

ON THE LIE ALGEBRA OF A TRANSVERSALLY
COMPLETE FOLIATION

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In this short note we consider the Lie algebra of all vector fields which preserve a transversally complete foliation. Our considerations are based on the structural theorem for a transversally complete foliation which was proved by Molino in [4]. The class of transversally complete foliations contains other important classes: fibrations with compact fibers and transversally oriented codimension one foliations without holonomy on compact manifolds (cf.[7], theorem 1.3). In [5], Omori gives an analogue of the classical theorem of Pursell and Shanks (cf.[6]) for a fibration with compact fibers. Recently Fukui and Tomita [2] proved another such an analogue for a transversally oriented codimension one foliation without holonomy. Our theorem 6 can be viewed as a generalization of the both above results.

In this note all objects are of class C^∞ . The manifolds are connected, Hausdorff and second countable.

§.1. The theorem of Amemiya for foliations.

For any manifold M we denote by $\mathfrak{X}(M)$ the Lie algebra of all vector fields on M . Let (M, F) be a foliated manifold and let $J(M)$ be the Lie algebra of all leaf preserving vector fields on M , i.e. vector fields which are tangent to the leaves of F . Let $n = \dim M$ and $q = \text{codim } F$. A local coordinate system $(U, x^1, \dots, x^{n-q}, y^1, \dots, y^q)$ is said to be distinguished by the foliation F , if for fixed y^1, \dots, y^q the coordinates x^1, \dots, x^{n-q} are coordinates of a leaf. Let us denote $x = (x^1, \dots, x^{n-q})$, $y = (y^1, \dots, y^q)$. The proof of the following proposition is obvious.

Proposition 1. For any $X \in \mathfrak{X}(M)$ the following conditions are equivalent:

- (1) $[X, Y] \in J(M)$, if $Y \in J(M)$,
- (2) the flow of X preserves F
- (3) $X = \sum \xi^i(x, y) \partial_i + \sum \eta^j(y) \tilde{\partial}_j$ on the domain of a distinguished chart, where $\partial_i = \partial/\partial x^i$, $\tilde{\partial}_j = \partial/\partial y^j$.

By \mathfrak{X}_F we denote the Lie algebra of all vector fields satisfying the above conditions.

Lemma 2. If $X \in \mathfrak{X}_F$ satisfies $X(p) \neq 0$, then there is a distinguished chart (U, x, y) at p such that one of the following identities is satisfied on U : (i) $X = \tilde{\partial}_1$, or (ii) $X = \partial_1 + \sum \eta^i(y) \tilde{\partial}_i$, where $\eta^i(0, \dots, 0) = 0$ for $i = 1, \dots, q$.

In fact, the vector $X(p)$ is either tangent to the leaf or not.

Theorem 3 (Amemiya [1]). Let (M, F) and (M', F') be foliated manifolds. If Φ is a Lie algebra isomorphism of $J(M)$ onto $J(M')$, then there is a foliation preserving diffeomorphism φ of M onto M' such that $\Phi = \varphi_*$ on $J(M)$.

The following corollary from Theorem 3 will be useful in the sequel.

Corollary 4. Let (M, F) and (M', F') be foliated manifolds. If Φ is a Lie algebra isomorphism of \mathfrak{X}_F onto $\mathfrak{X}_{F'}$, such that $\Phi(J(M)) = J(M')$, then there is a foliation preserving diffeomorphism φ of M onto M' such that $\Phi = \varphi_*$ on \mathfrak{X}_F .

Proof. Let φ be the diffeomorphism obtained in Theorem 3. We use Proposition 1. Since

$$\Phi[X, Y] = [\Phi X, \Phi Y] \in J(M'), \quad \text{if } X \in \mathfrak{X}_J, Y \in J(M)$$

$$\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y] \in J(M'), \quad \text{if } X \in \mathfrak{X}_F, Y \in J(M)$$

and $\Phi = \varphi_*$ on $J(M)$, we have

$$[\varphi_* X, Y'] = [\Phi X, Y'] \quad , \quad \text{if } Y' \in J(M'). \quad (\star)$$

In particular, $\Phi_* X \in \mathfrak{X}_{F'}$. Let us denote $X_1 = \varphi_* X - \Phi X$. We shall show that $X_1 = 0$. Suppose $X_1(p) \neq 0$. By Lemma 2 there is (U, x, y) a distinguished chart at p such that $X_1 = \tilde{\partial}_1$ or $X_1 = \partial_1 + \sum \eta^i(y) \tilde{\partial}_i$ in a neighborhood of p . Let $Y' = y^1 \partial_1$ in

the first case or $Y' = x^1 \partial_1$ in the second. In both cases we obtain $[X_1, Y'] \neq 0$ which contradicts (\star) .

