THE $p$-PERIOD OF AN INFINITE GROUP

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Abstract

For $\Gamma$ a group of finite virtual cohomological dimension and a prime $p$, the $p$-period of $\Gamma$ is defined to be the least positive integer $d$ such that Farrell cohomology groups $\tilde{H}^i(\Gamma; M)$ and $\tilde{H}^{i+d}(\Gamma; M)$ have naturally isomorphic $p$-primary components for all integers $i$ and $\mathbb{Z}\Gamma$-modules $M$.

We generalize a result of Swan on the $p$-period of a finite $p$-periodic group to a $p$-periodic infinite group, i.e., we prove that the $p$-period of a $p$-periodic group $\Gamma$ of finite vcd is $2\operatorname{LCM}([N(x)]/\operatorname{C}(x))$ if the $\Gamma$ has a finite quotient whose $p$-Sylow subgroup is elementary abelian or cyclic, and the kernel is torsion free, where $N(-)$ and $C(-)$ denote normalizer and centralizer, $(x)$ ranges over all conjugacy classes of $\mathbb{Z}/p$ subgroups. We apply this result to the computation of the $p$-period of a $p$-periodic mapping class group. Also, we give an example to illustrate this formula is false without our assumption.

For $\Gamma$ a group of virtual finite cohomological dimension (vcd) and a prime $p$, the $p$-period of $\Gamma$ is defined to be the least positive integer $d$ such that the Farrell cohomology groups $\tilde{H}^i(\Gamma; M)$ and $\tilde{H}^{i+d}(\Gamma; M)$ have naturally isomorphic $p$-primary components for all $i \in \mathbb{Z}$ and $\mathbb{Z}\Gamma$-modules $M$ [3].

The following classical result for a finite group $G$ was showed by Swan in 1960 [9].

Theorem (Swan).

a) If a 2-Sylow subgroup of $G$ is cyclic ($\neq \{1\}$), the 2-period of $G$ is 2. If a 2-Sylow subgroup of $G$ is a (generalized) quaternion group, the 2-period of $G$ is 4.

b) Suppose $p$ an odd prime and a $p$-Sylow subgroup of the finite group $G$ is cyclic ($\neq \{1\}$). Let $S_p$ denote the $p$-Sylow subgroup and $A_p$ the group of automorphisms of $S_p$ induced by inner automorphism of $G$. Then the $p$-period of $G$ is twice the order of $A_p$. 
Remark.

The group $A_p$ above is isomorphic to $N(S_p)/C(S_p)$, where $N(-)$ and $C(-)$ denote the normalizer and centralizer of $S_p$ in $G$.

It is very natural to ask a question: If $\Gamma$ is a $p$-periodic group of finite vcd, is a similar result still true? In other words, is it possible to describe the $p$-period of a $p$-periodic group $\Gamma$ of finite vcd by an algebraic non-homological invariant of the group $\Gamma$ itself?

In this paper, we generalize the result of Swan for a finite group to a $p$-periodic group $\Gamma$ of finite vcd which has a finite quotient whose a $p$-Sylow subgroup is elementary abelian or cyclic, and the kernel is torsion-free, i.e., we prove that the $p$-period of a $p$-periodic group $\Gamma$ of finite vcd is twice the least common multiple of $\{N((x))/C((x))\}$ in these two cases, where $\langle x \rangle$ ranges over all conjugacy classes of $Z/p$ subgroups of $\Gamma$. On the other hand, we give a group $\Gamma_0$ of finite vcd whose only finite subgroup is a $Z/2$, but the 2-period of $\Gamma_0$ is greater than $2|N(Z/2)/C(Z/2)|$. Finally, an application will be made for calculating the $p$-period of a mapping class group.

The following four theorems are our main results of this paper.

**Theorem 1.** Assume that $\Gamma$ is $p$-periodic. If $\Gamma$ has a normal subgroup of finite cohomological dimension so that the associated quotient is a finite group whose a $p$-Sylow subgroup is elementary abelian, then the $p$-period of $\Gamma$ is twice the least common multiple of $\{N((x))/C((x))\}$, where $\langle x \rangle$ ranges over all conjugacy classes of $Z/p$ subgroups of $\Gamma$.

