ON KÜNNETH RELATIONS

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Abstract. The aim of this note is to subsume a number of apparently quite distinct results in one general theorem. For a left exact functor $T : R-\text{Mod} \rightarrow \text{Ab}$ and a cochain complex $C^*$ we give a long exact sequence including the canonical map $H^n_T \rightarrow \text{TH}^n$, where $H^n$ is the $n$-th cohomology functor. Under the appropriate hypothesis the usual form of the Küneth relation (see [1], chap. VI) is a special case of our long exact sequence (Remark 1.2). Also the latest results of Coelho-Pezennc (see [2]) are contained in this long sequence (Proposition 2.3).

In particular, we obtain a simple proof of the following results of Osofsky on upper bounds of cohomological dimensions (see [7], [8]). If $I$ is a directed set and the cardinal number of it is no greater than $\kappa_m$, then:

1. $1.pd_{\colim I} R_i \colim M_i \leq \sup_I 1.pd_{R_i} M_i + m + 1$
   in the category of modules,

2. $cd_{\colim I} G_i \leq \sup_I cd_{R_i} G_i + m + 1$ in the category of
groups.

1. Main results. Let $R$ be a ring and let $R\text{-Mod}$ and $\text{Ab}$ denote the category of left $R$-modules and the category of abelian groups, respectively. For a left exact functor $T: R\text{-Mod} \to \text{Ab}$ denote by $T^n$ the right $n$-th derived functor of $T$ and for a cochain complex $C^* = (C^n,d^n)$ of $R$-modules we put $Z^n = \text{Ker } d^n$, $B^n = \text{Im } d^{n-1}$ and $H^n = Z^n/B^n$, where $n$ runs over the integers.

Theorem 1.1. (General theorem). If $T^kC^n = 0$ for $k \geq 1$ and all integers $n$, then there exists a long exact sequence of abelian groups

$$0 \to T^1Z^{n-1} \to H^n_T \to TH^n \to T^2Z^{n-1} \to \cdots$$

$$\cdots \to T^1Z^{n} \to T^1H^n \to T^2Z^{n-1} \to \cdots$$

$$\cdots \to T^1Z^{n} \to T^1H^n \to T^2Z^{n-1} \to \cdots .$$

Proof. The canonical short exact sequence of $R$-modules

$0 \to Z^n \overset{i^n}{\to} C^n \overset{d^n}{\to} B^{n+1} \to 0$ yields a long exact sequence of abelian groups

$$0 \to T^1Z^n \to T^1C^n \to T^1B^{n+1} \overset{a^n}{\to} T^1Z^n \to T^1C^n \overset{T^1d^n}{\to} \cdots$$

$$\cdots \to T^1B^{n+1} \overset{a^n}{\to} \cdots \to T^1Z^n \to T^1C^n \overset{T^1d^n}{\to} \cdots , \cdots$$

where $a^n_k : T^kB^{n+1} \to T^{k+1}Z^n$ is the connecting map.
By assumption $T^kC^n = 0$ for $k \geq 1$ and all integers $n$. So we get a short exact sequence

$$a) \quad 0 \rightarrow T^n Z \rightarrow T^n C \rightarrow \cdots \rightarrow T^n B \rightarrow 0$$

and a family of isomorphisms

$$b) \quad T^k B^{n+1} \rightarrow T^{k+1} Z^n \quad \text{for} \quad k \geq 1 \quad \text{and all integers} \quad n.$$ 

The sequence $a)$ yields an isomorphism

$$a') \quad \frac{T^n B^{n+1}}{\text{Im } T^n d} \rightarrow T^{1} Z^n.$$ 

Moreover, the canonical short exact sequence of $R$-modules

$$0 \rightarrow B^n \rightarrow Z^n \rightarrow H^n \rightarrow 0$$

yields a long exact sequence of abelian groups

$$c) \quad 0 \rightarrow T^n B \rightarrow T^n Z \rightarrow T^n H \rightarrow \cdots \rightarrow T^{1} B \rightarrow T^{1} Z \rightarrow T^{1} H \rightarrow \cdots,$$

where $\gamma^k_n : T^{k} H^n \rightarrow T^{k+1} B^n$ is the usual connecting map. Hence, by $b)$ we obtain the following long exact sequence of abelian groups