§2. The structural theorem for transversally complete foliations.

Let (M, F) denote a foliated manifold with codimension q . Suppose M is a compact manifold. The foliation F is said to be transversally complete, if for any $p \in M$ the evaluation map

$$ev_p: X \in \mathfrak{X}_F \longrightarrow X(p) \in T_p(M)$$

is a surjection. In particular, if M is connected, then the group of all foliation preserving diffeomorphisms acts transitively on M . The examples of the transversally complete foliations are the total spaces of fibrations, the transversally parallelisable foliations and the Lie foliations (c.f.[4]).

We need the following fundamental result :

Theorem 5 (Molino [4]). If (M, F) is a transversally complete foliated manifold, then the closures of the leaves of F are the fibers of a fibration $\pi: M \longrightarrow W$. Moreover, the local trivialisations $\psi: \pi^{-1}(U) \longrightarrow U \times F$ of the fibration π preserve foliation, if $U \times F$ is foliated by $\{pt\} \times L$, where L is a leaf of F .

Remark. The fibration $\pi: M \longrightarrow W$ is called basic. $r = \dim W$ is called the basic dimension of the transversally complete foliation..

Now we introduce a special kind of coordinates. A local coordinate system $(U, x^1, \dots, x^{n-q}, \bar{y}^1, \dots, \bar{y}^{q-r}, y^1, \dots, y^r)$ is called distinguished by the foliation F , if for any fixed (y^1, \dots, y^r) the coordinates $(x^1, \dots, x^{n-q}, \bar{y}^1, \dots, \bar{y}^{q-r})$ are coordinates of a fiber of π and for any fixed $(\bar{y}^1, \dots, \bar{y}^{q-r}, y^1, \dots, y^r)$ the coordinates (x^1, \dots, x^{n-q}) are coordinates of a leaf of F . The existence of the distinguished coordinates follows from Theorem 5. In the sequel, it shall be denoted $x = (x^1, \dots, x^{n-q}), \bar{y} = (\bar{y}^1, \dots, \bar{y}^{q-r}), y = (y^1, \dots, y^r)$.

§3. The analogue of Pursell-Shanks Theorem for transversally complete foliations.

We want to prove the following

Theorem 6. Let (M, F) and (M', F') be transversally complete foliated manifolds with M, M' compact and $\text{codim } F = q > 0$, $\text{codim } F' = q' > 0$.

If there is a Lie algebra isomorphism Φ of \mathfrak{X}_F onto $\mathfrak{X}_{F'}$, then there is a foliation preserving diffeomorphism φ of M onto M' such that $\Phi = \varphi_*$.

Example. Let $M = S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2$. For any $\alpha \in \mathbb{R}$ M admits a well-known linear foliation F_α which is generated by the differential form $\omega = \alpha dx^1 + dx^2$, where (x^1, x^2) are coordinates on \mathbb{R}^2 . If $\alpha \neq \beta$, then by the theorem of Denjoy there does not exist any diffeomorphism $\varphi: M \rightarrow M$ such that $\varphi_*(F_\alpha) = F_\beta$. Hence, the Lie algebras \mathfrak{X}_{F_α} and \mathfrak{X}_{F_β} are not isomorphic.

The proof of the theorem consists of several lemmas. Let $\pi : M \longrightarrow W$ be the basic fibration of F . By F_π we denote the foliation on M by the fibers of π . Let $J_\pi(M)$ denote $\ker \pi_*$ and let $\bar{J}(M)$ denote $J_\pi(M) \cap \mathfrak{X}_F$.

Lemma 7. Let (U, x, \bar{y}, y) be a distinguished chart of (M, F) . Then

$$(1) X = \sum \xi^i(x, \bar{y}, y) \partial_i + \sum \eta^j(x, \bar{y}, y) \bar{\partial}_j + \sum \zeta^k(y) \tilde{\partial}_k \quad \text{on } U \text{ for } X \in \mathfrak{X}_{F_\pi},$$

$$(2) X = \sum \xi^i(x, \bar{y}, y) \partial_i + \sum \eta^j(y) \bar{\partial}_j + \sum \zeta^k(y) \tilde{\partial}_k \quad \text{on } U \text{ for } X \in \mathfrak{X}_F,$$

where, $\partial_i = \partial/\partial x^i$, $\bar{\partial}_j = \partial/\partial \bar{y}^j$, $\tilde{\partial}_k = \partial/\partial y^k$.