**Theorem 2.** Let $\Gamma$ be a group which has a normal subgroup of finite cohomological dimension so that the associated quotient is a finite group whose a $p$-Sylow subgroup is cyclic, then the $p$-period of $\Gamma$ is twice the least common multiple of $\{N((x))/C((x))\}$, where $\langle x \rangle$ ranges over all conjugacy classes of $Z/p$ subgroups of $\Gamma$.

**Theorem 3.** There is a group $\Gamma_0$ of finite vcd whose only finite subgroup is a $Z/2$, but the 2-period is greater than $2|N(Z/2)/C(Z/2)|$.

**Theorem 4.** If the mapping class group $\Gamma_g$ is a $p$-periodic group and $g < p(p - 1)/2$, then the $p$-period of $\Gamma_g$ is $2\text{LCM}\{\gcd(p - 1, b_i)\}$, where $b_i \in B_{g, p}$ (cf. section 3).

The rest of this paper is organized as follows. In section 1, we prove Theorems 1 and 2. In section 2, we provide an example illustrating Theorem 3. Finally in section 3, we give a formula for the calculation of the $p$-period of a $p$-periodic mapping class group $\Gamma_g$. 
1. Proof of Theorems 1 and 2

Lemma 1.1. Let $H = \langle x, y/x^n = 1, yxy^{-1} = x^r \rangle$, where $q = 0$ or $q \neq 0 \mod(p)$. If $d$ is the minimal positive integer such that $r^d = 1 \mod(p)$, then the $p$-period of $H$ equals $2d$.

Proof: If $q \neq 0$, $H$ is a finite group, the proof is immediate by Swan Theorem. Otherwise, if $q = 0$, $H$ is infinite and we look at the short exact sequence $1 \to Z/p \to H \to Z \to 1$. The spectral sequence of Farrell cohomology associated to the exact sequence converges in the following way: $E_2^{ij} = H^i(Z; \tilde{H}^j(Z/p; Z)) \to \tilde{H}^{i+j}(H; Z)$ [2]. This spectral sequence collapses since $H^i(Z; \tilde{H}^j(Z/p; Z)) = 0$ when $i < 0$ or $i > 1$. Therefore, $1 \to \tilde{H}^{n-1}(Z/p; Z)_Z \to \tilde{H}^n(H; Z) \to \tilde{H}^n(Z/p; Z)_ZZ \to 1$ is an exact sequence. By looking at the $Z$ action on the subgroup $Z/p, u^d \in \tilde{H}^{2d}(Z/p; Z)$ is an invariant element of the $Z$ action on $\tilde{H}^{2d}(Z/p; Z)$. Here $u$ is a generator of $\tilde{H}^d(Z/p; Z)$. Therefore, there exists an element $h \in \tilde{H}^{2d}(H; Z)$ such that $Res(h) = u^d \neq 0$ on $\tilde{H}^{2d}(Z/p; Z)$. By Brown-Venkov theorem [2] and $\tilde{H}^{2kd}(H; Z) = Z/p, \tilde{H}^{2kd+1}(H; Z) = Z/p, \tilde{H}^i(H; Z) = 0$ for other $i$'s, the $p$-period of $H$ is $2d$. $\blacksquare$

Lemma 1.2. Let $Z/p$ be a normal subgroup of a group $\Gamma$ of finite index, and let $M$ be a finite quotient of $\Gamma$ with torsion free kernel. Then $\Gamma/C_{\Gamma}(Z/p) = N_{\Gamma}(Z/p)/C_{\Gamma}(Z/p) = N_{M}(Z/p)/C_{M}(Z/p) = M/C_{M}(Z/p)$. Here we still use $Z/p$ to stand for the image of $Z/p$ in $M$.

Proof: Let $pr : \Gamma \to M$ be the natural projection map. The map $pr$ maps $N_{\Gamma}(Z/p)$ onto $N_{M}(Z/p)$ and $C_{\Gamma}(Z/p)$ to $C_{M}(Z/p)$, so induced map $pr_* : N_{\Gamma}(Z/p)/C_{\Gamma}(Z/p) \to N_{M}(Z/p)/C_{M}(Z/p)$ is a well-defined surjective homomorphism. Let $(x) = Z/p$, if $yxy^{-1} = x^r$, then $pr(y)xpr(y)^{-1} = x^r$, i.e., $pr_*$ is an injective. $\blacksquare$

Lemma 1.3. Suppose a group $M$ contains a cyclic subgroup $Z/p^n \supset Z/p$ and $|N(Z/p^n)/C(Z/p^n)|$ is prime to $p$, then the homomorphism induced by inclusion $i_* : N(Z/p^n)/C(Z/p^n) \to N(Z/p)/C(Z/p)$ is injective.

