ABOUT SOME QUESTIONS OF DIFFERENTIAL ALGEBRA CONCERNING TO ELEMENTARY FUNCTIONS

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The study of elementary functions, that is, of those functions built up by using rational functions over $\mathbb{F}$, Exponentials, Logarithms and Algebraic operations, began somewhat systematically with the diverse works that Joseph Liouville did in the 1830's. Although his capital aim was to obtain some result about the integration by means of elementary functions, along his way he had to study aspects more circumscribed to the structure of these functions. About the first point we must say that, certainly, Liouville obtained a result that, even improved since that time, has not changed essentially. It suffices, for example, to compare the work of Liouville given in [2] and Rosenlicht's - in [6]. We can observe then that the introduction of new language and new technique has only linearized the problem, making clear which properties of elementary functions characterize them.

Using this new language and new technique it is possible deal more clearly with some questions about elementary functions. For example, their irredundancy, that was already established by Liouville himself in [2] and Hardy in [1]. Now, the irredundancy is a consequence of an Structure theorem (1.2) that we can quickly give by using Rosenlicht's techniques. Also, the not solvability by means of elementary functions of certain classical transcendental equations can be easily established. We study this in Section 2. Another treatment of the Structure theorem and irredundance questions (but not using Rosenlicht's techniques) can be found in the Risch's paper [5].
1.- AN STRUCTURE THEOREM.

First some definitions. All fields will be comutative and of characteristic zero.

1.A. Remember that if $E$ is a field, a map $D:E \rightarrow E$ is a derivation of $E$ if: (1) $\forall x, y \in E \ D(x+y) = D(x) + D(y)$, (2) $\forall x, y \in E \ D(xy) = xD(y) + yD(x)$. It follows from (2) that $D(1) = 0$, hence by (1) $D(z) = 0 \ \forall z \in E$. The set $C_D = \{ x \in E \mid D(x) = 0 \}$ is the set of constants of $D$. Given that $\forall x \in E \ D(x^n) = nD(x)x^{n-1}$, $D(x^{-1}) = -D(x)x^{-1}$ we have that $C_D$ is a subfield of $E$.

A field $E$ with a family of derivations $\Delta$ is a differential field; then, $C = \bigcap_{D \in \Delta} C_D$ is the field of constants of the differential field $E$.

Let $E \subset F$ two differential fields. The extension $E \subset F$ is differential if $\forall D \in \Delta_F, D|_E \in \Delta_E$. Although two different derivations of $\Delta_F$ can coincide over $E$, we won't distinguish between $\Delta_E$ and $\Delta_F$. Let $C_E, C_F$ be the respective constant fields. We have $C_E \subset C_F$. When the equality holds we say that the extension is with the same field of constants.

EXAMPLES: $C(X_1, \ldots, X_n)$ with $\Delta = (\delta/\delta x_i)_{i=1}^n$ is a differential field. $C(X) \subset C(X, e^x)$ is a differential extension with the same field of constants.

1.B. The elementary nature is then formulated in the next way: let $E$ be a differential field; $x, y \in E$. Then

$- y = \log(x) \iff Dy = Dx/x \ \forall D \in \Delta$ (y is Logarithm of x)
$- y = \exp(x) \iff Dy/y = Dx \ \forall D \in \Delta$ (y is Exponential of x)

If $E \subset F$ is a differential extension, $y \in F$ is elementary over $E$ if and only if

- either $y$ is algebraic over $E$
- or $y = \log(x)$ being $x \in E$
- or $y = \exp(x)$ being $x \in E$.

The differential extension $E \subset F$ is Elementary if $F = E(\theta_1, \ldots, \theta_n)$ with $\theta_1$ elementary over $E$, and $\theta_i$ elementary over $E(\theta_1, \ldots, \theta_{i-1}) \ \forall i \geq 2$. Then, $\text{Card} \Delta_E = \text{Card} \Delta_F$.

1.C. The tool which allow us to linearize the arguments is the Module of Differentials. A fast construction of it (sufficient for us) is the following:
let $E \subset F$ be fields and consider the $F$-vector space generated by the symbols $\{dx\}_{x \in F}$. Let us impose them the following relations:

1. $\forall x, y \in F \ d(x+y) = dx + dy$
2. $\forall x, y \in F \ d(xy) = xdy + ydx$
3. $\forall x \in E \ d(x) = 0$.

Then we get a $F$-vector space called the **Module of the Differentials of** $E \subset F$. Its symbol is $\Omega_{F/E}$.

