ON THE DOMAIN OF ATTRACTION OF STABLE LAWS

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Abstract:
Let $f$ be a function defined on $R$, and assume we may consider in $R$ a finite number $s+1$ of intervals such that $f$ is monotone in each of them. The minimum value $r$ that $s$ may assume is the variation index of $f$. Let $f$ be a non-negative, integrable, unimodal function possessing $k$-th order derivative. If the variation order of $f^{(i)}$ is $i+1$, $0 \leq i \leq k$, we shall say that $f$ is unimodal of order $k$. If $f$ is unimodal of order $k$ for all $k \in N$, we shall say that $f$ is totally unimodal. We shall prove that any stable distribution possesses a totally unimodal distribution in its domain of attraction.

Let $f(.)$ be a function defined on the real line, and assume that we may consider in $R$ a finite number $s+1$ of intervals such that in each of them $f(x)$ is monotone. The minimum value $r$ of the possible values $s$ may assume is called the variation index of $f$.

Let $f(.)$ be a non-negative, integrable, unimodal function possessing $k$-th order derivative. If the variation indices of $f', f'', \ldots, f^{(k)}$ are respectively $2, 3, \ldots, k+1$, we shall say that $f$ is unimodal of order $k$. If $f$ is unimodal of order $k$ for $k=1, 2, \ldots$, we shall say that $f$ is totally unimodal.

Consider the function $f_p(x; \alpha; c_1, c_2; a_1, a_2)$, where $c_1, c_2 > 0$ and $c_1 + c_2 \alpha$, $a_1, a_2 > 0$. defined as follows:

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i) For \( x < -a_1 \), \( f_p(x; \alpha; c_1, c_2; a_1, a_2) = c_1 |x|^{-\alpha + 1} \)

ii) For \( x > a_2 \), \( f_p(x; \alpha; c_1, c_2; a_1, a_2) = c_2 |x|^{-\alpha + 1} \)

iii) For \( x \in [a_1, a_2] \), \( f_p(x; \alpha; c_1, c_2; a_1, a_2) \) is identical to the polynomial (of degree 2p if \( a_1 = a_2 \) and \( c_1 = c_2 \), and otherwise of degree 2p+1) defined by the condition that \( f_p(x), f'_p(x), f''_p(x), \ldots, f^{(p)}_p(x) \) are continuous at \(-a_1\) and \(a_2\).

It is immediate that \( f_p(x; \alpha; c_1, c_2; a_1, a_2) \) is unimodal of order \( p \). On the other hand it is easy to check that

\[
f_p((x; \alpha; 1, 1; \sqrt{p}, \sqrt{p}) = p^{-\alpha + 1/2} f_p(x / \sqrt{p}; \alpha; 1, 1, 1, 1)
\]

and hence, when \( p \) is large,

\[
f_p(x; \alpha; 1, 1; \sqrt{p}, \sqrt{p}) \sim \sum_{k=0}^{p} \gamma_k \exp(-kx^2/p)
\]

where \( \gamma_k \) are the coefficients of the development in Taylor's series of \((1-x)^{-\alpha + 1/2}\) and hence

\[
\gamma_k = \frac{\Gamma\left(\frac{\alpha + 2k + 1}{2}\right)}{k! \Gamma\left(\frac{\alpha + 1}{2}\right)} \xrightarrow{k \to \infty} \frac{\Gamma\left(\frac{\alpha - 1}{2}\right)}{\Gamma\left(\frac{\alpha + 1}{2}\right)}
\]

From this, letting \( p \to \infty \), \( f_p(x; \alpha; 1, 1; \sqrt{p}, \sqrt{p}) \) converges to the integral

\[
f(x) = \frac{1}{\Gamma\left(\frac{\alpha + 1}{2}\right)} \int_0^1 y^{(\alpha-1)/2} \exp(-y \frac{x^2}{2}) \, dy =
\]

\[
= \frac{1}{\Gamma\left(\frac{\alpha + 1}{2}\right)} \int_0^x y^{(\alpha-1)/2} \exp(-\frac{x^2}{y}) \, dy
\]

Since \( f_p(x; \alpha; 1, 1; \sqrt{p}, \sqrt{p}) \) is unimodal of order \( k \) for any \( k < p \), \( f(x) \) is totally unimodal.

On the other hand

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\[ \lim_{x \to +\infty} |x|^{\alpha+1} f(x) = \frac{1}{\Gamma(\frac{\alpha+1}{2})} \int_0^\infty y^{(\alpha-1)/2} \exp(-y) \, dy = 1 \]

and hence \( f(x) \) is in the domain of attraction of the symmetric stable law with index parameter \( \alpha^* = \min(\alpha, 2) \).

Along similar lines, it is possible to prove that the function

\[ g(x) = |x|^{-(\alpha+1)} \int_0^{\frac{x}{|x| \Gamma((\alpha+1)/2) + \frac{1}{\Gamma((\alpha+2)/2)}}} y^{\alpha} \exp(-y^2) \, dy, \]

which belongs to the domain of attraction of the stable law with parameters parameters \( \alpha^*, \beta(0 < \alpha^* = \min(2, \alpha), |\beta| < 1) \), is totally unimodal.

References