Proof. (1) follows immediately from Proposition 1. In order to prove (2), let $X \in \mathfrak{X}_F$ and, in view of Proposition 1, $X = \sum \xi^i(x, \bar{y}, y) \partial_i + \sum \eta^j(\bar{y}, y) \bar{\partial}_j + \sum \zeta^k(\bar{y}, y) \tilde{\partial}_k$ on U . Note that $\mathfrak{X}_F \subset \mathfrak{X}_{F_\pi}$ since the leaves of F_π are the closures of the leaves of F . Hence, again in view of Proposition 1, we have $\zeta^k(\bar{y}, y) = \zeta^k(y)$. Next, let us denote by F an arbitrary fiber of the basic fibration of F and by F_F the foliation induced by F on F . Let \mathfrak{X}_F be the Lie algebra of all vector fields on F preserving F_F . Suppose L is a dense leaf in F . It is easily seen that if $X \in \mathfrak{X}_F$ is such that $X(p)$ is tangent to L , $p \in L$, then $X \in J(F)$, i.e. X is tangent to the foliation F_F . Therefore $\dim \mathfrak{X}_F/J(F) \leq \text{codim } F_F = q-r$. Hence $\eta^j(\bar{y}, y)$ cannot depend on \bar{y} and (2) is satisfied.

Lemma 8. Let m be a proper ideal of \mathfrak{X}_F such that for any

point $p \in M$ there is $X \in \mathfrak{X}_F$ satisfying $(\pi_* X)(x) \neq 0$, where $x = \pi(p)$. Then $m + \bar{J}(M) = \mathfrak{X}_F$.

Proof. Let $W = \bigcup_{\alpha=1}^s U'_\alpha$ be a finite open covering of W such that for each α the fibration $\pi: M \longrightarrow W$ is trivial over U'_α , there is $(U'_\alpha, y^1, \dots, y^r)$ a chart of W and there is $X \in \mathfrak{X}_F$ such that $\pi_* X = \tilde{\partial}_1$ on U'_α . Suppose $W = \bigcup_{\alpha=1}^s U_\alpha$ is a new open covering of W such that $U_\alpha \subset \bar{U}_\alpha \subset U'_\alpha$. We consider a distinguished coordinate system $(VxU'_\alpha, x, \bar{y}, y)$, where (V, x, \bar{y}) is a distinguished coordinate system for the foliation F_F . Let us define $X_1 \in \mathfrak{X}_F$ such that $X_1 = \tilde{\partial}_1$ on FxU_α and $X - X_1 \in \bar{J}(M)$. There is $\{\lambda, \mu\} \subset C^\infty(W)$ a partition of unity subordinated to $M = U'_\alpha \cup \bar{U}_\alpha$. Then we define $X_1 = \lambda \tilde{\partial}_1 + \mu X$. It is easily seen that $X - X_1 \in \bar{J}(M)$ and $X_1 \in m + \bar{J}(M)$.

Let Y be an arbitrary element of \mathfrak{X}_F . If $\{\psi_\alpha\}$ is a partition of unity on W subordinated to the covering $M = \bigcup U_\alpha$, then $Y = Y_1 + \dots + Y_s$, where $Y_\alpha = \psi_\alpha Y \in \mathfrak{X}_F$. Thus it suffices to show that $Y \in m + \bar{J}(M)$ for $Y \in \mathfrak{X}_F$ such that $\text{supp } Y \subset FxU = \pi^{-1}(U)$, where $U = U_\alpha$ for some α . Let $Z \in \mathfrak{X}_F$ be such that $Z = \pi_* Y$ on FxU and $\text{supp } Z \subset FxU$. Hence $Y - Z \in \bar{J}(M)$ and it suffices to show that $Z \in m + \bar{J}(M)$. Let χ denote a C^∞ -function on W such that $\text{supp } \chi \subset U$ and $\chi = 1$ on a neighborhood of $\pi(\text{supp } Z)$ and let μ be another C^∞ -function such that $\text{supp } \mu \subset U$ and $\mu = 1$ on a neighborhood of $\text{supp } \chi$. Let $Z = \sum \xi^j(y) \tilde{\partial}_j$, where $\text{supp } \xi^j \subset U$. Since $X_1 = \tilde{\partial}_1$ on FxU ,

$$[X_1, (\mu \int_{-\infty}^{y^1} \chi dy^1) \tilde{\partial}_j] = Z_1 + \chi \tilde{\partial}_j$$

for any j , where $\text{supp } z_1 \subset \text{supp } \chi$. Next

$$\begin{aligned} [z_1 + \chi \tilde{\partial}_j, (\chi \int_{-\infty}^{y^j} \xi^j dy^j) \tilde{\partial}_j] &= [\chi \tilde{\partial}_j, (\chi \int_{-\infty}^{y^j} \xi^j dy^j) \tilde{\partial}_j] = \\ &= \chi^2 \xi^j \tilde{\partial}_j = \xi^j \tilde{\partial}_j. \end{aligned}$$