Remember too that if $\{x_i\}_{i=1}^r$ are elements of $F$, then they are algebraically independent over $E$ if and only if the family $\{dx_i\}_{i=1}^r$ is $F$-linearly independent on $\Omega_{F/E}$. So $\text{Tr.deg.}_{F}E^F = \dim_F(\Omega_{F/E})$. (see [6], Prop.3)

The next result, due to Rosenlicht, is a fundamental one for this work:

**1.1.- THEOREM.** Let $E \subset F$ be a differential extension with the same field of constants. Let $C$ be this field and take $y_1, \ldots, y_n \in F$, $z_1, \ldots, z_r \in F\setminus\{0\}$ and $\{c_{ij}\}_{i=1}^n \subset C$ such that $\forall i = 1, \ldots, n, \forall D \in \Delta$

$$\frac{r}{j=1} c_{ij} Dz_j + Dy_i \in E$$

- either $\text{Tr.deg.}_{E}(y_1, \ldots, y_n, z_1, \ldots, z_r) \geq n$
- or the $n$ elements of $\Omega_{F/E}$:

$$\frac{r}{j=1} \frac{c_{ij}}{z_j} dz_j + dy_i, \quad i = 1, \ldots, n$$

are $C$-linearly dependent.

**Proof:** see Theorem 1. of [6].

**1.D.** Let $F$ be a differential field. We say that the equality $Y = \text{Log} X$ has a solution in $F$ if there are elements $x, y \in F$ verifying it. It is natural, then, to ask how many solutions of this equality there are in an elementary extension $E \subset F$. The following theorem, from which Risch gives another version in [5], answers this question. Previously some notation:

Let $E \subset F$ be an elementary differential field extension with the same field of constants: $E \subset F = E(\Theta_1, \ldots, \Theta_n)$. Let

$$y_1 = \text{Log} x_1, \ldots, y_r = \text{Log} x_r$$

the not algebraic cases among the $\Theta_i$'s; that is, $r = \text{Tr.deg.}_{E}F$ and for
each $\Theta_i$ not algebraic (over the preceeding subextension) there exists $x_j$ or $y_j$ such that $\Theta_i = x_j$ or $y_j$ depending on whether $\Theta_i$ is Exponential or Logarithm. Suppose they are arranged according to their order of appearance and that $\overline{E}$ is an algebraic closure of $E$.

1.2.** Theorem.** On the abovementioned hypothesis if the equality $Y = \log X$ holds in $F$, for any solution $x, y$ there exist $c_1, \ldots, c_r \in \mathbb{C}, f, g \in \overline{E} \cap F$, and $n_1, \ldots, n_r, n \in \mathbb{Z}$ such that

$$y + c_1 y_1 + \ldots + c_r y_r = f, \quad x^{n_1 x_1^{n_2^{\ldots^{n_r}}} = g}. $$

**Proof:** if the equality holds in $F$ we can consider the system

$$\begin{cases}
\forall i \quad D y_i - D x_i / x_i = 0 \in \overline{E} \\
D y - 1 / x = 0 \in \overline{E} \quad \forall D \in \Delta.
\end{cases}$$

By Theorem 1.1 we get

- either $\text{Tr.deg.}_E (y_1, \ldots, y_r, y, x_1, \ldots, x_r, x) \geq r + 1$

- or the elements of $\Omega_{F/E}$: $(dy_i - 1 / x_i dx_i), i = 1, \ldots, r$,

$(dy - 1 / x dx)$ are $E$-linearly dependent.

Here it is clear that only the second condition is possible.

So there exist $c_1, \ldots, c_r, c \in \mathbb{C}$ not all zero such that

$$\begin{align*}
(1) \quad c (dy - 1 / x dx) + \sum_{i=1}^r c_i (dy_i - 1 / x_i dx_i) &= 0.
\end{align*}$$

We can also take $c \neq 0$ since otherwise

$$\sum_{i=1}^r c_i (dy_i - 1 / x_i dx_i) = 0.$$  

But if $y_r = \Theta_j$ for some $j$, because of the elementarity of $E \subseteq F$, each $dy_i, dx_j$ except $dy_r$ is a linear combination of the preceding $r-1$ $d\Theta_s$ with coefficients in $F$. But they are $F$-linearly independent because of 1.C. So $c_r = 0$. The same happens if $x_r = \Theta_i$ for some $i$. Applying repeatedly this argument we get $c_1 = \ldots = c_r = 0$, not possible.

