A GENERALIZATION OF WRIGHT'S INEQUALITY

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Let $R$ be a commutative ring with identity, $E$ an $R$-module and $x_1, \ldots, x_r \in R$ a multiplicity system on $E$ (see [1], p.295). Then the length $\ell_R(E/(x_1^{n_1}, \ldots, x_r^{n_r})E)$ is finite for any positive integers $n_1, \ldots, n_r$, and Wright's inequality (see [1] p.296) says

$$\ell_R(E/(x_1^{n_1}, \ldots, x_r^{n_r})E) \leq n_1 \cdots n_r \cdot \ell_R(E/(x_1, \ldots, x_r)E),$$

for arbitrary $n_1, \ldots, n_r$.

This inequality can be written as

$$\ell_R \cdot H_0 K(x_1, \ldots, x_r|E) \leq n_1 \cdots n_r \cdot \ell_R \cdot H_0 K(x_1, \ldots, x_r|E),$$

where $K(x_1, \ldots, x_r|E)$ denotes the Koszul complex defined by $E$ and the elements $x_1, \ldots, x_r$.

In this paper we establish that for a Noetherian module similar inequalities hold for the higher Koszul homology modules, i.e., for $i \geq 0$, $\ell_R H_i K(x_1, \ldots, x_r|E) \leq n_1 \cdots n_r \cdot \ell_R H_i K(x_1, \ldots, x_r|E)$. Moreover, the same is true for the higher Euler-Poincaré characteristics of the Koszul complexes $K(x_1, \ldots, x_r|E)$, i.e., for $i \geq 0$, we have $\chi_i(x_1, \ldots, x_r|E) \leq n_1 \cdots n_r \cdot x_i(x_1, \ldots, x_r|E)$, where, by definition,

$$\chi_i(x_1, \ldots, x_r|E) = \sum_{j \geq i} (-1)^{j-i} \ell_R H_j K(x_1, \ldots, x_r|E).$$

Actually the inequality for $\chi_i$ is an equality if $i=0$ (see [1] p.311).
We also prove that the functions \( l H_1^1 K(x_1, \ldots, x_r | E) \) and \( x_1(x_1, \ldots, x_r | E) \) increase with the exponents \( n_1, \ldots, n_r \).

In what follows \( R \) denotes a commutative ring with identity, \( E \) is a Noetherian module over \( R \) and \( x_1, \ldots, x_r, y \in R \) is a system of multiplicity on \( E \). This will ensure that all the lengths which appear are indeed finite, though some of the results would also hold without assuming the lengths to be finite. We denote the length by \( l \) or \( l_R \).

**Lemma 1.** Let \( E \) be an \( R \)-module and \( x_1, \ldots, x_r, y \) be elements of \( R \). Then, for any \( i \geq 0 \),

\[
\ell_R H_1^1 K(x_1, \ldots, x_r | E) \leq \ell_R H_1^1 K(x_1 y, x_2, \ldots, x_r | E).
\]

**Proof.** The inequality follows from the exact sequence (cf. [2] p.IV-2)

\[
0 \to \frac{H_1^1 K(x_2, \ldots, x_r | E)}{x_1 H_1^1 K(x_2, \ldots, x_r | E)} \to H_1^1 K(x_1, x_2, \ldots, x_r | E) \to \\
\to (0 : x_1) \to H_{i-1}^1 K(x_2, \ldots, x_r | E) \to 0,
\]

and the corresponding one for \( H_1^1 K(x_1 y, x_2, \ldots, x_r | E) \), by observing that both \( (0 : x_1) \subset H_{i-1}^1 K(x_2, \ldots, x_r | E) \subset (0 : x_1 y) \subset H_{i-1}^1 K(x_2, \ldots, x_r | E) \)

and \( x_1 H_1^1 K(x_2, \ldots, x_r | E) \supset x_1 y H_1^1 K(x_2, \ldots, x_r | E) \).

Bearing in mind that the Koszul homology modules do not depend on the order of the elements defining it, we get the following
Proposition 2. For any \( i \geq 0 \), the mapping from \( \mathbb{N}^r \) to \( \mathbb{N} \) defined by

\[
(n_1, \ldots, n_r) \mapsto \varepsilon H_1 K(x_1, \ldots, x_r | E)
\]

is increasing, i.e., \( n_1 \leq m_1, \ldots, n_r \leq m_r \) imply

\[
\varepsilon H_1 K(x_1, \ldots, x_r | E) \leq \varepsilon H_1 K(x_1, \ldots, x_r | E).
\]

Lemma 3. If \( a \in \mathbb{R} \), then we have:

i) \( \varepsilon(0:a^n) \leq n \cdot \varepsilon(0:a) \), and

\[
\varepsilon(\frac{E}{a^n}) \leq n \cdot \varepsilon(\frac{E}{a}).
\]

Proof. By induction on \( n \). From the exact sequence

\[
\begin{array}{cccccc}
R/\mathbb{R} & \overset{a^{n-1}}{\longrightarrow} & R/\mathbb{R} & \longrightarrow & R/\mathbb{R} & \longrightarrow 0,
\end{array}
\]

if we apply \( \text{Hom}_R(,E) \), we get i), for \( \text{Hom}_R(R/\mathbb{R},E) \cong O:a^n \),

and if we apply \( \mathbb{R} \), we get ii), for \( (R/\mathbb{R}) \otimes E \cong E/\mathbb{R} \). #