Since $x_1 \in m + \bar{J}(M)$, we have $\xi^j \tilde{\partial}_j \in m + \bar{J}(M)$ and $z \in m + \bar{J}(M)$. Lemma 8 is then proved.

Note that the assumption of Lemma 8 is satisfied only if the basic dimension r is nonzero.

Lemma 9. Under the assumption of Lemma 8, if m is maximal then m is one codimensional.

Proof. First we show that $J(M) \subset m$. Let $M = \cup V_\alpha$ be a finite open covering of M such that $(V_\alpha, x, \bar{y}, y)$ is a distinguished chart of M and a suitable extension of $\tilde{\partial}_1 = \partial/\partial y^1$ is contained in m . Let $\{\psi_\alpha\}$ be a partition of unity subordinated to the above covering. We define $X_\alpha = \psi_\alpha X$ for $X \in J(M)$. By an argument similar to that in the proof of Lemma 8, one can see that $X_\alpha \in m$. Hence $X \in m$ and $J(M) \subset m$. Next observe that $\bar{J}(M)/J(M)$ is abelian. Indeed, every $X \in \bar{J}(M)$ satisfies

$$X = \sum \xi^i(x, \bar{y}, y) \partial_i + \sum \eta^j(y) \bar{\partial}_j$$

on the domain of a distinguished chart. Hence, it is easily seen that $[X, Y] \in J(M)$ for $X, Y \in \bar{J}(M)$. In particular, $[X, Y] \in m$ for $X, Y \in \bar{J}(M)$. Thus, in view of Lemma 8, m is a maximal

vector subspace of \mathfrak{X}_F .

Lemma 10. If m is a maximal ideal of \mathfrak{X}_F satisfying the assumption of Lemma 8, then $\bar{J}(M) \subset m$.

Proof. Let $Y \in \bar{J}(M)$. As in the proof of Lemma 8, we can assume that $\text{supp } Y \subset FxU = \pi^{-1}(U)$, where (U, y) is a chart of W . Let $\partial \in \mathfrak{X}_F$ denote some extension of $\tilde{\partial} = \partial/\partial y^1$ on FxU . We can assume $\partial \in m$. In fact, if $\partial \notin m$ and $y^1 \partial \notin m$ (otherwise $[\partial, y^1 \partial] = \partial \in m$), then there is $\alpha \neq 0$ such that $\alpha \partial + y^1 \partial \in m$ (Lemma 9) and we get

$$[\partial, \alpha \partial + y^1 \partial] = \partial \in m.$$

Next we have

$$[\partial, Y] = \partial Y \in m,$$

$$[\partial, y^1 Y] = y^1 \partial Y \in m,$$

$$[y^1 \partial Y + y^1 \partial Y] = 2y^1 \partial Y + (y^1)^2 \partial^2 Y \in m,$$

$$[(y^1)^2 \partial, \partial Y] = (y^1)^2 \partial^2 Y \in m.$$

Hence we have $y^1 \partial Y \in m$ and $Y \in m$. Lemma 10 is then proved.

Suppose $\dim W = r > 0$ and $x \in W$. By m_x we denote the ideal of \mathfrak{X}_F which consists of all $X \in \mathfrak{X}_F$ such that $\pi_* X$ and all its derivatives vanish at x .

Lemma 11. Let m be an ideal of \mathfrak{X}_F such that $\pi_* m$ vanishes

at $x \in W$. Then the ideal $\pi_* m$ vanishes at x with all its derivatives, i.e. $m \subset m_x$.

Proof. Let (V, y^1, \dots, y^r) be a chart at x such that π is trivial over V . Let $X \in m$ be such that $\pi_* X = \sum \eta^j(y) \tilde{\delta}_j$ and suppose there is $k > 0$ such that

$$\partial^k \eta^i(x) / \partial y^I \neq 0 \quad \text{for some } i \text{ and } I = (i_1, \dots, i_k).$$

Then $[\pi_* X, \tilde{\delta}_{i_k}] = -\sum (\eta^j(x) / \partial y^{i_k}) \tilde{\delta}_j \in \pi_* m$. This vector field has a coefficient with derivative of order $k-1$ which does not vanish at x . Repeating this procedure $k-1$ times we get $X' \in m$ such that $\pi_* X'$ does not vanish at x .