Hence, dividing by $c$, we can assume

$$\begin{align*}
(2) \quad dy + c_1 dy_1 + \ldots + c_r dy_r &= 1 / x dx + c_1 / x_1 dx_1 + \ldots + c_r / x_r dx_r.
\end{align*}$$

Consider now a maximal $Q$-linearly independent system among the
\{1, c_1, \ldots, c_r\} : \{e_1, \ldots, e_k\} \text{ such that } e_1 = 1. \text{ Then}

\forall i : c_i = \sum_{j=1}^{k} q_{ij} e_j, q_{ij} \in Q \forall i, j. \text{ Therefore}

\frac{1}{x} dx + c_1 \frac{1}{x} dx_1 + \ldots + c_r \frac{1}{x_r} dx_r = e_1 \frac{1}{x} dx + \sum_{j=1}^{k} q_{ij} e_j dx_1 + \ldots +

\sum_{j=1}^{k} q_{rj} e_j \frac{1}{x_r} dx_r = e_1 (\frac{1}{x} dx + \sum_{i=1}^{r} q_{i1} \frac{1}{x_i} dx_i) + \ldots + e_k (\sum_{i=1}^{r} q_{ik} \frac{1}{x_i} dx_i) =

= e_1 f_1 + \ldots + e_k f_k, \text{ being } f_1 = x x_1 \ldots x_r;

\vdots

f_k = x_1 \ldots x_r.

Then

\left(2'\right) \; d(y + c_1 dy_1 + \ldots + c_r dy_r) = e_1 f_1 + \ldots + e_k f_k.

By Prop. 4. of [6] we have

- y + c_1 y_1 + \ldots + c_r y_r = g \in E \cap F

- f_i \in E \cap F \forall i.

So \, x x_1 \ldots x_r \in E \cap F. \text{ But if } \forall i \; q_{il} = m_{il}/m, \; m_{il}, \; m \in \mathbb{Z} \text{ we get}

x^{m_{il}} x^{m_{r1}} = f \in \overline{E} \cap F, \text{ q.e.d.}

Sometimes it is possible to give a complete description for the solution of \( Y = \text{Log} \, X \). This happens when \( E \) is a classical differential field:

1.3.- Theorem. \quad On the hypothesis of Theorem 1.2, suppose moreover that \( E = C(z) \), \( C \) the field of constants of \( E \) and \( z \in C \) such that \( \forall D \in \Delta \; Dz \in C \).

Then, any solution of the equality can be written in the form

\[ y = c_1 y_1 + \ldots + c_r y_r + c \]

\[ x = x_1 \ldots x_r c', \text{ being } c_1, \ldots, c_r \in Q, c, c' \in C. \]

Proof: applying the same argument used in 1.2 and taking the system

\[
\begin{align*}
\forall i \; Dy_i - Dx_i/x_i &= 0 \in C \\
Dy - Dx/x &= 0 \in C \forall D \in \Delta,
\end{align*}
\]
we get there exist q₁, ..., qᵣ ∈ Q such that

\[ \frac{q₁}{x₁} \ldots \frac{qᵣ}{xᵣ} \in C \cap F. \]

But any derivation has only one extension for an algebraic extension of E ([8] Cap. 2, 17, Cor. 2). So C ∩ F is a field of constants and given that ECF is an extension with the same field of constants we have C = C ∩ F. Therefore

(1) \[ x = \frac{q₁}{x₁} \ldots \frac{qᵣ}{xᵣ} c', \quad c' ∈ C. \]

Deriving (1) yields

\[ \forall D ∈ Δ, Dy = \frac{Dx'}{x} = \frac{D(x₁ \ldots xᵣ)}{(x₁ \ldots xᵣ)} = \frac{q₁Dx₁'}{x₁} + \ldots + \frac{qᵣDxᵣ'}{xᵣ}. \]

So

\[ y = q₁y₁ + \ldots + qᵣyᵣ + c, \quad c ∈ C, \quad q.e.d. \]

Remark: it can happen that \[ \frac{q₁}{x₁} \ldots \frac{qᵣ}{xᵣ} \notin F. \] However, it is an algebraic point that doesn't disturb the elementarity of the process.

2. SOME CONSEQUENCES.

2.A. The first conclusion we draw from 1. is that we'll name The **Irredundance of Elementary Functions**. This means that building up elementary extensions by means of algebraic elements, logarithm elements or exponential elements are completely independent processes: no one of them can be obtained from the others.

