ON BLOCK-QUASI-TRIDIAGONAL MATRICES

José Vitória

Dpto. de Matemática
Universidade de Coimbra

Abstract - We intend to invert (and to study the eigenvalues of) block-quasi-tridiagonal matrices. We also mention ways for handling the problem of eigenvalues of a block-quasi-tridiagonal matrix and we obtain upper bounds for the spectral radius of a (certain) block-quasi-tridiagonal matrix which arises in the discretization of partial differential equations of elliptic type, self-adjoint case.

1. About the inversion of block-quasi-tridiagonal matrices

1.1 - In this section we interest us for inverting block-quasi-tridiagonal matrices, that is to say, matrices of the form

\[
T = \begin{pmatrix}
A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & A_{1n} \\
A_{21} & A_{22} & A_{23} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & A_{n-1,n-2} & A_{n-1,n-1} & A_{n-1,n} \\
A_{n1} & 0 & 0 & \cdots & 0 & 0 & A_{n,n-1} & A_{nn}
\end{pmatrix}
\]

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where all blocks are of the same order and commute in pairs. Such matrices arise in the discretization of elliptic partial differential equations.

We need a result involving a matrix obtained from \( T \) by taking its determinant considering the (commuting) blocks as elements. Let \( \Delta_T = \text{det} T \) be such a matrix; \( \Delta_T \) is the formal determinant of \( T \). It is known that \( \text{det} \Delta_T = \text{det} T \). This result allows us to manipulate only matrices of low order, when solving large systems of linear equations and, like in our case, inverting large block-matrices.

1.2 - To invert the matrix \( T \) we shall use an hybrid method: classical partitioning in four blocks plus a recurring procedure.

Let us partition and note as follows

\[
T = \begin{pmatrix}
\begin{array}{c|c}
A_{1n} & \# \\
\hline
0 & \ddots \\
\hline
0 & 0 & \ddots & 0 \\
A_{n1} & 0 & \ldots & A_{nn}
\end{array}
\end{pmatrix} = \begin{pmatrix}
P & \cdots & 0 \\
\hline
R & \ddots & \vdots \\
\hline
\vdots & \ddots & D \\
S & \cdots & Q
\end{pmatrix}
\]

where \( P \) is a block-tridiagonal matrix.

It is known that the inverse of a matrix partitioned in that manner is

\[
T^{-1} = \begin{pmatrix}
K & M \\
\hline
L & N
\end{pmatrix}
\]

with the same partitioning and where

\[
K = P^{-1} - P^{-1}QM, \quad M = -NP^{-1}, \quad L = -P^{-1}QN \quad \text{and} \quad N = (S-\text{RP}^{-1}Q)^{-1}.
\]

In this way we need only to invert two matrices: \( P \) and \( (S-\text{RP}^{-1}Q) \). So we have to invert a large matrix \( P \), with commuting blocks. For achieving this, one can use a recurring procedure. We obtain a matrix \( P^{-1} = (X_{ij}) \) partitioned in the same way as \( P \) and where the blocks \( X_{ij} \) also commute in pairs.

Let us denote by \( P_I(i,j) \) the matrix obtained from \( P \) by replacing the \( j \)th block-column with the matrix \( \begin{pmatrix} 0 \\ \vdots \\ I \\ \vdots \\ 0 \end{pmatrix} \) the \( i \)th block-line.

Then, if we put

\[
\Delta = \text{det} P \quad \text{and} \quad \eta_{ij} = \text{det} P_I(i,j), \quad (i,j=1,2,\ldots,n-1),
\]

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we obtain

\[ x_{ij} = \Delta^{-1} y_{ij}, \quad (i,j=1,2,\ldots,n-1) \]

1.3 - As we have seen, for inverting a block-quasi-tridiagonal matrix we need only to invert two matrices: \( S^{-1}RP^{-1}Q \) and \( \Delta: = \text{dev} P \). But, in practice, we invert two matrices of the same order, instead of inverting a low order matrix \( S^{-1}RP^{-1}Q \) and a high order matrix \( P \). We remark that the matrices \( S^{-1}RP^{-1}Q \) and \( \Delta \) have the order of the original blocks \( A_{ij} \).

2. About the eigenvalues of a (certain) block-quasi-tridiagonal matrix

In this section we interest us for the eigenvalue problem in block-quasi-tridiagonal matrices. Here we get upper bounds for the absolute value of the eigenvalues by using a matricial norm:

Upper bounds for the spectral radius of a block-quasi-tridiagonal matrix, which arise in the discretization of partial differential equations of elliptic type, self-adjoint case.

Given the matrix

\[
T = \begin{pmatrix}
A & B & 0 & 0 & \ldots & 0 & 0 & S \\
B & A & B & 0 & \ldots & 0 & 0 & 0 \\
0 & B & A & B & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & B & A \\
B & 0 & 0 & 0 & \ldots & 0 & B & A
\end{pmatrix}
\]

we look for upper bounds of \( \rho(T) \), where \( \rho(T) \) is the spectral radius of the matrix \( T \), that is to say, \( \rho(T): = \text{Max} |\lambda(T) | \), where \( \lambda(T) \) is any eigenvalue of \( T \).

We take the (scaler) norm \( ||.||_1^{(1)} \), \( (i=1,\infty) \), of each block, so obtaining a matricial norm \( M_1(T), (i=1,\infty) \). It is known, that \( \rho(T) \leq \rho(M_1(T)), \quad (i=1,\infty) \). And it is also known that \( \rho(A) \leq ||A||_j, \quad (j=1,\infty) \), for any matrix \( A \), and any (subordinate) matrix norm \( ||.||_1 \).

(1) For \( B=(B_{ij}) \in M_{r,s}(K), K= \mathbb{R} \) (or \( \mathbb{C} \)), we let

\[ ||B||_1: = \text{Max}_{j=1,2,\ldots,s} \{ \sum_{i=1}^{r} |B_{ij}| \}, \quad ||B||_{\infty}: = \text{Max}_{i=1,2,\ldots,r} \{ \sum_{j=1}^{s} |B_{ij}| \}. \]
Hence we have

\[ p(T) \leq \| A \|_1 + 2 \| B \|_1, \quad (i=1, \omega) \]

a very simple upper bound for the eigenvalues of \( T \).

Remark

The complete version of this paper is to appear in "Revista de Universidade de Coimbra".

PRINCIPAL REFERENCES


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Inversion of matrices partitioned in commuting blocks.

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