DUALITY OF INFINITE-DIMENSIONAL SUBSPACES
IN AN INDEFINITE INNER PRODUCT SPACE
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Abstract: A characterization of the subspaces of an inner product space which admit a dual companion and the relation between duality and orthogonal projections are given.

Following Bognár [1] we mean, by "inner product space", a complex vector space $E$, endowed with a sesquilinear form $(\cdot | \cdot)$ - the "inner product on $E"- not necessarily positive defined. If $A$ is a subset of $E$, we symbolize by $A^\perp$ the set of all vectors $x \in E$ such that $(a|x)=0$ for every $a \in A$. Two subspaces $L$ and $M$ are "dual companions" ($L\#M$) if $L \cap M^\perp = L^\perp \cap M =0$. A locally convex topology $\tau$ on $E$ is "admissible" if the inner product is separately $\tau$-continuous and, for every linear form $\tau$-continuous $\varphi$ there exists a vector $x_0$ with $\varphi(y) = (y|x_0)$ for every $y$.

In [1], [2] and [3] duality between finite-dimensional subspaces is studied.

The purpose of this note is to give a characterisation of the subspaces $L$ of $E$ which admit a dual companion (Corollary 1) that generalizes the results of the quoted works. Theorem 2 gives a constructive method of dual companions in a particular case, which extends a result of [1]. Finally, if the subspaces considered are orthocomplemented ($L^\perp_1=E$), we express the duality by means of orthogonal projections.

**THEOREM 1:** Let $L$ and $M$ be subspaces of the inner product space $E$ with $L \cap M^\perp =0$. Then, $M$ contains a dual companion of $L$.

**Proof:** The result is obvious when $L = 0$. In the other case, let $\Omega$ be the family of the pairs $(L',M')$ of subspaces of $E$ such that $L'\#M'$, $L' \subset L$ and $M' \subset M$. Because $0\#0$, $\Omega$ is not void.
Besides, the relation
\[(L', M') \subseteq (L'', M'') \iff L' \subseteq L'' \text{ and } M' \subseteq M''\]
determines a partial ordering of \(\Omega\). If
\[\Omega_1 = \{(L_i, M_i) : i \in I\}\]
is a totally ordered subset of \(\Omega\), let be
\[L' = \bigcup_{i \in I} L_i \quad \text{and} \quad M' = \bigcup_{i \in I} M_i.\]
Since \(\Omega_1\) is totally ordered, we get the identities
\[L' = \bigcup_{i \in I} L_i = \bigcup_{i \in I} M_i.\]
Consequently, if \(z\) is a vector of \(L' \cap M'^\perp\), there must exist an index \(i_0\) in \(I\) such that
\[z \in L_{i_0} \cap M'^\perp = L_{i_0} \cap (\bigcap_{i \in I} M_i^\perp) \subseteq L_{i_0} \cap M_{i_0}^\perp,\]
i.e., \(z = 0\), because \(L_{i_0} \neq M_{i_0}\).

Similarly, \(L'^\perp \cap M' = 0\), so \((L', M')\) is an upper bound of \(\Omega_1\).

Thus, from the Zorn Lemma, we infer the existence of a maximal element \((L_1, M_1)\) in \(\Omega\). The proof will conclude if we get the identity \(L = L_1\). Let is assume now \(L \subseteq L_1\); if so, we can consider a vector \(x\) in \(L - L_1\) and the subspace \(L_2 = L_1 + \langle x \rangle \subseteq L\) (\(\langle x \rangle\) symbolizes the linear span of \(x\)). By virtue of maximality of \((L_1, M_1)\), there must be \(u \neq 0\) in \(L_2 \cap M_1^\perp\) since \(L_2 \cap M_1 \subseteq L_1^\perp \cap M_1 = 0\). Thus, \(u = z + ax\), with \(z \in L_1\) and \(a \neq 0\), which implies \(L_2 = L_1 + \langle u \rangle\).

On the other hand, since \(L \cap M^\perp = 0\), it is possible to get \(y \in M\) with
\[(u | y) = 1\] \tag{1}
and consider the subspace of \(M\), \(M_2 = M_1 + \langle y \rangle\).

Then immediately follows \(M_2^\perp \cap L_2 = 0\), so, \(M_2 \cap L_2^\perp \neq 0\) and there exists \(v \neq 0\) in \(M_2 \cap L_2^\perp\). So, \(M_2 = M_1 + \langle v \rangle\), since
\[v = m + dy, \quad m \in M_1, \quad d \neq 0\] \tag{2}
By considering a not vanishing vector \( m_1 + bv \) in \( M_2 \cap L_2^\perp \) (consequently, \( b \neq 0 \)), for every \( t \in L_1 \),
\[
(m_1 + bv | t) = 0,
\]
and so,
\[
m_1 = 0.
\]

Since \( L_2^\perp \subseteq L_1^\perp \). From it,
\[
(m_1 | t) = 0,
\]
Finally, since \( u \in L_2 \), \( v \in L_2^\perp \),
\[
(m_1 + bv | u) = b(v | u) = 0.
\]
But, taking into account (1) and (2),
\[
(v | u) = (m_1 + bv | u) = b \neq 0.
\]

COROLLARY 1: A subspace \( L \) of the inner product space \( E \) admits a dual companion if and only if \( L \cap E^\perp = 0 \).

COROLLARY 2: Let \( E \) be an inner product space and let \( \zeta \) be an admissible topology on \( E \). The following propositions are equivalent:

i) \( E \) is non degenerate (i.e., \( E^\perp = 0 \))

ii) Every subspace of \( E \) admits a dual companion

iii) There exists a subspace \( L \), \( \zeta \)-closed in \( E \), which admits dual companion.

