The purpose of this talk is to report on some results concerning a classification problem for a special kind of cohomology theories. For the beginning, however, I would like to consider a different and perhaps more concrete problem which may serve as motivation for the rest.

Let $K^*(-)$ denote ordinary complex $K$-theory. We consider it as a $\mathbb{Z}/2$-graded theory defined on the category $CW_*$ of pointed spaces of the homotopy type of a CW-complex. Recall that $K^0(S^0) \cong \mathbb{Z}$, $K^1(S^0) = 0$ and that there is a natural equivalence $\beta: K^0(X) \sim K^0(S^2 \wedge X)$, the Bott isomorphism. $K^*(-)$ is usually considered as a multiplicative theory, the product being induced by the tensor product operation of complex vector bundles. We may ask the following

**Question 1:** Are there products in $K^*(-)$ different from the ordinary one and, if so, can one describe the set $\text{Prod}(K)$ of all isomorphism classes of such products in some reasonable way?
Note that all products we consider here are assumed to be with unit, associative and commutative in the graded sense. Moreover, two products $\mu, \mu' : K^*(X) \otimes K^*(X) \to K^*(X)$ are called isomorphic if there is an isomorphism $\theta : K^*(-) \to K^*(-)$ of cohomology theories with values in the category $\text{Ab}$ of abelian groups (an additive isomorphism) such that the following diagram commutes:

\[
\begin{array}{ccc}
K^*(-) \otimes K^*(-) & \xrightarrow{\mu} & K^*(-) \\
\downarrow \theta \otimes \theta & & \downarrow \theta \\
K^*(-) \otimes K^*(-) & \xrightarrow{\mu'} & K^*(-)
\end{array}
\]

To answer the question above one could certainly try to construct elements $\mu \in K^0(\text{BU} \times \text{BU})$ with appropriate properties and then determine the set of all such elements. Here, however, we will adopt a different point of view.

Let $A$ be an ungraded commutative ring with unit. For any such ring $A$ we consider the set $C(A)$ of all isomorphism classes $[T]$ of $\mathbb{Z}/2$-graded multiplicative cohomology theories $T^*(-)$ with coefficient ring $T^*(S^0)$ of the form $T^0(S^0) = A, T^1(S^0) = 0$ ($\mathbb{Z}/2$-graded ring theories with coefficients $A$ for short). Clearly, $[K] \in C(\mathbb{Z})$. Using this notation we ask the following unprecise question:

**Question 2:** Given a ring $A$, can one describe the set $C(A)$ or at least some interesting subsets of $C(A)$ in an explicit way?

Of course, this is just the classification problem for $\mathbb{Z}/2$-graded ring theories with coefficients $A$.

Now we remark that there is a connection between Question 1 and Question 2. Put $A = \mathbb{Z}$ and let $C_K(\mathbb{Z})$ denote the subset of
C(\mathbb{Z}) whose elements are all isomorphism classes \([T] \in C(\mathbb{Z})\) with the property that \(T^*(-)\) is additively isomorphic to \(K^*(-)\), i.e. \(T^*(-) \cong K^*(-)\) as \(\mathbb{Z}/2\) -graded cohomology theories with values in the category \(\mathbf{Ab}\). Suppose \([T] \in C_K(\mathbb{Z})\), let \(\theta: T^*(X) \sim K^*(X)\) be an additive equivalence and suppose \(\alpha: T^*(X) \otimes T^*(X) \to T^*(X)\) is a product on \(T^*(-)\). Then \(\theta \circ \alpha \circ (\theta^{-1} \otimes \theta^{-1})\): \(K^*(X) \otimes K^*(X) \to K^*(X)\) defines a product on \(K^*(-)\). Moreover, different equivalences \(\theta\) and isomorphic products on \(T^*(-)\) produce isomorphic products on \(K^*(-)\) and one sees easily that there is a bijection

\[
(*) \quad C_K(\mathbb{Z}) \sim \text{Prod}(K)
\]

defined by \(\alpha \mapsto \theta \circ \alpha \circ (\theta^{-1} \otimes \theta^{-1})\). This leads us to study the problem raised by Question 2 in more detail.