$$0 \rightarrow T^n Z \rightarrow T^n B \rightarrow T^n H \rightarrow \cdots \rightarrow T^{1} B \rightarrow T^{1} Z \rightarrow T^{1} H \rightarrow \cdots.$$
The functor \( T : R-\text{Mod} \rightarrow \text{Ab} \) is left exact, hence \( \text{Ker } Td^n = Tz^n \) and the commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & \text{Im } Td^{n-1} & \xrightarrow{j'} & \text{Ker } Td^n & \xrightarrow{\beta'} & H^nT & \rightarrow & 0 \\
& & \downarrow{\varphi} & & \downarrow & & \downarrow{\psi} & \\
0 & \rightarrow & TB^n & \xrightarrow{Tj^n} & Tz^n & \xrightarrow{\delta} & Tz^n/\text{TB}^n & \rightarrow & 0 \\
\end{array}
\]

yields \( \text{Ker } \psi = \text{coker } \varphi = TB^n/\text{Im } Td^{n-1} \), by the Snake Lemma. Let \( \eta : \text{Ker } \psi \rightarrow H^nT \) be the canonical inclusion. Then, finally, we obtain the long exact sequence

\[
0 \rightarrow T_{Z^n-1}^{1} \eta \cdot (\alpha_{0}^{n-1})^{-1} \rightarrow H^nT \xrightarrow{T_{\beta}^{n,\psi}} TH^n \xrightarrow{\alpha_1^{n-1} \circ \gamma_0^n} T_{Z^n-1}^{2}j_0(\alpha_{0}^{n-1})^{-1} \\
\rightarrow T_{Z^n}^{1} \beta^n \rightarrow T_{H^n}^{1} \alpha_{2}^{n-1} \gamma_1^n \rightarrow T_{Z^n-1}^{2}j_0(\alpha_{0}^{n-1})^{-1} \\
\rightarrow T_{Z^n}^{k}j_0(\alpha_{k}^{n-1})^{-1} \rightarrow T_{Z^n}^{k} \beta^n \rightarrow T_{H^n}^{k} \alpha_{k+1}^{n} \gamma_k^n \rightarrow T_{Z^n-1}^{k+1}j_0(\alpha_{k+1}^{n-1})^{-1} \rightarrow \ldots \\
\]

As a corollary we get the usual form of the Künneth relation (see [1], chap. VI).

**Remark 1.2.** If \( T_{k} = 0 \) for \( k \geq 2 \), then \( T_{Z^n}^{1} \beta^n \approx T_{H^n}^{1} \) and the short sequence

\[
0 \rightarrow \eta \cdot (\alpha_{0}^{n-1})^{-1} \circ (T_{\beta}^{n-1})^{-1} \rightarrow H^nT \xrightarrow{T_{\beta}^{n,\psi}} TH^n \rightarrow 0
\]
is exact.

Moreover, the above Theorem yields the following result.

**Corollary 1.3.** If the maps \( T^k z^n \overset{T^k \beta^n}{\longrightarrow} T^k h^n \) induced by the canonical map \( z^n \overset{\beta^n}{\longrightarrow} h^n \) are left split (i.e. there exists a map \( \rho^n : T^k h^n \longrightarrow T^k z^n \) such that \( \rho^n \circ T^k \beta^n = \text{id}_{T^k z^n} \)), then there exists a long exact sequence of abelian groups

\[
\begin{align*}
\cdots & \longrightarrow T^{2k+1} h^{n-k-1} \longrightarrow T^3 h^{n-2} \longrightarrow T^1 h^{n-1} \longrightarrow h^n_{T} \longrightarrow T^n_{h} \longrightarrow \\
& \longrightarrow T^{2} h^{n-1} \longrightarrow T^{4} h^{n-2} \longrightarrow \cdots \longrightarrow T^{2k} h^{n-k} \longrightarrow \cdots.
\end{align*}
\]

**Proof.** In virtue of assumption the sequence c) from the proof of Theorem 1.1 determines the short exact sequence

\[
0 \longrightarrow T^*_z^n \longrightarrow \overline{T^*_\beta^n} \quad T^n_{h} \quad \gamma^n_0 \longrightarrow T^1_{T^n_{h}} \longrightarrow 0
\]

and the split short exact sequences

\[
0 \longrightarrow T^k z^n \overset{T^k \beta^n}{\longrightarrow} T^k h^n \quad \gamma^n \overset{\delta^n}{\longrightarrow} T^{k+1} h^n \longrightarrow 0
\]

for \( k \geq 1 \) and all integers \( n \).