Proof: By using Corollary 1 it is enough to prove that i) follows from iii). If \( M \) is a dual companion of the \( \zeta \)-closed subspace \( L \), we obtain,
\[
E^\perp \subseteq (L^\perp + M)^\perp = L^\perp + M^\perp = L \cap M^\perp = 0
\]
since \( L^\perp \) coincides with the \( \zeta \)-closure of \( L \) given that \( \zeta \) is admissible.

In Corollary 2, iii) the hypothesis of \( L \) be closed is necessary as the following example proves: Let \( E = \langle e, f \rangle \), \( (e | e) = (e | f) = (f | e) = 0 \), \( (f | f) = 1 \). Then \( \langle f \rangle \# \langle f \rangle \), but \( E \) is degenerate.
DEFINITION: Two families of vectors \( \{ e_i : i \in I \} \) and \( \{ f_i : i \in I \} \) in the inner product space \( E \) form a "dual pair" if, for every \( i, j \in I \), with \( i \neq j \), the relations
\[
(e_i | f_j) = 1 \quad (e_i | f_j) = 0,
\]
are verified.

As is easily checked, if two families of vectors form dual pair each of them is linearly independent and their linear envelopes are dual companions.

THEOREM 2: Let \( L \) and \( M \) be subspaces of the inner product space \( E \) such that \( L \cap M^\perp = 0 \). If \( L \) admits a countable Hamel basis then it exists a dual pair of families of vectors, their linear envelopes being \( L \) and a subspace of \( M \).

Proof: We will construct recurrently the dual pair.

Let \( \{ g_n : n=1,2, \ldots \} \) a Hamel basis of \( L \). Since \( g_1 \notin M^\perp \), it is possible to find \( f_1 \) in \( M \) with \( (g_1 | f_1) = 1 \). Let \( e_1 = g_1 \).

Assuming that the vectors \( e_1, e_2, \ldots, e_n \) in \( L \) and \( f_1, f_2, \ldots, f_n \) in \( M \) are given verifying
\[
(e_i | f_j) = \begin{cases} 
1, & \text{if } i=j, i \neq j, j \neq n \\
0, & \text{if } i \neq j
\end{cases} \tag{4}
\]
and
\[
\langle e_1, e_2, \ldots, e_n \rangle = \langle g_1, g_2, \ldots, g_n \rangle, \tag{4'}
\]
we define
\[
e_{n+1} = g_{n+1} - \sum_{k=1}^{n} (g_{n+1} | f_k)e_k,
\]
and, because \( e_{n+1} \in M \), it exists \( h \in M \) such that \( (e_{n+1} | h) = 1 \). Let
\[
f_{n+1} = h - \sum_{k=1}^{n} (h | e_k)f_k.
\]

Now, the relations (4),(4') also follow when the indexes \( i,j \) range
form 1 to n+1.

The countable families obtained by means of this process form dual pair and, obviously, the former one is a Hamel basis for L.\[1\]

If the subspace L is orthocomplemented, every vector of E can be expressed (in a not necessarily only way) as the sum of one of L and another of $L^\perp$. Thus, in a natural way, the "orthogonal projection" of a subspace M on L, $P_LM$, can be defined as the set of the vectors $x \in L$ such that $x-z \in L^\perp$ for some $z$ in M.

For a Hilbert space it is well known the fact that, given two closed subspaces L and M, $L \cap M^\perp = 0$ if and only if $P_LM$ is dense in L. The following lemma expresses the best possible generalization of this result for an inner product space.

**LEMMA 1:** Let L be an orthocomplemented subspace of the inner product space E such that $L \cap L^\perp = 0$ (i.e., L is nondegenerate). Then, for every subspace M in E and for every admissible topology $\mathcal{Z}$ on E,

$$M^\perp \cap L = 0 \iff P_LM \text{ is } \mathcal{Z}\text{-dense in } L.$$  

**Proof:** Since the closures of the subspaces are the same for every admissible topologies it is enough to work with one of them, in particular with the weak topology $\sigma(E)$. Following a result of Scheibe (see [3]) if L is orthocomplemented then the weak topology of L, $\sigma(L)$ coincides with the relative one of $\sigma(E)$. Besides, in [1] it is proved that ($P_LM^\perp \cap L = 0$ if and only if $P_LM$ is $\sigma(L)$-dense in L. Finally, it is easily checked that (for L orthocomplemented) $M^\perp \cap L = (P_LM^\perp \cap L$, which concludes the proof.\[2\]

Lemma 1 is a particular case of the following fact: if L is an orthocomplemented subspace of E, then for every admissible topology and for every subspace M, $P_LM$ is dense in L if and only if $L \cap M^\perp = L \cap L^\perp$, result established in [4].

From Lemma 1 the proof of the next theorem follows straightforwardly
THEOREM 3: Let $L$ and $M$ be subspaces of the inner product space $E$ and assume that $L$ is orthocomplemented and nondegenerate. Then, if $\mathcal{Z}$ is an admissible topology on $E$, 

$$L \# M \iff P_L M \text{ is } \mathcal{Z}\text{-dense in } L \text{ and } M \cap L^\perp = 0.$$ 

COROLLARY 3: If $L$ and $M$ are orthocomplemented nondegenerate subspaces of the inner product space $E$, then $L \# M$ if and only if $P_L M$ is weakly dense in $L$ and $P_M L$ is weakly dense in $M.$

REFERENCES


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