CENTRAL EXTENSION AND COVERINGS
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The theory of central extensions has a lot of analogy with the theory of covering spaces. It is mentioned for example in [1]. In this paper we show that the category of central extensions of a perfect group and a certain category of covering spaces of a certain space are equivalent (see Theorem 1). Then the facts about central extensions will follow from the corresponding facts about coverings (see Corollaries 1-3).

We start with some definitions to make this work self-contained.

Definition 1. (see [2] § 5) A pair \((X;\psi)\) is called a central extension of a group \(G\) if \(\psi : X \to G\) is an epimorphism and \(\ker(\psi) \subseteq \text{center} X\).

Definition 2. (see [2] § 5) The central extension \((X;\psi)\) of a group \(G\) is called universal if for every central extension \((Y;\psi)\) of \(G\) there is one and only one homomorphism \(h : X \to Y\) such that \(\psi h = \psi\).

It follows from [2] Theorem 5.3 that if a group \(G\) has universal central extension \((X;\psi)\) then \(G\) and \(X\) are perfect.

We shall denote by \(E(G)\) the category of central extensions of \(G\). Morphisms in this category are homomorphisms over \(G\).

Now we describe a category \(\text{Cov}^{ab}(X)\) of pointed abelian coverings over a connected space \(X\) with a base point. Objects of \(\text{Cov}^{ab}(X)\) are principal \(G\)-fibrations over \(X\) with a base point in the fibre over the base point of \(X\). \(G\) is a discrete abelian group. Such principal \(G\)-fibrations are regular coverings and they are induced from the universal covering of \(BG\) by a map \(f : X \to BG\). If \(E_1\) and \(E_2\) are

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two coverings induced respectively by \( f_1 : X \to BG_1 \) and \( f_2 : X \to BG_2 \) then morphisms of \( E_1 \) in \( E_2 \) in the category. \( \text{Cov}^{ab}(X) \) are those pointed maps from \( E_1 \) in \( E_2 \) over \( X \) which are induced by maps \( h : BG_1 \to BG_2 \) such that \( h \circ f_1 \) is homotopic to \( f_2 \). The category \( \text{Cov}^{ab}(X) \) has an initial object. It is the universal, pointed covering.

Let us suppose now that \( G \) is a perfect group. Then the fundamental group of \( BG \) is perfect and we can apply the "+" construction to get \( BG^+ \). \( BG^+ \) is simply-connected and therefore \( \Omega(BG^+) \) is connected.

**Theorem 1.** Let \( G \) be a perfect group. Then the categories \( \text{Cov}^{ab}(\Omega(BG^+)) \) and \( E(G) \) are equivalent. The full subcategory of \( \text{Cov}^{ab}(\Omega(BG^+)) \) which objects are connected coverings and the category of central extensions \((X,\varphi)\) of \( G \) such that \( X \)'s are perfect, are also equivalent.

**Proof.** We shall define two functors \( F : E(G) \to \text{Cov}^{ab}(\Omega BG^+) \) and \( J : \text{Cov}^{ab}(\Omega BG^+) \to E(G) \) such that the compositions \( F \circ J \) and \( J \circ F \) are natural isomorphic to the identity functors.

Let \( 1 \to H \to X \xrightarrow{\varphi} G \to 1 \) be a central extension. Then \( BH \to BX \to BG \) is a fibration. Let \( \text{tr} : H_2(BG) \to H_1(BH) \) be a transgression homomorphism in the Serre spectral sequence of this fibration. The homomorphism \( \text{tr} \) we can consider as an element \( t \in H^2(BG,H) = H^2(BG^+,H) \). We have the following long sequence of fibrations

\[
(*_X) \to \Omega g(X) \to \Omega BG^+ \xrightarrow{\delta = \Omega t} K(H,1) \to g(X) \to BG^+ \xrightarrow{t} K(H,2),
\]

where \( g(X) \) is a homotopy fibre of \( t \).