Hence, using the isomorphisms b) from Theorem 1.1 we obtain the following diagram
\[
\begin{align*}
\ldots & \longrightarrow T^{2k+3} \Rightarrow T^{2k+1} \Rightarrow T^{2k+2} \Rightarrow T^{2k+3} \Rightarrow 0 \\
0 & \longrightarrow T^2 \Rightarrow T^3 \Rightarrow T^4 \Rightarrow 0 \\
0 & \longrightarrow T^1 \Rightarrow T^2 \Rightarrow T^3 \Rightarrow 0 \\
0 & \longrightarrow T^0 \Rightarrow 0
\end{align*}
\]

Composing the above short exact sequences we obtain the announced long exact sequence of abelian groups.\[\]

2. Applications. Let \( I \) be a directed set. It is well known that the functor \( \text{colim}_I \) is exact. Moreover, if the cardinal number of \( I \) is no greater than \( k \), then \( \text{lim}^m_I = 0 \) for \( k \geq 2 \) (see [4]).

Let \( \{ R_i, \varphi_{ij} \}_{i,j \in I} \) and \( \{ M_i, \psi_{ij} \}_{i,j \in I} \) be directed systems of rings and abelian groups respectively, such that
each $M_i$ is a left $R_i$-module and $\psi_{ij}(r_i m_j) = \varphi_{ij}(r_i) \psi_{ij}(m_j)$ for $r_i \in R_i$, $m_j \in M_j$ and $i < j$.

(Such systems will be called consistent).

Then, $M = \text{colim}_i M_i$ is a left $R = \text{colim}_i R_i$-module and $M \cong \text{colim}_i R \otimes_{R_i} M_i$ in the category of $R$-$\text{Mod}$.

For further purposes the following lemmas will be useful.

**Lemma 2.1.** If each $M_i$ is a (pure) projective $R_i$-module for all $i \in I$, then

$$\text{lim}_i^n \text{Hom}_{R_i}(M_i N) \cong \text{Ext}_R^n(\text{colim}_i M_i, N) \cong \text{Pext}_R^n(\text{colim}_i M_i, N)$$

for any $R$-module $N$.

**Proof.** A directed system $\{M_i, \psi_{ij}\}_{i, j \in I}$ yields an exact sequence of $R$-modules (see [3], Appendix I)

$$\ldots \rightarrow i_0 \otimes_{R_{i_0}} R_{i_0} \otimes_{R_{i_0}} M_{i_0} \rightarrow \ldots \rightarrow i_0 \otimes_{R_{i_0}} R_{i_0} M_{i_0} \rightarrow i_0 \otimes_{R_{i_0}} R_{i_0} M_{i_0} \rightarrow \text{colim}_i R \otimes_{R_i} M_i \cong \text{colim}_i M_i.$$

If each $M_i$ is a (pure) projective $R_i$-module for all $i \in I$ then the above sequence is an $R$-(pure) projective resolution of $\text{colim}_i M_i$.

Applying the functor $\text{Hom}_R(-, N)$ we obtain the following chain complexes:
\[ 0 \longrightarrow \text{Hom}_R(\bigoplus_{i \in I} R_{R_i} \otimes M_i, N) \longrightarrow \text{Hom}_R(\bigoplus_{i \leq 1} R_{R_i} \otimes M, N) \longrightarrow \ldots \]
\[ \longrightarrow \bigoplus_{i \in I} \text{Hom}_R(M_i, N) \longrightarrow \bigoplus_{i \leq 1} \text{Hom}_R(M_i, N) \longrightarrow \ldots \]

Consequently, \( \lim^N_i \text{Hom}_R(M_i, N) \approx \text{Ext}_R^N(\text{colim}_I M_i, N) \)
\[ (\approx \text{Pext}_R^N(\text{colim}_I M_i, N)). \]

Let \( F_{M_i} \) denotes the free \( R_i \)-module generated by the elements of \( M_i \), then \( F_{\text{colim}_I M_i} \approx \text{colim}_I F_{M_i} \). Hence, we obtain the following generalization of Lemma 9.5 from [1].