In order to set the problem we'll use an adequate language; we say that the differential extension ECF is **Algebraic** if the field extension ECF so is; it is **Logarithmic** if \[ F = E(θ₁, ..., θₙ) \] such that \[ θ₁ = \log \psi₁, θᵢ = \log \psiᵢ \], \[ θᵢ ∈ E, \psiᵢ ∈ E(θ₁, ..., θᵢ₋₁) \] \( ∀ i ≥ 2 \). Changing Log by Exp we have an **Exponential** extension.

2.1. **Lemma.** Let ECF = E(θ) be a differential extension with the same field of constants C and θ ∉ E.

(1) If \[ \forall D ∈ Δ, Dθ ∈ E, \] then θ is transcendental over E.

(2) If \[ \forall D ∈ Δ, Dθ/θ ∈ E, \] then θ is algebraic over E if and only if there exists \( n ∈ N \) such that \( θⁿ ∈ E \), and the irreducible polynomial of θ over E is \( xⁿ - θⁿ \), n being the least of these naturals.
Proof: assume $\Theta$ to be algebraic over $E$ and let $P(X) = X^n + a_1X^{n-1} + \ldots + a_{n-1}X + a_n$ be the irreducible polynomial of $\Theta$. Then

\begin{equation}
(*) \quad \Theta^n + a_1\Theta^{n-1} + \ldots + a_{n-1}\Theta + a_n = 0.
\end{equation}

(1) Deriving (*) we get $\forall D \in \Delta, (Da_1 + n\Theta)\Theta^{n-1} + \ldots = 0$.

Given that $P(X)$ is the irreducible polynomial of $\Theta$ over $E$ we have that $\forall D \in \Delta Da_1 + n\Theta = 0$. So $\forall D \in \Delta D\Theta = D(-a_1/n)$ and $\Theta + a_1/n$ is a constant. Due to $E \subset F$ is with the same field of constants we get $\Theta \in E$, not possible.

(2) Now it suffices to prove that $\Theta^n \in E$. Deriving (*) we get

$\forall D \in \Delta n\Theta\Theta^n + (Da_1 + (n-1)D\Theta/\Theta)\Theta^{n-1} + \ldots + Da_n = 0$. But $a_n \neq 0$, so

$Da_n = n\Theta\Theta^n/\Theta a_n \forall D \in \Delta$. Hence $Da_n/a_n = n\Theta\Theta^n/\Theta \Rightarrow Da_n/a_n = D\Theta^n/\Theta n \Rightarrow D(a_n/\Theta n) = 0$

$\forall D \in \Delta$. So $a_n/\Theta n \in C \subset E$, and $\Theta^n \in E$, q.e.d.

2.2. Theorem. Let $E$ be a differential field with field of constants $C$. Let $E \subset F = E(\Theta_1, \ldots, \Theta_r)$ be an elemental differential extension with the same field of constants. Then:

(a) When $F$ is Logarithmic, $E \subset F$ is a purely transcendental extension. If $E \subset F$ is Exponential, $E \subset F$ is purely transcendental unless there exist $n_1, \ldots, n_r \in \mathbb{Z}$ such that $\Theta_1^{n_1} \ldots \Theta_r^{n_r} \in E$.

Let $x \in E$.

(b) The equality $Y = \log(x)$ never holds in $F - E$ if $E \subset F$ is Algebraic or Exponential.

(c) The equality $Y = \exp(x)$ never holds in $F - E$ if $E \subset F$ is Logarithmic, and if there is a solution when $E \subset F$ is Algebraic then there exists $n \in \mathbb{N}$ such that $y^n \in E$.

Moreover, if $E = C(x)$, $z \notin C$, $Dz \in C \forall D \in \Delta$ being $C$ the field of constants of $E$, $C$ algebraically closed, then there are not exceptions for the case (c).

