Realizability of localized groups and spaces

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The theory of localization of nilpotent groups and spaces (see [4] for a reference) associates to each nilpotent group (space) $G$, a family $\{G_p\}$ of nilpotent groups (spaces), $G_p$ $p$-local. In this paper we study the problem of deciding if given a family $\{G(p)\}$ of groups (spaces) there is a group (space) $G$ such that $\{G(p)\}$ coincides with the family of localizations of $G$. We obtain necessary and sufficient conditions for an affirmative answer (see § 3 for a precise definition).

In the last section of this paper we apply the preceding results to the problem of fibering a space by a subspace. We show that under certain conditions it is a "local" problem in the sense that a space $E$ can be fibered by a subspace $F$ if and only if the localizations $E_p$ can be fibered by $F_p$ for all $p$.

All spaces are assumed to be of the homotopy type of CW complexes.

1. Realizability of localized groups

In this section we consider the following problem: Let $\{G(p)\}$ be a family of nilpotent groups of class $\leq c$, $G(p)$ $p$-local, and let $G(p) \rightarrow G(o)$ be $\omega$-localization (i.e. all groups $G(p)$ have isomorphic rationalizations).

We want to obtain necessary and sufficient conditions in order to insure the existence of a group $G$ with $p$-localizations isomorphic to $G(p)$. More precisely, we say that a nilpotent group $G$ of class $\leq c$ solves the problem if:

a) There are isomorphisms $G_p \cong G(p)$ and $G_o \cong G(o)$;
b) the following diagram is commutative:

\[
\begin{array}{ccc}
G_p & \xrightarrow{\cong} & G(p) \\
\downarrow & & \downarrow \\
G_o & \xrightarrow{\cong} & G(o)
\end{array}
\]

Notice that the homomorphisms \( G(p) \longrightarrow G(o) \) are data of the problem. This is important because it is known that there are non-isomorphic groups with isomorphic localizations (see [4], p.33), whereas, at least if the group \( G \) is finitely generated, \( G \) is completely determined by the homomorphism \( G_p \longrightarrow G_o \). Note also that the problem does not always have a solution. A counterexample can be constructed by taking \( G(p) = \mathbb{Z}(p) \), \( G(o) = \mathbb{Q} \) and \( G(p) \longrightarrow G(o) \) multiplication by \( p \). We will see later that there is no group \( G \) solving the problem in this case. Clearly, if we omit the condition b), we can take \( G = \mathbb{Z} \).

**Theorem 1.1** With the above notations let us consider the following conditions:

i) the problem has a solution;

ii) there exists \( \rho: G(o) \longrightarrow (\prod G(p))_o \) such that if \( h_p \) is the rationalization of the canonical projection \( \prod G(p) \longrightarrow G(p) \), then the following diagram is commutative:

\[
\begin{array}{ccc}
G(o) & \longrightarrow & G(p) \\
\downarrow & & \downarrow h_p^{-1}
\end{array}
\]

iii) let us denote \( H_p = \text{Im}(G(p) \longrightarrow G(o)) \), \( H = \cap H_p \). Given \( x \in G(o) \) there exists \( n \) such that \( x^n \in H \).
Then we have: $i \Rightarrow ii \Rightarrow iii$ and if the groups $G(p)$ are torsion free abelian groups then all three conditions are equivalent.

Proof: $i \Rightarrow ii$. Let $G$ be a group solving the problem. We can define $\rho$ as the composition $G(o) \xrightarrow{\cong} G_o \rightarrow (\Pi G(p))_o$ where the second map is the rationalization of the composition $G \rightarrow \Pi G \xrightarrow{\cong} \Pi G(p)$.

$i \Rightarrow iii$. It suffices to prove $iii$ for $G_p$ and $G_o$ instead of $G(p)$ and $G(o)$. Given $x \in G_o$, there exist $n$ such that $x^n = ry$, $y \in G$, $r: G \rightarrow G_o$ the rationalization. Let us consider the $p$-localizations of $y$, $x_p \in G_p$. Then $x_p$ rationalizes to $x$ and $x^n \in H$.