**Lemma 2.2.** There exist \( R_i \)-(pure) projective resolutions \( \mathcal{P}_i \) of \( M_i \) forming a consistent directed system \( \{ \mathcal{P}_i, \mathcal{P}_{ij} \}_{i,j \in I} \) such that \( \mathcal{P} = \text{colim}_I \mathcal{P}_i \) is an \( R \)-(pure) projective resolution of \( \text{colim}_I M_i \).

The two lemmas stated above will be used in the sequel.

Let \( \{ C^n, \psi_{ij} \}_{i,j \in I} \) be a consistent directed system of chain complexes such that \( C^n_i \) are \( R_i \)-modules for all \( i \in I \). Put \( C^* = \text{colim}_I C^*_i \) and \( z'^n_i = \text{coKer} \, d^n_i \).

Then the following generalization of the Coelho-Pezennec result is a simple consequence of Theorem 1.1 and Lemma 2.1.

**Proposition 2.3.** (see [2]). If \( C^n_i \) are (pure) projective \( R_i \)-modules for all integers \( n \), then the following long
sequence

\[ 0 \rightarrow \lim_1 \text{Hom}_{R_i} (\mathbb{Z}_n^{-1}, N) \rightarrow H^n (C_*, N) \rightarrow \lim_1 H^n (C_*, N) \rightarrow \lim_2 \text{Hom}_{R_i} (\mathbb{Z}_n^{i-1}, N) \rightarrow \lim_1 \text{Hom}_{R_i} (\mathbb{Z}_n^{i-1}, N) \rightarrow \lim_1 H^n (C_i, N) \rightarrow \lim_3 \text{Hom}_{R_i} (\mathbb{Z}_n^{i-1}, N) \rightarrow \ldots \rightarrow \lim_k \text{Hom}_{R_i} (\mathbb{Z}_n^{i-1}, N) \rightarrow \lim_1 H^n (C_i, N) \rightarrow \lim_2 \text{Hom}_{R_i} (\mathbb{Z}_n^{i-1}, N) \rightarrow \lim_1 \text{Hom}_{R_i} (\mathbb{Z}_n^{i-1}, N) \rightarrow \lim_2 \text{Hom}_{R_i} (\mathbb{Z}_n^{i-1}, N) \rightarrow \ldots \text{ is exact}. \]

Moreover, as direct consequences of this Proposition we obtain the results of Osofsky (see [7] and [8]) and Kielpiński-Simson (see [6]).

Let \( 1.pd_R M (1.p.pd_R M) \) denote the left (pure) projective dimension of an \( R \)-module \( M \) and let \( 1.gl \dim_R (1.p.gl \dim_R) \) denote the left (pure) global dimension of a ring \( R \).

**Corollary 2.4.** i) \( 1.pd_{\text{colim}_I R_i} \text{colim}_I M_i \leq \)

\[ \leq \sup_I 1.pd_{R_i} M_i + m + 1 \]

\( (1.p.pd_{\text{colim}_I R_i} \text{colim}_I M_i \leq \sup_I 1.p.pd_{R_i} M_i + m + 1) \)

and

ii) \( 1.gl \dim \text{colim}_I R_i \leq \sup_I 1.gl \dim_{R_i} + m + 1 \)
\[(1.\text{P.gl dim } \text{colim}_I R_i \leq \sup_I \text{l.gl dim} R_i + m + 1)\].

**Proof.** Applying Proposition 2.3 to the directed system \(\{P_i, \psi_{ij}\}_{i, j \in I}\) of projective resolutions of \(\{M_i, \psi_{ij}\}_{i, j \in I}\) given by Lemma 2.2 we obtain the exact sequence

\[
0 \to \lim_1^1 \text{Hom}_{R_i} (Z_{i, n-1}, N) \to \text{Ext}_R ^1 (\text{colim}_I M_i, N) \to \lim_1^1 \text{Ext}_R ^2 (M_i, N) \to
\]

\[
\lim_1^2 \text{Hom}_{R_i} (Z_{i, n-1}, N) \to \lim_1^1 \text{Hom}_{R_i} (Z_{i, n}, N) \to \lim_1^1 \text{Ext}_R ^2 (M_i, N) \to
\]

\[
\lim_1^3 \text{Hom}_{R_i} (Z_{i, n-1}, N) \to \cdots \to
\]

\[
\lim_1^k \text{Hom}_{R_i} (Z_{i, n}, N) \to \lim_1^k \text{Ext}_R ^2 (M_i, N) \to \lim_1^k \text{Ext}_R ^2 (M_i, N) \to
\]

\[
\lim_1 \text{Hom}_{R_i} (Z_{i, n-1}, N) \to \cdots , \text{ where } R = \text{colim}_I R_i.
\]