We set \( F(X;\varphi) = (\delta!(EH) \to \Omega(BG^+)) \) where \( \delta!(EH) \to \Omega(BG^+) \) is a
covering induced by $\delta$ from the universal covering over $BH$. The
base point of $\delta!EH$ we choose in the fibre over the base point of
$\Omega(BG^+)$. The homomorphism $f : (X_1, \varphi_1) \to (X_2, \varphi_2)$ of central extensions
induces a map between sequences of fibrations $(\pi_{X_1})$ and $(\pi_{X_2})$. As
a part of this map we get a commutative diagram

\[ \begin{array}{ccc}
\Omega(BG^+) & \xrightarrow{\delta_1} & BH_1 \\
& \downarrow{f_*} & \\
& \delta_2 & BH_2
\end{array} \]

This diagram induces a morphism between coverings $\delta_1!(EH_1) \to \Omega(BG^+)$
and $\delta_2!(EH_2) \to \Omega(BG^+)$ in the category $\text{Cov}_{\ab}^\ab(\Omega(BG^+))$.

Now we shall define a functor $J : \text{Cov}_{\ab}^\ab(\Omega(BG^+)) \to E(G)$. Let
$(E \to \Omega BG^+) \in \text{Cov}_{\ab}^\ab(\Omega BG^+)$ and let us suppose that $p : E \to \Omega BG^+$ is
a principal $K$ fibration. $(p : E \to \Omega BG^+)$ is induced from the uni-
versal covering over $BK$ by a map $x : \Omega(BG^+) \to BK$. We have the follow-
ing isomorphisms

$$H^1(\Omega(BG^+); K) \cong \text{Hom}(\pi_1(\Omega(BG^+); K) \cong \text{Hom}(\pi_2(BG^+); K) \cong H^2(BG^+; K).$$

Therefore there is $y \in H^2(BG^+; K)$ which corresponds to $x$ by these
isomorphisms. Let us form the following sequence of fibrations

\[ (***) \quad \Omega BG^+ \xrightarrow{\Omega y = x} K(H; 1) \to Y = \text{Fibre}(y) \to BG^+ \xrightarrow{y} K(H; 2). \]

Let $i : BG \to BG^+$ be a natural map in the "+" construction. Let

\[ (***) \quad K(H; 1) \to S = i!Y \to BG \to BG^+ \]

be a fibration induced by $i$ from the fibration

$$K(H; 1) \to Y \to BG^+. \]
After applying functor $\pi_1$ to the fibration (***), we get an exact sequence

\[(***) \quad 1 \to H \to \pi_1(S) = T \to G \to 1.\]

The action of $\pi_1(BG)$ on the fibre in the fibration (***), is trivial because this fibration is induced from the fibration over the simply-connected space $BG^+$. Therefore, the extension (***) is central.

A map in the category $\text{Cov.}^{ab}(\Omega BG^+)$ induces a homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega BG^+ & \longrightarrow & BH_1 \\
\downarrow & & \downarrow \\
\Omega BG^+ & \longrightarrow & BH_2.
\end{array}
\]

Hence we get a homotopy commutative diagram

\[
\begin{array}{ccc}
K(H_1,1) & \longrightarrow & Y_1 \longrightarrow BG^+ \\
\downarrow & & \downarrow \quad \quad \downarrow \\
K(H_2,1) & \longrightarrow & Y_2 \longrightarrow BG^+.
\end{array}
\]

and consequently a map between central extensions

\[
\begin{array}{ccc}
H_1 & \longrightarrow & T_1 \longrightarrow G \\
\downarrow & & \downarrow \quad \quad \downarrow \\
H_2 & \longrightarrow & T_2 \longrightarrow G.
\end{array}
\]

The proof that the compositions $F \circ J$ and $J \circ F$ are natural isomorphic to the identities follows immediately from definitions of $F$ and $J$, and I omit it.