$ii \Rightarrow i$. If there exists $\rho$, we define $G$ as the pullback

\[
\begin{array}{ccc}
G & \rightarrow & \Pi G(p) \\
\downarrow & & \downarrow r \\
G(o) & \xrightarrow{\rho} & (\Pi G(p))_o 
\end{array}
\]

$G$ is a nilpotent group of class $\leq c$. Composing the top homomorphism with the canonical projections $\Pi G(p) \rightarrow G(p)$ we obtain homomorphisms $g_p: G \rightarrow G(p)$. We will show that $g_p$ is a $p$-localization i.e. $g_p$ is a $p$-isomorphism. From the hypothesis on $\rho$ we obtain the commutativity of the diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{g_p} & G(p) \\
\downarrow & & \downarrow \\
G(o) & & 
\end{array}
\] (1)

We have

\[ G = \{((x_q), y)| x_q \in G(q), y \in G(o) \text{ and } r((x_q)) = \rho y \}. \]
Let us assume \( g_p((x_q), y) = 1 \), i.e. \( x_p = 1 \). Then the above diagram yields \( y = 1 \) and so \( r((x_q)) = \rho y = 1 \). Since \( r \) is a 0-isomorphism, there exists \( n \) such that \( (x_q^n) = 1 \). But \( x_q \) belongs to the q-local group \( G(q) \), hence we can assume \((n, p) = 1\) and so we have proved that \( g_p \) is a p-monomorphism.

Let \( x_p \in G(p) \). We have to see that there exists \( m \) such that \((m, p) = 1\) and \( x_p^m = g_p a \) for some \( a \in G \). Let \( y = rx_p \in G(o) \), \( z = \rho y \in (\pi G(p))_0 \). Then, \( h_p z = y \). Since \( r: \pi G(p) \to (\pi G(p))_0 \) is a 0-isomorphism, there exists \( n \) such that \( z^n = r((x_q)) \). Since \( x_q \in G(q) \) and this group is q-local, if \( q \neq p \) we can take \( x_q = x_q^{p^k} \) with \( h = p^k m \) and \((p, m) = 1\). On the other hand \( x_q^{p^k} \) goes to \( y^n = rx_p^n \). Since \( G(p) \to G(o) \) is a q-isomorphism, we have \( x_p^{p^k m} = x_p^{p^k + t m} \) and we take \( x_p^{p^k} = x_p^m \). Let us consider \((x_q') \in \pi G(p) \). We have:

\[
(z_p^{p^k + t m} = r((x_q')^{p^k + t m}) = r((x_q')^{p^k + t}) = r((x_q'))^p \in (\pi G(p))_0
\]

Since \((\pi G(p))_0 \) is o-local, we obtain \( r((x_q')) = z^m \) and \( g_p((x_q'), y^m) = x_p^{p^k} = x_p^m \) with \((m, p) = 1\). This proves that \( g_p \) is a p-epimorphism.

Let us see now that the group \( G \) solves the problem. Since we have proven that \( g_p: G \to G(p) \) is a p-localization, we have an isomorphism \( G_p \cong G(p) \). Moreover, since the diagram \((1)\) is commutative, the homomorphism \( G \to G(o) \) is a 0-isomorphism and we have an isomorphism \( G_o \cong G(o) \). We only have to see that the diagram

\[
\begin{array}{ccc}
G_p & \cong & G(p) \\
\downarrow & & \downarrow \\
G_o & \cong & G(o)
\end{array}
\]

is commutative, but this follows from the fact that it is obtained from

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by localization. This ends the proof of \( ii \Rightarrow i \). Let us assume now that the groups \( G(p) \) are torsion free abelian groups and let us show that \( iii \Rightarrow ii \).

Given \( x \in G(o) \), let \( n \) be such that \( nx \in H \). Then for each \( p \) there is a uniquely determined \( x_p \in G(p) \) such that \( nx = rx_p \). We take \( z = (x_p) \in \Pi G(p) \) and we define \( px = z' \in (\Pi G(p))_0 \) where \( z' \) is such that \( nz' = rz \). It is then clear that \( z' \) does not depend on the \( n \) we have chosen. In this way we obtain an homomorphism \( \rho : G(o) \rightarrow (\Pi G(p))_0 \).

This ends the proof of the theorem. \( \Box \)

Now we can see that if we take \( G(p) = \mathbb{Z}(p), G(o) = \mathbb{Q} \) and \( G(p) \rightarrow G(o) \) multiplication by \( p \), then there is no group \( G \) solving the problem because condition \( iii \) in the above theorem is not satisfied.

Theorem 3.1 in [3] proves that for a given \( \rho \) the solution is uniquely determined.

We will study now under what conditions a family \( \{ G \rightarrow H_p \}_p \) of homomorphisms, where \( G \) and \( H \) are nilpotent groups, comes from a homomorphism \( G \rightarrow H \). A necessary condition is that the family \( \{ G \rightarrow H_p \}_p \) should be rationally coherent i.e. for all primes \( p, q \) the diagram

\[
\begin{array}{ccc}
G & \longrightarrow & H_p \\
\downarrow & & \downarrow \\
H_q & \longrightarrow & H_o
\end{array}
\]

should be commutative. If \( H \) is finitely generated this condition is also sufficient ([4], p.26). In general we have:
Proposition 1.2 A rationally coherent family of homomorphism \( \{ G \to H_p \}_p \) comes from a homomorphism \( G \to H \) if and only if the induced diagram

\[
\begin{array}{ccc}
H_0 & \xrightarrow{p} & (\pi H_p)_0 \\
\uparrow & & \uparrow \\
G_0 & \xrightarrow{p} & (\pi G_p)_0
\end{array}
\]

is commutative.