Hence, for \(n > \sup_I 1.\text{pd}_{R_i} M_i\) we have the following isomorphisms

\[
\lim_1^2 \text{Hom}_{R_i} (Z_{i, n-1}, N) \cong \lim_1^1 \text{Hom}_{R_i} (Z_{i, n}, N)
\]

\[
\cdots \cdots \cdots \cdots
\]

\[
\lim_1^k \text{Hom}_{R_i} (Z_{i, n-1}, N) \cong \lim_1^{k-1} \text{Hom}_{R_i} (Z_{i, n}, N).
\]

Therefore, for \(n-k > \sup_I 1.\text{pd}_{R_i} M_i\)

\[
\lim_1^1 \text{Hom}_{R_i} (Z_{i, n-1}, N) \cong \cdots \cong \lim_1^k \text{Hom}_{R_i} (Z_{i, n-k}, N).
\]
But $\lim_{I}^{k} = 0$ for $k > m + 1$. Consequently,

$$\lim_{I}^{1} \text{Hom}_{R_{i}}(Z_{i}^{n-1}, N) = 0 \text{ and } \text{Ext}_{R}^{n}(\text{colim}_{i} M_{i}, N) = 0 \text{ for }$$

$$n > \sup_{I} 1 \cdot \text{pd}_{R_{i}} M_{i} + m + 1.$$ Hence,

$$1 \cdot \text{pd}_{\text{colim}_{i}R_{i}} \text{colim}_{i} M_{i} \leq \sup_{I} 1 \cdot \text{pd}_{R_{i}} M_{i} + m + 1.$$ ii) For any $R$-module $M$ we have $M = \text{colim}_{i} M_{i}$, where $M_{i} = M$ are $R_{i}$-modules for all $i \in I$. Therefore, by i)

$$1 \cdot \text{pd}_{\text{colim}_{i}R_{i}} M \leq \sup_{I} 1 \cdot \text{pd}_{R_{i}} M + m + 1 \leq \sup_{I} 1 \cdot \text{gl \ dim}_{R_{i}} M + m + 1$$

and hence $1 \cdot \text{gl \ dim}_{\text{colim}_{i}R_{i}} M \leq \sup_{I} 1 \cdot \text{gl \ dim}_{R_{i}} M + m + 1$.

The analogous results for the left (pure) projective and global dimension are obtained by the same methods.

In particular, if $\{G_{i}, \varphi_{ij}\}_{i,j} \in I$ is a directed system of groups, then for group-rings over a ring $R$ we have $R[\text{colim}_{i} G_{i}] \approx \text{colim}_{i} R[G_{i}]$.

So, by the above Corollary $\text{pd}_{\text{colim}_{i}R[G_{i}]^{\Delta R}} \leq \sup_{I} \text{pd}_{R[G_{i}]^{\Delta R}} M + m + 1$, where $\Delta R$ denotes the trivial module over the appropriate group-ring.

Therefore, we get another result due to Osofsky (see [8])
cd_R \text{colim}_i C_i \leq \sup_I cd_R C_i + m + 1, \text{ where}

cd_R \text{ denotes the R-cohomological dimension.}

More generally, if \{C_i, \varphi_{ij}\}_{i,j \in I} \text{ is a directed system of small categories, then using methods similar to those above, we obtain}

cd_R \text{colim}_i C_i \leq \sup_I cd_R C_i + m + 1.

Remark 2.5. By results from [5] and [9] we can replace the directed set \(I\) by any small category such that the functor \(\text{colim}_i\) is exact.

Now let \(R\) be a hereditary ring and let \{C^i_*, \psi_{ij}\}_{i,j \in I}\) be a directed system of chain complexes over the category \(R\)-mod.

