A GEOMETRICAL CHARACTERIZATION
OF REFLEXIVITY IN BANACH SPACES

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Summary: The main result in this paper is the equivalence, for any Banach space \( B \), between

(i) "Every normalized basic sequence \( \{a_n\}_{n \in \mathbb{N}} \) in \( B \) is weakly null",

and

(ii) "For every normalized basic sequence \( \{a_n\}_{n \in \mathbb{N}} \) in \( B \),

\[
a_1 \in \overline{\text{span}} \left( a_n - a_{n+1} \right)_{n \in \mathbb{N}}
\]

Pelczyński proved that (i) characterizes the fact of \( B \) being reflexive. So, the same holds for (ii) and we have a "geometrical" characterization of reflexivity.

We finish quoting some equivalent version of the above result.

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1. **Previous Concepts**

Let $B$ denote a Banach space and $K$ its scalar field, $N$ the set of natural numbers, $[\ldots]$ "closed linear span", and $f = (a_n)_{n \in N}$ be a linearly independent sequence of vectors in $B$.

Call $K(f) = \bigcap_{n \in N} [a_n, a_{n+1}, \ldots]$ (kernel of $f$) and

$$K_s(f) = \{K(f') ; f' \text{ is a subsequence (infinite) of } f \} \text{ (strict kernel of } f)$$

$f$ is normalized if $||a_n|| = 1 \ (n \in N)$

$f$ is basic if there is a unique sequence of scalars $(\lambda_n)_{n \in N}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n a_n \ , \text{ for every } x \in \text{[f].}$$

The sequence $(a_n - a_{n+1})_{n \in N}$ is called sequence of differences of $f$.

$f$ is said to be weakly convergent to $x \in B$ if $\lim_{n} f(a_n) = f(x)$, for every $f \in B^*$ (dual of $B$).

$f$ is said to be minimal if there exists a sequence $(a^*_n)_{n \in N}$ in $[f]^*$ with $a^*_n(a_m) = \delta_{nm}$ (Kronecker indices), and uniformly minimal if it also verifies $\sup_{n} ||a_n|| \cdot ||a^*_n|| < \infty$.

2. **The main result**

The result leans on the following two lemmas:

**Lemma 1**: Every subsequence $f'$ of a given sequence $f = (a_n)_{n \in N}$ has zero strict kernel if and only if the normalized sequence $f_N = (a_n/||a_n||)_{n}$ has no subsequence weakly convergent to some vector distinct from zero.

Lemma 2: Let \( f = (a_n)_{n \in \mathbb{N}} \) be a minimal sequence with zero kernel. Let \( x \in [f] \) such that the set \( S_x = \{ k \in \mathbb{N} ; a_k^*(x) \neq 0 \} \) is infinite. We note \( S_x = (p_n)_{n \in \mathbb{N}} \). Then
\[
x \in K_s \left( \left( \sum_{h=1}^{n} a_h^*(x) a_h \right) \quad p_n \quad n \in \mathbb{N} \right)
\]
if and only if the sequence
\[
( \sum_{h=1}^{n} a_h^*(x) a_h ) \quad p_n \quad n \in \mathbb{N}
\]
is weakly convergent to \( x \).

Proof: (See |I-T|). It follows from lemma 1 and the third Fréchet' s axiom of convergence (see |K|).

Now, we finally have the

Theorem: Let \( B \) be a Banach space. Then the following statements are equivalent:

(i) \( B \) is reflexive,

(ii) Every normalized basic sequence \( (a_n)_{n \in \mathbb{N}} \) in \( B \) is weakly convergent to zero,

(iii) Every normalized basic sequence \( (a_n)_{n \in \mathbb{N}} \) in \( B \) verifies
\[
a_1 \in [a_n - a_{n+1} ; n \in \mathbb{N}]
\]

Proof: In \(|P|\) has been proved that (i) is equivalent to (ii).

-(ii) implies (iii) is obvious, considering
\[
a_1 - a_n = \sum_{i=1}^{n-1} (a_i - a_{i+1})
\]

(iii) implies (ii):

-Suppose that for every normalized basic sequence \( f = (a_n)_{n \in \mathbb{N}} \),
\[
a_1 \in [a_n - a_{n+1} ; n \in \mathbb{N}].
\]
Notice that \( a_1 \in [a_n - a_{n+1} ; n \in \mathbb{N}] \) if and only if \( a_1 \in K((a_1 - a_n)_n) \) (see, for instance, \(|R|\), proposition 2.2)

Take \((p_n)_n \in \mathbb{N}\) a subsequence of \( \mathbb{N} \), with \( p_1 = 1 \). By hypothesis, the sequence \((a_{p_n})_n \in \mathbb{N}\) also verifies \( a_1 \in [a_{p_n} - a_{p_{n+1}} ; n \in \mathbb{N}] \), so, it follows that \( a_1 \in K_s((a_1 - a_n)_n \in \mathbb{N}) \).

Now, applying lemma 2 to \( a_1 \) and \((a_n - a_{n+1})_n \in \mathbb{N}\), we have that \((a_1 - a_n)_n \in \mathbb{N}\) is weakly convergent to \( a_1 \), and therefore \((a_n)_n \in \mathbb{N}\) is weakly convergent to zero.

3. Equivalent versions.

-In \(|CH-I|\) (preprint of this paper) the following equivalent versions of the theorem are given:

(iv) \([a_n ; n \in \mathbb{N}] = [a_n - a_{n+1} ; n \in \mathbb{N}]\), for every normalized basic sequence \((a_n)_n \in \mathbb{N}\) in \(B\),

(v) Let \((a_n)_n \in \mathbb{N}\) be a normalized basic sequence in \(B\). Then, its sequence of differences cannot be uniformly minimal,

(vi) For every normalized basic sequence \((a_n)_n \in \mathbb{N}\) in \(B\),

\([a_n - a_{n+1} ; n \in \mathbb{N}]\) cannot be an hyperplane in \([a_n ; n \in \mathbb{N}]\).

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4. References


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