If a principal $H$-fibration $E \to \Omega BG^+$ is connected then $\pi_1(\Omega BG^+) \to \pi_1(BH)$ is an epimorphism. This implies that $\pi_1(Y) = 0$ and
therefore \( H_1(T) = 0 \). Consequently \( J(E \to \Omega BG^+) = (1 + H \to T \to G + 1) \)
is an extension of \( G \) such that \( T \) is perfect.

If \( 1 \to H \to X \to G \to 1 \) is a central extension with \( X \) perfect
then \( \pi_1(\Omega BG^+) \to \pi_1(BH) \) is an epimorphism and consequently the induced
covering over \( \Omega BG^+ \) is connected.

The following corollaries, usually proved in an algebraic way,
follow immediately from Theorem 1.

**Corollary 1.** There exists a universal central extension of a perfect
group \( G \).

**Proof.** The universal central extension is an initial object in the
category \( \text{E}(G) \). The category \( \text{Cov.}^\text{ab}(\Omega BG^+) \) has an initial object. It
is a universal covering. Therefore there is an initial object in \( \text{E}(G) \).

**Corollary 2.** \((X;\varphi)\) is a universal extension iff \( H_1(X) = 0 \) and
\( H_2(X) = 0 \). Then we have \( \ker \varphi = H_2(G) \).

**Proof.** The principal fibration corresponding to \((X;\varphi)\) is
\( \Omega BX^+ \to \Omega BG^+ \). This covering is universal if and only if \( \pi_0(\Omega BX^+) = 0 \)
and \( \pi_1(\Omega BX^+) = 0 \). Hence we have that \((X;\varphi)\) is universal if and
only if \( H_1(BX^+) = H_1(X) = 0 \) and \( H_2(BX^+) = H_2(X) = 0 \). The fibration
\( \Omega BX^+ \to \Omega BG^+ \) is induced from the universal covering over \( B(\ker \varphi) \) by
a map \( \Omega BG^+ \to B(\ker \varphi) \). If it is universal then
\( \ker \varphi = \pi_1(\Omega BG^+) = \pi_2(BG^+) = H_2(BG^+) = H_2(G) \).

**Corollary 3.** The isomorphism classes of central extensions \((X,\varphi)\) of
\( G \) such that \( X \)'s are perfect, are in one to one correspondence with
subgroups of \( H_2(G) \).

**Proof.** The isomorphism classes of connected coverings over \( \Omega BG^+ \) are
in one to one correspondence with subgroups of \( \pi_1(\Omega BG^+) = H_2(G) \).
Some steps in the proofs given below can be shown using the following proposition which itself seems to be interesting.

**Proposition 1.** Let us suppose that \( 0 \to H \to X \to G \to 1 \) is a central extension of a perfect group \( G \) by a group \( H \). Then \( BH \to BX^+ \to BG^+ \) is a fibration. (The "+" construction is done with respect to a maximal perfect subgroup of \( X \).)

**Proof.** Let us assume first that \( X \) is perfect. Let \( F \) be a fibre of \( BX^+ \to BG^+ \). There is a map of a fibration \( BH \to BX \to BG \) into a fibration \( F \to BX^+ \to BG^+ \). This map induces a map of Serre spectral sequences. This map is an isomorphism on \( E^2_{*,0} \) and on \( E^\infty_{*,*} \)-terms. Therefore it is isomorphism on \( E^2_{0,*} \)-terms. This means that a map \( H_*(BH;Z) \to H_*(F;Z) \) is an isomorphism. \( F \) is a fibre of a map between nilpotent spaces therefore it is nilpotent. It implies that \( BH \to F \) is a homotopy equivalence.

Let now \( X \) be arbitrary and let \( X' \) be a maximal, perfect subgroup of \( X \). The extension \( 0 \to H' = \text{Ker}(i) \to X' \xrightarrow{i} G \to 1 \) is also central. Moreover \( BX'^+ \) is a universal cover of \( BX^+ \). If \( F \) is a fibre of \( BX^+ \to BG^+ \) then only \( \pi_1(F) \) is non-zero and it appears in the following exact sequence

\[
0 \to \pi_2(BG^+) \to \pi_1(F) \to \pi_1(BX^+) \to 1
\]

\( \pi_1(BX^+) \) is abelian. This implies that \( \pi_1(F) \) is nilpotent. Repeating once more arguments with the Serre spectral sequence we get that \( F \) is homotopically equivalent to \( K(H,1) \).

