim_{n \to \infty} x_n \). \( a_n^{(k)} = 0 \) for \( k = 1, 2, \ldots \), it follows that \( \sum_{n=0}^{\infty} |x_n| a_n^{(k)} < \infty \), for every \( k = 1, 2, \ldots \) [3].
On the other hand, the Grothendieck-Pietsch criterium establishes that if and only if for every $k$, there exists an $N(k)$ such that $\sum_{n=0}^{\infty} a_n^{(k)} / a_n^{(N(k))} < \infty [2, p.98].$

The previous theorem is, then, an immediate consequence of the following proposition.

 Proposition. Let $k_0$ be such that for each $j = 1, 2, \ldots$, we have $\sum_{n=0}^{\infty} a_n^{(k_0)} / a_n^{(j)} = \infty$. There exists a sequence $(a_n)$, $a_n > 0$, $n = 1, 2, \ldots$, such that $\lim_{n \to \infty} a_n^{(k_0)} / a_n^{(k)} = 0$, $k = 1, 2, \ldots$ while $\sum_{n=0}^{\infty} a_n^{(k_0)} a_n^{(j)} = \infty$.

 Proof. $\sum_{n=0}^{\infty} a_n^{(k_0)} / a_n^{(j)} = \infty$, $j = 1, 2, \ldots$ implies that there exists $0 < n_1 < n_2 < \ldots$ such that

$$\sum_{n=0}^{n_1} a_n^{(k_0)} / a_n^{(1)} \geq 2$$

$$\sum_{n=n_i+1}^{n_{i+1}} a_n^{(k_0)} / a_n^{(i+1)} \geq 2^{i+1}, \quad i = 1, 2, \ldots$$

Consider, now, the sequence

$$\frac{1}{2} (a_n^{(1)})^{-1} \quad , \quad 0 < n < n_1$$

$$a_n^* = \frac{1}{2^{i+1}} a_n^{(i+1)}^{-1} \quad , \quad n_i + 1 < n < n_{i+1}, \quad i = 1, 2, \ldots$$
It is obvious that \( \lim_{n \to \infty} a_n^{(k)} = 0 \), \( k = 1, 2, \ldots \) because given \( k \), we have
\[
a_n^{(j-1)} \cdot a_n^{(k)} \leq 1, \quad j \geq k, \quad n = 1, 2, \ldots .
\]
However, \( \sum_{n=0}^{\infty} a_n \cdot a_n^{(k)} = \infty \).

REFERENCES


3 M. Tort Pinilla, Consideraciones sobre la topología normal en los espacios de Köthe, Actas de las primeras jornadas matemáticas luso-espai-

Appendix. The referee has kindly pointed out to us that it is not necessary to take \( a_n^{(k)} > 0 \) for each \( k \) and \( n \). Supposing \( a_n^{(k)} \geq 0 \) for each \( k \) and \( n \), and that for each \( k \in \mathbb{N} \) there exists an \( n \in \mathbb{N} \) such that \( a_n^{(k)} \neq 0 \), then we may proceed in an analogous way finding the same results.