ONE MORE FACET OF A MAPPING THEOREM
FOR LIJSTERNIK SCHNIRELMANN CATEGORY

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Let \( f: X \to Y \) be a map of simply connected CW-spaces. When is \( \text{cat}(X) \leq \text{cat}(Y) \)? In [2] an answer has been given within rational homotopy category and in [3] within rational or tame homotopy theory. Here we prove a corresponding result using the theory of [1].

Let \( R \) be a subring of \( \mathbb{Q} \) containing \( 1/2, 1/3 \) throughout.

Proposition: Let \( f: X \to Y \) be a map of simply connected \( R \)-local CW-spaces. Let \( \Omega Y \) be decomposable over \( R \); assume that the homomorphism \( (\Omega f)_*: \pi_*(\Omega X) \to \pi_*(\Omega Y) \) of \( M \)-Lie algebras (in the sense of [1]) has a left inverse in the category of \( M \)-Lie algebras.

Then \( \text{cat}(X) \leq \text{cat}(Y) \).

We first explain the notions used in the proposition.

We work in the homotopy category of pointed spaces.

A connected complex \( X \) is called "\( R \)-local", if the reduced homology \( \tilde{H}_*(X;\mathbb{Z}) \) is an \( R \)-module. For \( X \) nilpotent we denote by \( X X_R \) the localization of \( X \) with respect to the set of primes not invertible in \( R \).

For \( n > 0 \) we set

\[
\Omega^n_R := \begin{cases} 
S^n_R & \text{n odd}, \\
\Omega \Sigma S^n_R & \text{n even}, 
\end{cases}
\]
where $\Sigma, \Omega$ denotes the suspension resp. loop space functor.

For any connected finite dimensional complex $X$ let $M^i(X) := R$ for $i = 0$ and $M^i(X) := [X, \Omega^1_R]$ for $i > 0$.

An $H$-space $E$ is called "decomposable over $R"$, if $E$ is homotopy equivalent to a weak direct product $\bigoplus_{i \in I}^\sim \Omega^1_{R_i}$.

Let $E$ be a connected grouplike $R$-local $H$-space. Then the Lie algebra $\pi_*(E)$ (with the Samelson product as Lie bracket) has an additional structure as an $M$-Lie algebra (see [1], chap. V, (2.12) and [4], section 7), i.e. there is an operation $(i, r > 0)$

$$\pi_i(E) \times M^i(S^r) \to \pi_r(E), \ (\alpha, \zeta) \to \alpha \cdot \zeta,$$

defined by the formula

$$\alpha \cdot \zeta := \begin{cases} \alpha \zeta & \text{for } i \text{ odd}, \\ r_E(\Omega \alpha) \zeta & \text{for } i \text{ even}, \end{cases}$$

where $r_E: \Omega \Sigma E \to E$ is an $H$-retraction (see [4], section 7). (Note that we do not notationally distinguish between maps and there homotopy classes). This operation obeys certain laws ([1], loc.cit.) which we do not need here.

Proof of the proposition: Let $\Omega Y$ be homotopy equivalent to $\bigoplus_{i \in I}^\sim \Omega^1_{R_i}$. Let $\alpha_i: S^i \to \Omega Y$ represent a generator (over $R$) of the direct summand $\pi^{n_i}_{1}(\Omega^1_{R_i})$ of $\pi^{n_i}_{1}(\Omega Y)$.

For $m > 0$ let $M^{ni,*} := \bigoplus_{j > 0}^\sim M^m(S_j)$; the maps $\alpha_i$ induce maps $M^{ni,*} \to \pi_*(\Omega Y)$, $\zeta \to \alpha_i \cdot \zeta$, such that the map

$$\bigoplus_{i \in I}^\sim M^{ni,*} \to \pi_*(\Omega Y), \ (\zeta_i)_{i \in I} \to \Sigma \alpha_i \cdot \zeta_i,$$

is an isomorphism of $R$-modules. (This is proved in [1], chap. V, (3.13) in case $H_*(\Omega Y; R)$ is of finite type over $R$; but this assumption is not needed).

Let now $\phi$ be a left inverse to $(\Omega f)_*: \pi_*(\Omega X) \to \pi_*(\Omega Y)$.
We define maps \( \beta_i : \Omega_R^{n_i} \to \Omega X \) using \( \phi(a_i) \) as follows:

\[
\beta_i := \begin{cases} 
\phi(a_i) & \text{for } n_i \text{ odd,} \\
r_{\Omega X}(\Omega X(\phi(a_i))) & \text{for } n_i \text{ even.}
\end{cases}
\]

Since \( \Omega X \) may be thought of as an associative H-space with unit, the maps \( \beta_i \) can be multiplied together to a map \( \beta : \Omega R \to \Omega X \) with \( \beta|_{\Omega R^{n_i}} \) homotopic to \( \beta_i \). We have \( \beta_i^*(a_i) = \phi(a_i) \), hence \( \beta_i^*(a_i \circ \zeta_i) = \beta_i^*(a_i) \circ \zeta_i \). For \( n_i \) even this equation follows (see [4], section 7) from the fact that \( \beta_i \) is an H-map; for \( n_i \) odd the equality is trivial. (Note that also for \( n_i \) odd \( \pi_*^*(\Omega_R^{n_i}) \) is an \( M \)-Lie algebra, because \( \Omega_R^{n_i} \) is group-like (compare [1], chap. V, (1.9)).) We now deduce the formula \( \beta_i(a_i) \circ \zeta_i = \phi(a_i) \circ \zeta_i = \phi(a_i \circ \zeta_i) \), because \( \phi \) commutes with the operation "\( \circ \)" by assumption. Hence \( \beta_i^* : \pi_*^*(\Omega_R^{n_i}) \to \pi_*^*(\Omega X) \) coincides with \( \phi|_{\pi_*^*(\Omega_R^{n_i})} \), hence \( \beta_* = \phi \). It follows that there is a left inverse to \( \Omega f \) up to homotopy. This implies \( \text{cat}(X) \leq \text{cat}(Y) \) by [3], lemma 2.

Remark: By [4], appendix, the H-space \( \Omega X \) is also decomposable over \( R \).

References:


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