Corollary 12. Every ideal m_x is maximal. Every proper ideal of \mathfrak{X}_F must be contained in some m_x .

Corollary 13. The ideal $\bar{J}(M)$ is the intersection of all maximal ideals of \mathfrak{X}_F .

Lemma 14. The ideal $J(M)$ can be characterized as a minimal ideal of $\bar{J}(M)$ such that $\bar{J}(M)/J(M)$ is abelian.

Proof. We stated that $\bar{J}(M)/J(M)$ is abelian in the proof of Lemma 9. It suffices to show that $J(M)$ is a unique ideal which is minimal with this property. Suppose a is an ideal of $\bar{J}(M)$ such that $a \not\supset J(M)$ and $\bar{J}(M)/a$ is abelian. Then $J(M)/a \cap J(M) \neq 0$ is also abelian. So it suffices to prove that there does not exist any ideal a such that $a \subsetneq J(M)$ and $J(M)/a$ is abelian. Assume a is an ideal with the above property. Let $X \in J(M)$ and, by a partition-of-unity argument, let $\text{supp } X$ be

contained in a chart domain U . Let $Z \in J(M)$ be such that $Z = \partial/\partial x^1$ on a neighborhood of $\text{supp } X$ and $\text{supp } Z \subset U$. Repeating the reasoning from the proof of Lemma 8, one can see that there are $I(X), J(X) \in J(M)$ such that

$$[[Z, I(X)], J(X)] = X \quad \text{on } M.$$

Since $J(M)/a$ is abelian, we have $X \in a$ and $J(M) \subset a$. This contradiction proves the uniqueness of $J(M)$.

Corollary 15. \mathfrak{X}_F has an ideal with finite codimension if and only if $r = 0$.

In fact, for $r = 0$ $J(M)$ is q codimensional in \mathfrak{X}_F .

Now we are in a position to conclude the proof of Theorem 6. Let Φ be a Lie algebra isomorphism of \mathfrak{X}_F onto $\mathfrak{X}_{F'}$. If \mathfrak{X}_F and $\mathfrak{X}_{F'}$ have finite codimensional ideals, then $\mathfrak{X}_F' = \bar{J}(M)$, $\mathfrak{X}_{F'} = \bar{J}(M')$ and trivially $\Phi(\bar{J}(M)) = \bar{J}(M')$. Otherwise, we get $\Phi(\bar{J}(M)) = \bar{J}(M')$ by Corollary 13. Next Lemma 14 implies $\Phi(J(M)) = J(M')$. Finally we apply Corollary 4.

Remark. Let us discuss briefly the assumption of Theorem 6. The theorem is no longer true for the trivial case of codimension 0 foliation i.e. the foliation with one leaf M . The foliations $F = \{M\}$, $F' = \{\text{points}\}$ give a simple counterexample to Theorem 6. Lecomte in his thesis [3] considered the Lie algebra of all infinitesimal automorphisms of a vector bundle E with a base B and standard fibre F . It was proved a Pursell-Shanks type theorem. However, if $\dim F = 1$ then the Lie algebra of all infinitesimal automorphisms of E is isomorphic to $\mathfrak{X}(B) \times C^\infty(B)$

and obviously it does not determine E. This result suggest that Theorem 6 cannot be proved for the non-compact case.

References

- [1] I. Amemiya, Lie algebra of vector fields and complex structure, J. Math. Soc. Japan 27(1975), p.545.
- [2] K. Fukui, N. Tomita, Lie algebra of foliation preserving vector fields, J. Math. Kyoto Univ. 22(1983), p.685.
- [3] P. Lecomte, Algèbres de Lie d'ordre zero sur une variété, Thesis, Université de Liège, 1980.
- [4] P. Molino, Etude des feuilletages transversalement complets et applications, Ann. Sci.Ec.Norm.Sup. 10(1977), p.289.
- [5] H. Omori, Infinite dimensional Lie transformation groups, Springer Lecture Notes in Math., New York 1976.
- [6] L.E. Pursell, M.E. Shanks, The Lie algebra of smooth manifold, Proc.Amer.Math.Soc. 5(1954), p.468.
- [7] H. Rosenberg, R. Roussarie, Reeb foliations, Ann. of Math. 91 (1970), p.1.

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