Proof: (a) The statement is an easy consequence of Lemma 2.1 for the Logarithmic case. Assume that $E \subset F$ is Exponential and not purely transcendental extension. By Lemma 2.1 there exists $\Theta_s$, $p \in \mathbb{N}$ such that $\Theta_s^p \in E(\Theta_1, \ldots, \Theta_{s-1})$. Let $\Theta_k$ the first of them, that is, $\Theta_1, \ldots, \Theta_{k-1}$ are algebraic independent over $E$ and $\Theta_k^p \in E(\Theta_1, \ldots, \Theta_{k-1})$. Then by Theorem 1.2 we get the statement.
(b) Let \( y \) be a solution. Then, \( \forall D \in \Delta \, Dy = Dx/x \). Since \( x \in E \) we can take the differential extension \( E \subset E(y) \). By 2.1 \( y \) is not algebraic over \( E \). Suppose now \( E \subset F \) is Exponential. Then, by 1.2 we get there exist \( c_{i_1}, \ldots, c_{i_k} \in C, n, n_{i_1}, \ldots, n_{i_k} \in Z \) such that
\[
y + c_{i_1} \psi_{i_1} + \ldots + c_{i_k} \psi_{i_k} \in \overline{E} \cap F, \quad x^{n_{i_1}} \theta_{i_1} \ldots \theta_{i_k} \in \overline{E} \cap F,
\]
being \( \theta_{i_1}, \ldots, \theta_{i_k} \) a maximal algebraically independent system over \( E \) among \( \theta_{i_1}, \ldots, \theta_{i_k} \) like in 1.D.
But \( x \in E \): so \( n_{i_1}, \ldots, n_{i_k} \) are 0, and looking in 1.2 for the construction of these naturals we have \( c_{i_1} = \ldots = c_{i_k} = 0 \). Hence \( y \in \overline{E} \cap F \), not possible as we have proved above.

(c) The Lemma 2.1 assure us that if \( y \) is algebraic over \( E \) then there exist \( n \in N \) such that \( y^n \in E \). Suppose \( E \subset F \) is Logarithmic. By 1.2 we have there exist \( n_{i_1}, \ldots, n_{i_r} \), \( n \in Z \), \( c_{i_1}, \ldots, c_{i_r} \in C \) such that
\[
x + c_{i_1} \theta_{i_1} + \ldots + c_{i_r} \theta_{i_r} \in \overline{E} \cap F, \quad y^{n_{i_1}} \psi_{i_1} \ldots \psi_{i_r} \in \overline{E} \cap F.
\]
Now, by Lemma 2.1, \( \theta_{i_1}, \ldots, \theta_{i_r} \) is an algebraically independent system over \( E \), so \( c_{i_1} = \ldots = c_{i_r} = 0 \), and \( n_{i_1} = \ldots = n_{i_r} = 0 \) (look for the construction of \( n_{i_1}, \ldots, n_{i_r} \) in 1.2). Hence \( y \in \overline{E} \cap F = E \), not possible.

On the assumption that \( E = C(z) \ldots \), the statement is consequence of applying Theorem 1.3 to \( y^n \in E = C(z) \).

Remark: an example that give us an exception for (a) is:
\[
E = C(z)(\text{Exp}(2z+2z^2)), \quad F = E(\text{Exp}z, \text{Exp}z^2). \quad \text{Then, Exp}(z+z^2) \in F-E
\]
and is algebraic over \( E \).

2.B. The question of whether some transcendental equations can be solved by means of elementary functions sometimes can be answered using the Structure theorem 1.2. Let us see two classical examples:

assume \( E \subset F \) is a differential extension with the same field of constants. Let \( C \) be this field and \( E = C(z) \) such that \( \forall D \in \Delta \, Dz \in C, \, z \notin C \). Suppose \( C \) is algebraically closed and \( E \subset F \) Elementary.

Consider the equation
\[
\alpha Y = \text{Log(} \beta Y) \quad \alpha, \beta \in E.
\]
Suposse there is a solution in \( F, y \). Using the same notation of 1.3 we get

\[
y = c x_1 \cdots c_n, \quad y = c + c_1 y_1 + \cdots + c_n y_n,
\]

\( c, \bar{c} \in \mathbb{C}, c_1, \ldots, c_n \in \mathbb{Q} \). Passing to the Module of differentials, \( \Omega_{F/E} \), we have

\[
a'((x_1 \ldots x_n) c_1^{1/}x_1 dx_1 + \ldots + (x_1 \ldots x_n) c_n^{1/}x_n dx_n) =
\]

\[
= c_1 dy_1 + \ldots + c_n dy_n, \quad a' = ac/\beta.
\]

But taking the Module of differentials respect on the penultima-
tute subextension not algebraic and taking into account 1.C we have that

\[
a'(x_1 \ldots x_n) c_1^{1/}x_1 dx_1 = c_n dy_n, \text{ where } dx_n = 0 \text{ or } dy_n = 0 beca-
use of the elementarity of \( E \subset F \), being one of them not zero. Therefore \( c_n =
0 \); repeating this argument we have \( c_i = 0 \forall i \). Consequently, any solu-
tion is trivial.

As a particular case and taking \( E = \mathbb{Q}(z) \) we have that the equa-
tion \( \log(Y) = Y/z \) has no solution by means of elementary functions.