Proof: The "only if" part is trivial. If the above diagram commutes we have:

\[
\begin{array}{ccc}
G & \xrightarrow{\Phi} & \pi H_p \\
\downarrow \phi & & \downarrow \\
H & \to & (\pi H_p)_p
\end{array}
\]

and we get \( \phi \) because the square is a pullback \((\mathbb{I}3)\).

2. Realizability of localized spaces

Let \( \{ B(p) \} \) be a family of nilpotent connected spaces, \( B(p) \) \( p \)-local, and let \( B(p) \to B(o) \) be rationalizations (i.e. all spaces \( B(p) \) have homotopy equivalent rationalizations). We ask for the existence of a nilpotent space \( B \) and homotopy equivalences \( B_p \sim B(p) \), \( B_o \sim B(o) \) such that the following diagram is homotopy commutative:

\[
\begin{array}{ccc}
B_p & \xrightarrow{\sim} & B(p) \\
\downarrow & & \downarrow \\
B_o & \xrightarrow{\sim} & B(o)
\end{array}
\]
If such a space $B$ exists we say that $B$ solves the problem. First of all, a necessary condition for the existence of a solution is that $\pi B(p)$ must be a nilpotent space. It is not difficult to see that this is equivalent to say that there exist integers $c_n$, $n \geq 1$ such that $\pi_1 B(p)$ is a nilpotent group of class $\leq c_1$ and $\pi_1 B(p)$ is a nilpotent $\pi_1 B(p)$-module of class $\leq c_n$, for all $p$. From now on we assume $\pi B(p)$ nilpotent.

We have seen in the last section that the realizability problem for groups does not always have a solution. The same holds for spaces because if $G(p) \rightarrow G(o)$ is a counterexample for groups, we can consider $K(G(p),1) \rightarrow K(G(o),1)$.

**Theorem 2.1** There exists a nilpotent space $B$ solving the problem if and only if there is a map $\rho : B(o) \rightarrow (\pi B(p))_o$ such that if $h_p$ is the rationalization of the map $\pi B(p) \rightarrow B(p)$, then the following diagram commutes up to homotopy:

$$
\begin{array}{ccc}
B(o) & \rightarrow & B(p) \\
\downarrow_{h_p} & & \downarrow \\
B(p)_o & & \\
\end{array}
$$

Proof: If $B$ is given we take $\rho$ to be the rationalization of the composition $B \rightarrow \pi B_0 \rightarrow \pi B(p)$. Conversely, let us assume that there exists a map $\rho$ satisfying the hypothesis of the theorem. For each $i \geq 1$, we define the group $G_i$ as the pullback

$$
\begin{array}{ccc}
G_i & \rightarrow & \pi_i B(p) \\
\downarrow & & \downarrow_{r_*} \\
\pi_i B(o) & \rightarrow & (\pi_i B(p))_o
\end{array}
$$
Then $G^i$ is a nilpotent group (abelian if $i > 1$) whose localized groups coincide with the $\pi_1 B(p)$. By [3], the diagram is bicartesian and we have exact sequences:

\[
G^i \longrightarrow \pi_1 B(o) \oplus \pi_1 B(p) \xrightarrow{<\rho^*, -r^*>} (\pi_1 B(p))_0
\]

(2)

\[
G^1 \longrightarrow \pi_1 B(o) \times \pi_1 B(p) \longrightarrow (\pi_1 B(p))_0
\]

We define the space $B$ as the (weak) pullback

\[
\begin{array}{ccc}
B & \longrightarrow & \pi B(p) \\
\downarrow & & \downarrow r \\
B(o) & \longrightarrow & (\pi B(p))_0
\end{array}
\]

(3)

If we apply ([3], 3.4) to the diagram (1) we see that every $z \in (\pi_1 B(p))_0$ can be expressed as $z = r^* x \cdot \rho^* y$ and this implies, by [4], II. 7.11, that $B$ is connected. Since $\pi B(p)$ is nilpotent, [4], II.7.6 implies that $B$ is also nilpotent.

The homotopy Mayer-Vietoris exact sequence of the (weak) pullback (3) yields ([2]):

\[
\ldots \longrightarrow \pi_1 B \longrightarrow \pi_1 B(o) \oplus \pi_1 B(p) \xrightarrow{<\rho^*, -r^*>} (\pi_1 B(p))_0 \longrightarrow \ldots
\]

(4)

Since (1) is a pullback we have a canonical homomorphism $\pi_1 B \longrightarrow G^i$ and it follows from (2) and (4) that it is an isomorphism. Then $B \longrightarrow B(p)$ is a $p$-localization because $\pi_1 B \longrightarrow \pi_1 B(p)$ is also a $p$-localization. The rest of the proof is formally analogous to that of 1.1. □

Theorem 3.3 in [3] proves that for a given map $\rho$ the solution is uniquely determined up to homotopy.