Proposition 2.6. If \(C^i_*\) and \(C_* = \text{colim}_i C^i_*\) are chain complexes of projective \(R\)-modules for all \(i \in I\), then the following long sequence

\[
\ldots \longrightarrow \lim_I^{2k+1} H^{n-k-1}(C^i_*, N) \longrightarrow \ldots \longrightarrow \lim_I^3 H^{n-2}(C^i_*, N) \longrightarrow \\
\longrightarrow \lim_I^1 H^{n-1}(C^i_*, N) \longrightarrow H^n(C_*, N) \longrightarrow \lim_I H^n(C^i_*, N) \longrightarrow \\
\longrightarrow \lim_I^2 H^{n-1}(C^i_*, N) \longrightarrow \lim_I^4 H^{n-2}(C^i_*, N) \rightarrow \ldots \rightarrow \lim_I^k H^{n-k}(C^i_*, N) \longrightarrow \\
\longrightarrow \ldots \text{ is exact for all integers } n \text{ and any } R\text{-module } M.
\]
Proof. Because \( \{Z^n \text{Hom}_R(C_i^*, N)\}_{i \in I} = \{\text{Hom}_R(C^n_{i/B_i}, N)\}_{i \in I} \) and the sequence

\[
0 \rightarrow H_n C_i^* \rightarrow C_i^{n/B_i} \rightarrow C_i^{n/Z_i} \rightarrow 0 \quad \text{splits,}
\]

therefore

\[
\{\text{Hom}_R(C^n_{i/B_i}, N)\}_{i \in I} = \{\text{Hom}_R(H_n C_i^*, N)\}_{i \in I} \oplus \{\text{Hom}_R(C^n_{i/Z_i}, N)\}_{i \in I}
\]

and

\[
\lim_k Z^n \text{Hom}_R(C_i^*, N) = \lim_k \text{Hom}_R(H_n C_i^*, N) \oplus \lim_k \text{Hom}_R(C^n_{i/Z_i}, N).
\]

But \( C_i^{n/Z_i} \) are projective \( R \)-modules and \( \text{colim}_I C_i^{n/Z_i} = C_i^{n/Z_n} \) is a projective \( R \)-module, so by Lemma 2.1

\[
\lim_k \text{Hom}_R(C^n_{i/Z_i}, N) = \text{Ext}_R^n(C^n_{n/Z_n}, N) = 0 \quad \text{for} \quad k \geq 1.
\]

Moreover, by Universal Coefficient Theorem (see [1] chap. VI) we have natural epimorphisms \( H^n(C_i^*, N) \rightarrow \text{Hom}_R(H_n C_i^*, N) \) for all \( i \in I \). Consequently, the map

\[
\lim_k Z^n \text{Hom}_R(C_i^*, N) = \lim_k \text{Hom}_R(H_n C_i^*, N) \rightarrow \lim_k H^n(C_i^*, N)
\]

splits and an appropriate long exact sequence is determined by Corollary 1.2. 

**Corollary 2.7.** If \( \{X_i, \phi_{ij}\}_{i, j \in I} \) is a directed system of compact topological spaces, then the cochain functor commutes...
with limits. Thus, the following sequence of singular cohomology groups

\[ 0 \rightarrow \lim_{I}^{2n-1} H_{0}(X_{i}, A) \rightarrow \ldots \rightarrow \lim_{I}^{n-2} H_{n-2}(X_{i}, A) \rightarrow \]

\[ \rightarrow \lim_{I}^{n-1} H_{n-1}(X_{i}, A) \rightarrow H^{n}(X, A) \rightarrow \lim_{I} H^{n}(X_{i}, A) \rightarrow \]

\[ \rightarrow \lim_{I}^{2} H^{n-1}(X_{i}, A) \rightarrow \lim_{I}^{4} H^{n-2}(X_{i}, A) \rightarrow \ldots \rightarrow \lim_{I}^{2n-0} H^{0}(X_{i}, A) \rightarrow 0 \]

is exact for any abelian group \( A \), where \( X = \text{colim}_{I} X_{i} \).

Similarly, if \( \{ G_{i}, \phi_{ij} \}_{i,j \in I} \) and \( \{ C_{i}, \phi_{ij} \}_{i,j \in I} \) are directed systems of groups and small categories respectively, then we obtain the appropriate long exact sequence as above.

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