With the same hypothesis consider now the equation

\[
(\ast) \quad Y + \alpha = \beta \exp(\gamma Y) + \beta' \exp(-\gamma Y), \quad \alpha, \beta, \beta', \gamma \in E.
\]

Let \( y \) be a solution, \( y \in F \). We can suppose also that \( \exp(\gamma y) \in F \). Then by

\[
(\ast\ast) \quad \gamma y = c + c_1 y_1 + \ldots + c_n y_n, \quad c \in \mathbb{C}, c_1, \ldots, c_n \in \mathbb{Q}.
\]

Substituting for \( \gamma y \) in (\ast) we get that

\[
y + \alpha = c x_1 \cdots x_n + \beta x_1 \cdots x_n - c_1 \cdots c_n
\]

(where we have operated


d\text{adequately } \beta, \beta').

Passing now to the Module of differentials \( \Omega_{F/E} \) we get

\[
dy = \sum_{i=1}^{c_1} \beta(x_1 \ldots x_n) c_1^{1/}x_1 dx_i + \sum_{i=1}^{c_n} \beta(x_1 \ldots x_n) c_n^{1/}x_n dx_i.
\]

But taking into account (\ast\ast) we have that

\[
1/(\gamma(c_1 dy_1 + \ldots + c_n dy_n)) = \sum_{i=1}^{c_1} \beta(x_1 \ldots x_n) c_1^{1/}x_i dx_i + \sum_{i=1}^{c_n} \beta(x_1 \ldots x_n) c_n^{1/}x_i dx_i.
\]
If as above we take now the Module of differentials respect on the penultimate subextension not algebraic we get only
\[ c_1 \frac{dy_n}{x_n} + \ldots + c_n \frac{dy_n}{x_n} - \beta(x_1 \ldots x_n) c_1 \frac{dx_n}{x_n} - \beta'(x_1 \ldots x_n) c_n \frac{dx_n}{x_n}. \]

It follows from the elementarity of \( E \subset F \) that either \( dx_n \) or \( dy_n = 0 \), one of them being not zero. Then

\[ -dx_n = 0 \Rightarrow c_n = 0. \]
\[ -dy_n = 0 \Rightarrow \text{either } c_n = 0 \text{ or } x_1 \ldots x_n = \beta'/\beta \in E. \]

The last equality can hold in \( F \), but if we assume that \( \beta, \beta', \gamma \in C \) then

\[ \frac{2c_1}{x_1 \ldots x_n} \in C \Rightarrow \gamma y \in C \Rightarrow y \in C. \]

Repeating this argument we conclude that if \( \beta, \beta', \gamma \in C \) any solution of (*) is constant, that is, trivial.

Taking \( E = \mathbb{C}(z) \), \( \alpha = -\beta \), \( \beta' = -\beta = -h/2i \), \( \gamma = i \) we get that the equation \( Y = z + h \sin(Y) \), \( h \in \mathbb{C} \) (Kepler's equation) has not a solution by means of elementary functions.

2. C

As a final application of 1.1 we give a result of Ostrowski proved in [4]; this is an example of how the methods purposed by Rosenlicht simplify the arguments. The result permit, under certain conditions, to transform algebraic relations into linear relations.

2.3.-Proposition. Let \( E \subset F \) be a differential extension with the same field of constants, \( C \). Let \( y_1, \ldots, y_n \) be elements of \( F \) such that \( \forall D \in \Delta \) \( Dy_i \in E \) \( \forall i \). Then, if \( y_1, \ldots, y_n \) are algebraically dependent over \( E \) there are \( c_1, \ldots, c_n \in C \) such that \( c_1 y_1 + \ldots + c_n y_n \in E \).

Proof: given that \( y_1, \ldots, y_n \) are algebraically dependent over \( E \), by 1.1 we get that \( dy_1, \ldots, dy_n \in \Omega_{E/F} \) are \( F \)-linearly dependent. So there exist \( c_1, \ldots, c_n \in C \) such that \( c_1 dy_1 + \ldots + c_n dy_n = 0 \). So \( d(c_1 y_1 + \ldots + c_n y_n) = 0 \) and by 1.1 \( c_1 y_1 + \ldots + c_n y_n \in CE \). But \( D(c_1 y_1 + \ldots + c_n y_n) \in E \) (hyp), so by Lemma 2.1 we get that \( c_1 y_1 + \ldots + c_n y_n \in E \) q.e.d.

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