Proposition 4. The following inequality holds for all \( i \geq 0 \),

\[
\varepsilon H_1 K(x_1, x_2, \ldots, x_r | E) \leq n \cdot \varepsilon H_1 K(x_1, x_2, \ldots, x_r | E).
\]

Proof. From the exact sequences (see [2] p.IV-2)

\[
0 \longrightarrow H_0 K(a|H_1 K(x_2, \ldots, x_r | E)) \longrightarrow H_1 K(a, x_2, \ldots, x_r | E)
\]

\[
\longrightarrow H_1 K(a|H_{i-1} K(x_2, \ldots, x_r | E)) \longrightarrow 0,
\]

with \( a = x_1 \) or \( x_1^n \), we get

\[
\varepsilon H_1 K(x_1^n, x_2, \ldots, x_r | E) = \varepsilon \left( \frac{H_1 K(x_2, \ldots, x_r | E)}{x_1 H_1 K(x_2, \ldots, x_r | E)} \right) + \varepsilon(0:x_1^n)
\]

\[
\varepsilon_{H_1 K(x_2, \ldots, x_r | E)}
\]

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\[
\begin{align*}
&\leq n \cdot \varepsilon_1 (\frac{H_i K(x_1, \ldots, x_r | E)}{x_1 H_i K(x_2, \ldots, x_r | E)}) + n \cdot \varepsilon_1 (0 : x_1) \\
&= n \cdot \varepsilon_1 H_i K(x_1, x_2, \ldots, x_r / E),
\end{align*}
\]

the inequalities being by virtue of lemma 3. #

Again by the independence of the Koszul homology modules with respect to the order of the elements, Proposition 4 yields the following theorem which generalizes Wright's inequality.

**Theorem 5.** For any \( i \geq 0 \), and any \( n_1, \ldots, n_r \geq 0 \), we have

\[
\varepsilon_1 H_i K(x_1, \ldots, x_r | E) \leq n_1 \ldots n_r \cdot \varepsilon_1 H_i K(x_1, \ldots, x_r | E). #
\]

Let us consider now the higher Euler-Poincaré characteristics. Observe first that \( \chi_0(x_1, \ldots, x_r | E) = n_1 \ldots n_r \cdot \chi_0(x_1, \ldots, x_r | E) \) and that \( \chi_0(x_1, \ldots, x_r | E) \geq 0 \) (cf. [1] p.311). For the higher characteristics we have

**Proposition 6.** For all \( i \geq 0 \), the mapping

\[
(n_1, \ldots, n_r) \mapsto \chi_i(x_1, \ldots, x_r | E)
\]

from \( \mathbb{N}^r \) to \( \mathbb{N} \) is increasing, i.e., \( n_1 \leq m_1, \ldots, n_r \leq m_r \), imply

\[
\chi_i(x_1, \ldots, x_r | E) \leq \chi_i(x_1, \ldots, x_r | E).
\]

**Proof.** By [2] p.IV-56, we have

\[
\chi_i(a, x_2, \ldots, x_r | E) = \varepsilon H_i K(a | H_i-1 K(x_2, \ldots, x_r | E)) +
\]

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\[ + x_0(a|x_1(x_2, \ldots, x_r|E)). \]

Setting \( a = x_1 \) or \( x_1y \) and using the multiplicativity of \( x_0 \) (see [1] p.309 Thm.7), we deduce

\[ x_i(x_1, x_2, \ldots, x_r|E) \leq x_i(x_1y, x_2, \ldots, x_r|E). \]

From this we get, for \( n < m \), that

\[ x_i(x_1^n, x_2, \ldots, x_r|E) \leq x_i(x_1^m, x_2, \ldots, x_r|E). \]

Proceeding equally with the other variables (\( x_i \) does not depend on the order of the elements), we get the result.

We finish with a theorem on higher Euler-Poincaré characteristics similar to theorem 5.

**Theorem 7.** For any \( i > 0 \), and any \( n_1, \ldots, n_r > 0 \), we have

\[ x_i(x_1^{n_1}, \ldots, x_r^{n_r}|E) \leq n_1 \cdot \ldots \cdot n_r \cdot x_i(x_1, \ldots, x_r|E). \]

**Proof.** It is enough to prove

\[ x_i(x_1^n, x_2, \ldots, x_r|E) \leq n \cdot x_i(x_1, x_2, \ldots, x_r|E), \]

and this can be done by considering the formula used in the proof of the preceding proposition with \( a = x_1 \) or \( x_1^n \). We get

\[ x_i(x_1^n, x_2, \ldots, x_r|E) = \]

\[ + n \cdot x_i(x_1, x_2, \ldots, x_r|E) \]

\[ = x_0(n|x_1^{n}) \quad \text{or} \quad x_0(n|x_1) + n \cdot x_0(x_1| x_1(x_2, \ldots, x_r|E)) \]

\[ \leq n \cdot x_0(n|x_1) \quad \text{or} \quad x_0(n|x_1) + n \cdot x_0(x_1| x_1(x_2, \ldots, x_r|E)) = \]

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\[ n \cdot x_i(x_1, x_2, \ldots, x_r | E), \]
the inequality being justified by lemma 3. #

References