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There is also an analogous of proposition 1.2.:

**Proposition 2.2** A rational cohomology of maps \( (X \to \to Y)_p \) comes from a map \( X \to \to Y \) if and only if the induced diagram

\[
\begin{array}{ccc}
Y_0 & \to & (\prod Y)_0 \\
\uparrow & & \uparrow \\
X_0 & \to & (\prod X)_0
\end{array}
\]

commutes up to homotopy. \( \square \)

3. **The problem of fibering a space by a subspace**

Let \((E,F)\) be a couple of nilpotent spaces, i.e. \(F\) is a subspace of \(E\). We say that \((E,F)\) is a fiber couple if there exists a nilpotent space \(B\) and a map \(E \to \to B\) such that \(F \to \to E \to \to B\) is homotopically equivalent to a fibration. In other words, there is a homotopy commutative diagram

\[
\begin{array}{ccc}
F & \to & E & \to & B \\
\downarrow & & \downarrow & & \nearrow \\
F & \to & \tilde{E} & \to & B
\end{array}
\]

where \(F \to \to \tilde{E} \to \to B\) is a fibration and the vertical arrows are homotopy equivalences. By [1] p.60, the fibration \(F \to \to \tilde{E} \to \to B\) turns out to be nilpotent.

To characterize fiber couples is one of the problems listed in [5].

It is not difficult to prove the following result:

**Lemma 3.1** \((E,F)\) is a fiber couple if and only if there exists a nilpotent space \(B\) and a map \(p : E \to B\) such that i) \(p_F \sim \ast\); ii) \(p_\ast : \pi_i(E,F) \to \pi_iB\) is an isomorphism for all \(i\). \( \square \)

Our goal is to relate the fact that \((E,F)\) is a fiber couple to the fact that \((E_p,F_p)\) are fiber couples for all primes \(p\). The equivalence of
bouth assertions will be obtained only under certain hypothesis.

We say that \((E, F)\) is a nice couple if \(F_0 = \ast\) or \(F \longrightarrow E\) is a rational homotopy equivalence. Recall that a space \(X\) is called quasifinite if the homotopy groups \(\pi_n X\) are finitely generated for all \(n \geq 1\) and \(H_n X = 0\) for \(n\) sufficiently large.

**Theorem 3.2** Let \(F\) be a quasifinite space and let \((E, F)\) be a nice couple. \((E, F)\) is a fiber couple if and only if \((E_p, F_p)\) is a fiber couple for all primes \(p\).

**Proof:** Since localization preserves fibrations, only the part "if" of the theorem needs a proof. Let us assume we have nilpotent fibrations \(F_p \longrightarrow E_p \longrightarrow B(p)\) for all \(p\). The exact homotopy sequence of these fibrations yields that \(B(p)\) is a \(p\)-local space. Since \((E, F)\) is a nice couple we have homotopy equivalences \(B(p)_0 \sim B(q)_0\). In order to construct a space \(B\) whose localizations coincide with the \(B(p)\), we have to see that \(\pi B(p)\) is nilpotent but since we have fibrations \(F_p \longrightarrow E_p \longrightarrow B(p)\), the nilpotency class of the homotopy groups of \(B(p)\) is bounded because the same holds for \(E_p\) and \(F_p\). Let us consider the diagram:

\[
\begin{array}{ccc}
F_0 & \longrightarrow & E_0 \\
\rho & \downarrow & \rho \\
(\pi F)_0 & \longrightarrow & (\pi E)_0 \\
& \rho & \longrightarrow \pi B(p)_0
\end{array}
\]

and the existence of the dotted map \(\rho\) follows from the fact that the couple \((E, F)\) is a nice one. Moreover the hypothesis of theorem 2.1 are fulfilled and we obtain a space \(B\) such that \(B_p \sim B(p)\).

We have to construct a map \(E \longrightarrow B\). Since we have compatible maps \(E_p \longrightarrow B_p\) we can apply proposition 2.2 and we get a map \(E \longrightarrow B\). It remains only to show that \(E \longrightarrow B\) is homotopy equivalent to a fibration.
Since $F$ is quasifinite, the composition $F \rightarrow E \rightarrow B$ is homotopically trivial ([4], p. 89) and since $\pi_i(E_p, F_p) \rightarrow \pi_i(B_p)$ is an isomorphism for all $p$, all $i$, then $\pi_i(E,F) \rightarrow \pi_i B$ is also an isomorphism. Hence $(E,F)$ is a fiber couple. □

References


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