In [3] we have introduced "+" construction in the case if \( H_1(X;\mathbb{Z}_p) = 0 \). (\( \mathbb{Z}_p \) is a ring of integers localized outside \( p \).)
Definition 3. We say that $G$ is P-perfect if $H_1(G;\mathbb{Z}_p) = 0$.

We shall study central extensions of a P-perfect group $G$ by finitely generated $\mathbb{Z}_p$-modules. We shall denote this category by $E_p(G)$. We have the following proposition.

Proposition 2. Let $0 \to H \to X \to G \to 1$ be a central extension of a P-perfect group $G$ by a finitely generated $\mathbb{Z}_p$-module $H$. Then $BH \to BX^+P \to BG^+P$ is a fibration. (The $^+_p$-construction is done with respect to a maximal P-perfect subgroup of $X$).

The proof of Proposition 2 is exactly the same as the proof of Proposition 1.

Let $X$ be a P-local space. We define a category $\text{Cov}_{ab}^p(X)$. Objects of $\text{Cov}_{ab}^p(X)$ are principal $M$-fibrations over $X$ with a fixed base point in the fibre over the base point of $X$. $M$ is a $\mathbb{Z}_p$-module.

Theorem 2. Let $G$ be a P-perfect group. Then the categories $\text{Cov}_{ab}^p(\Omega G^+P)$ and $E_p(G)$ are equivalent. The full subcategory of $\text{Cov}_{ab}^p(\Omega G^+P)$ which objects are connected coverings and the category of central extensions $(X,\phi)$ of $G$ such that $X$'s are P-perfect, are also equivalent.

Corollary 4. i) There exists a universal central extension of a P-perfect group $G$ in the category $E_p(G)$.

ii) $(X,\phi)$ is a universal central extension of a P-perfect group $G$ in the category $E_p(G)$ if and only if $H_1(X;\mathbb{Z}_p) = 0$ and $H_2(X;\mathbb{Z}_p) = 0$. Then we have that $\ker \phi = H_2(G;\mathbb{Z}_p)$.

iii) The isomorphism classes of central extensions of $G$ by P-perfect groups in the category $E_p(G)$ are in one to one correspondence with
$\mathbb{Z}_p$-submodules of $H_2(G, \mathbb{Z}_p)$.

The proofs are the same as before.

**Proposition 3.** Let $\mathbb{H} \to X \to G$ be a central extension of $G$. Then there is a central extension of $G$ by $H \otimes \mathbb{Z}_p$ together with a natural map

$$O \to H \to X \to G \to 1$$

$$i\downarrow \quad \downarrow \quad \quad \quad \quad \downarrow$$

$$O \to H \otimes \mathbb{Z}_p \to X_p \to G \to 1$$

where $i$ is $\mathbb{Z}_p$-localization ($i(a) = a \otimes 1$).

**Proof.** We have a fibration

$$(*): \quad BH \to BX \to BG.$$  

Bousfield and Kan have introduced the fibrewise localization functor. After applying it to a fibration $(*$) we obtain a fibration

$$(**): \quad (BH)_p \to (BX)_p^+ \to BG$$

and a fibre map of $(*$) into $(**$).

From the fibration $(**$) we get the following exact sequence

$$O \to H \otimes \mathbb{Z}_p \to \pi_1(BX^f_p) := X_p \to G \to 1.$$  

The action of $\pi_1(BG) = G$ on fibres of $(*$) and $(**$) are compatible therefore $(**$)is a central extension. □

Proposition 3 is of course a special case of a more general result proved by algebraic method in [0]. Proposition 1 is of course well known. The related results about $"+"$ construction are also in A.J. Berrick "An Approach to Algebraic K-theory", Pitman research notes in Math. 56 (London, 1982).

**References**