Observe that any \(\mathbb{Z}/2\) - graded ring theory \(T^*(-)\) with coefficients an ungraded ring \(A\) is automatically a complex-orientable theory, i.e. the canonical complex line bundle \(\eta_\infty\) over \(\mathbb{C}P_\infty\) is \(T^*(-)\)-orientable. This follows immediately from [1], p.399. Let \(m: \mathbb{C}P_\infty \times \mathbb{C}P_\infty \to \mathbb{C}P_\infty\) be the classifying map of the bundle \(\eta_\infty \times \eta_\infty\) and let \(x \in T^0(\mathbb{C}P_\infty)\) be an Euler class of \(\eta_\infty\) (a \(\mathbb{C}\)-orientation of \(T^*(-)\)). Then, as is well known, the formal power series

\[F(x_1, x_2) := m^*(x) \in A[[x_1, x_2]]\]

is a one-dimensional commutative formal group law on \(A\) (a formal group on \(A\) for short), where \(x_i = \text{pr}_i^*(x) \in T^*(\mathbb{C}P_\infty \times \mathbb{C}P_\infty)\). Now formal groups corresponding to different \(\mathbb{C}\) - orientations of the same theory are isomorphic and isomorphic theories with the same coefficient ring produce isomorphic formal groups, so if we associate to any \(\mathbb{Z}/2\)-graded ring theory \(T^*(-)\) with coefficients \(A\) its formal group

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we get a map
\[ \phi : C(A) \rightarrow FG(A) \]
where \( FG(A) \) denotes the set of (strict) isomorphism classes of formal groups over \( A \). We will use this map to get an answer to Question 2 in some particular cases.

Suppose first that \( A = k \) is a field. If the characteristic of \( k \) is 0, classical results imply that \( C(k) \) consists of only one element, namely \( H^{**}(-;k) \). For fields of positive characteristic, however, the situation changes. First we have:

**Theorem 1:** Let \( k \) be a field of characteristic \( p > 2 \). Then the map \( \phi : C(k) \rightarrow FG(k) \) is a bijection.

**Remark:** For \( p=2 \) we have only partial results. In this case, the map \( \phi \) is surjective but not injective. Difficulties arise from the fact that all elements of \( C(k) \) different from \( H^{**}(-,k) \) are non-commutative.

Formal groups over fields of positive characteristic are rather well understood (see for example the book [3]). In particular, there is an important isomorphism invariant for such formal groups \( F \), their height \( h_F \in \mathbb{N} \cup \{\infty\} \). Briefly, \( h_F = n \) if \( [p]_F(x) = ax^n + \text{terms of higher order}, a \neq 0 \), and \( h_F = \infty \) if \( [p]_F(x) = 0 \). Let \( FG(k)^n \) denote the subset of \( FG(k) \) of formal groups of height \( n \) and put \( C(k)^n = \phi^{-1}(FG(k)^n) \). Then \( FG(k) = \bigcup_{n=1}^{\infty} FG(k)^n \) and \( C(k) = \bigcup_{n=1}^{\infty} C(k)^n \). The next theorem tells us that \( \mathbb{Z}/2 \)-graded ring theories with coefficients \( k \) and formal groups of equal height are very strongly related, in fact they only differ by their multiplicative structure:
Theorem 2: Let \( p \) be any prime and suppose \( T_1^*(-), T_2^*(-) \) are \( \mathbb{Z}/2 \)-graded ring theories with coefficients \( k \), a field of characteristic \( p \). Then \( T_1^*(-) \) and \( T_2^*(-) \) are isomorphic as cohomology theories with values in the category of \( k \)-vector-spaces if and only if their formal groups are of the same height.

Recall from [2],[4] that for any prime \( p \) and any positive integer \( n \) the \( \mathbb{Z}/2 \)-graded version of the \( n \)-th Morava K-theory with coefficients \( k \), \( K(n)^*(-;k) \), represents an element of \( C(k)^n \). For \( n = \infty \) we set \( K(\infty)^*(-;k) = H^\infty*(-;k) \). Note also that \( K(1)^*(-;\mathbb{F}_p) = K^*(-;\mathbb{F}_p) \). Using the same argument which lead to the bijection (*) we get from theorems 1 and 2 the

Corollary 3: Let \( k \) be a field of characteristic \( p > 2 \). Then for all \( n \in \mathbb{N} \cup \{ \infty \} \) there are bijections

\[
C(k)^n \leftrightarrow \text{Prod}(K(n)^*(-;k)) \leftrightarrow FG(k)^n.
\]

It should be noted that for \( FG(k)^n \), there are several more or less explicit descriptions available (see e.g. [3]). Let us recall very briefly one of them. Consider the power series

\[
\log_n(x) = \sum_{i \geq 0} p^{-i} x^{p^i} \in \mathbb{Q}[x]
\]

and put \( \overline{F}_n(x,y) = \log_n^{-1}(\log_n(x) + \log_n(y)) \). \( \overline{F}_n(x,y) \) is a formal group over \( \mathbb{Z}(p) \). \( \overline{F}_n(x,y) \), its reduction mod \( p \), is defined over \( \mathbb{F}_p \) and so over every field of characteristic \( p \). Let \( \overline{k}_{\text{sep}} \) be a separable closure of \( k \) and \( \overline{S}_n = \text{Aut}_{k_{\text{sep}}}^{-}(F_n) \) the automorphism group of \( F_n \) over \( \overline{k}_{\text{sep}} \). A classical result of Dieudonné-Lubin tells us that \( \overline{S}_n \) is isomorphic to the group of units of the maximal order in the central division algebra \( D_n \) of invariant
1/n and rank $n^2$ over $\mathbb{Q}_p$. Let $\Gamma$ be the Galois group $\text{Gal}(\overline{k}_{\text{sep}}: k)$.

Then $\Gamma$ acts on $S_n$ (by acting on the coefficients of power series) and there is a bijection

\[(**): \quad \text{FG}(k)^n \sim H^1(\Gamma, S_n).\]

This bijection together with the fact that formal groups of infinite height over a ring of prime characteristic are isomorphic to the additive formal group imply the following

**Corollary 4:** If $k$ is a separably closed field of odd characteristic and $n^{< \infty}$ or if $k$ is an arbitrary field of positive characteristic and $n^{= \infty}$, then, up to isomorphism, $K(n)^*\langle - , k \rangle$ is the only $\mathbb{Z}/2$-graded ring theory with coefficients $k$ and formal group of height $n$.

If $n = 1$, $S_1$ is isomorphic to the group $\mathbb{Z}_p^*$ of $p$-adic units. If $k = \mathbb{F}_p$ is a finite field, $\Gamma$ is topologically generated by the Frobenius homomorphism and one obtains $H^1(\Gamma, \mathbb{Z}_p^*) \cong \mathbb{Z}_p^*$. So corollary 3 implies a bijection

\[C(\mathbb{F}_p) \cong \text{Prod}(K^*\langle - , \mathbb{F}_p \rangle) \cong \mathbb{Z}_p^*.\]

For more general rings $A$ we have only very partial results to offer for the moment and the question seems to be difficult. To end this talk, let me just describe some results for the case $A = \mathbb{Z}$. This will be enough to answer our initial Question 1.

Let $F$ denote the set of all primes and let $F(x, y)$ be a formal group over $\mathbb{Z}$. Define the height function of $F$, $\text{ht}_F: F \to \mathbb{N} \cup \{ \infty \}$, by setting $\text{ht}_F(p) = \text{height of } F \text{ mod } p$ over $\mathbb{F}_p$. It is an isomorphism invariant of $F$. Using this notion we get some sort of global version of Theorem 2.
Theorem 5: Let $T_1^*(-)$ and $T_2^*(-)$ be $\mathbb{Z}/2$-graded ring theories with coefficients $\mathbb{Z}$ and formal groups $F_1$ resp. $F_2$. Then $T_1^*(-)$ and $T_2^*(-)$ are additively isomorphic if and only if $ht_{F_1}(p) = ht_{F_2}(p)$ for all primes $p$.

We do not know if the map $\phi: C(\mathbb{Z}) \rightarrow FG(\mathbb{Z})$ is surjective or injective in general although we have some partial results which we will not describe here. However, if we restrict our attention to the subset $C_K(\mathbb{Z})$ of $C(\mathbb{Z})$, we can be more precise.

Let $FG(\mathbb{Z})^1$ be the set of all isomorphism classes of formal groups $F$ over $\mathbb{Z}$ of height 1 at any prime, i.e. $ht_F(p) = 1$ for all $p$, and let $\phi_K$ denote the restriction of $\phi$ to $C_K(\mathbb{Z})$.

Theorem 6: There are bijections

$$\text{Prod}(K) \overset{\sim}{\longrightarrow} C_K(\mathbb{Z}) \overset{\phi_K}{\sim} FG(\mathbb{Z})^1 \overset{\sim}{\longrightarrow} \prod_{p \in \mathbb{P}} \mathbb{Z}_p^*.$$ 

One may ask what all these new products on $K^*(-)$ described by theorem 6 are good for. It turns out that there are interesting connections between them and characteristic classes $c \in H^{**}(BU, \mathcal{O})$ associated to certain integral Hirzebruch genera (i.e. ring homomorphisms)

$$\chi : U^*_x \rightarrow \mathbb{Z} \subset \mathcal{O}$$

which can be described in terms of Riemann-Roch relations. Also, to any exotic product on $K^*(-)$ their corresponds a set of "exotic Adams operations" with interesting properties.
References