Proof: Notice $N(Z/p) \supset N(Z/p^n)$ and the inclusion $i$ maps $C(Z/p^n)$ to $C(Z/p)$, i.e., the induced map by inclusion $i_* : N(Z/p^n)/C(Z/p^n) \to N(Z/p)/C(Z/p)$ is a well-defined homomorphism. Now let $(x) = Z/p^n$, then $(x^{p^n-1}) = Z/p$, if $y \in C(Z/p), yxy^{-1} = x^k$, then $yx^{p^n-1}y^{-1} = x^{kp^n-1} = x^{p^n-1}$, so $(k-1)p^n-1 = 0 \mod(p^n)$, i.e., $k = 1 \mod(p)$. Let $k = Ap^n + 1, A$ is prime to $p$ and $1 \leq m < n$, $k^d = 1 \mod(p^n)$, $d$ divides $p-1$.
by assumption. Hence \( k^d = (Ap^m + 1)^d = B + Ap^m + 1 = 1 \mod(p^n) \), where \( p^{2m} \) divides \( B \). This implies \( Ad = 0 \mod(p) \), a contradiction unless \( A = 0 \). 

Lemma 1.4 (Swan) [9]. Suppose the \( p \)-Sylow subgroup \( S_p \) of a finite group \( M \) is abelian. Let \( A_p \) be the group of automorphisms of \( S_p \) induced by inner automorphisms of \( M \). Then an element \( a \in H^4(S_p; \mathbb{Z}) \) is stable if and only if it is fixed under the action of \( A_p \) on \( H^4(S_p; \mathbb{Z}) \).

Proof: See [9].

Proof of Theorem 1: A theorem of Brown [3, p. 293] states that if \( \Gamma \) is \( p \)-periodic, then \( \hat{H}^*(\Gamma; \mathbb{Z})_{(p)} = \prod_{P_i \in S} \hat{H}^*(N(P_i); \mathbb{Z})_{(p)} \), where \( S \) is the set of all conjugacy classes of \( Z/p \) of \( \Gamma \). Therefore, the \( p \)-period of \( \Gamma \) is the least common multiple of the \( p \)-periods of \( N_{\Gamma}(P_i) \).

1) Lower bound. Let \( |N_{\Gamma}(P_i)/C_{\Gamma}(P_i)| = d_i, \langle x \rangle = P_i \). There exists \( y \in \Gamma \), such that \( yxy^{-1} = x^r \), \( r^{d_i} = 1 \mod(p) \). Let \( H = \langle x, y \rangle \) be a subgroup of \( \Gamma \) generated by elements \( x \) and \( y \). Then the \( p \)-period of \( H \) is \( 2d_i \) by Lemma 1.1, i.e., the \( p \)-period of \( N_{\Gamma}(P_i) \) is a multiple of \( 2d_i \).

2) Upper bound. Let \( pr : \Gamma \rightarrow M \) be a projection onto the finite quotient \( M \) whose \( p \)-Sylow subgroup is elementary abelian, and \( pr_i : N_{\Gamma}(P_i) \rightarrow M_i \) be the restriction map of \( pr \), where \( M_i \) is the image of \( pr_i \). Then \( M_i = \text{Im}N_{\Gamma}(P_i) = N_{M_i}(P_i) \) normalizes \( P_i \) (\( P_i \) also denotes the image of \( P_i \)), the group \( A_p \) of automorphisms of \( S_p \) induced by inner automorphisms of \( M_i \) maps \( P_i \) to itself.

Let \( u \in H^2(S_p; \mathbb{Z}) = \text{Hom}(P_i \times Z/p \times \ldots \times Z/p, \mathbb{Z}^*) \) be a cohomology element such that \( u(x) \neq 1 \) and \( u(y) = 1 \) if \( \langle x \rangle = P_i, \langle y \rangle = Z/p, \mathbb{Z}^* \) is the multiplicative group of nonzero complex numbers. Then \( \text{Res}(u) \neq 0 \) in \( H^2(P_i; \mathbb{Z}) \). Now we claim that \( u^{d_i} \in H^{2d_i}(S_p; \mathbb{Z}) \) is a stable element for \( S_p \) in \( M_i \). In fact, \( d_i = |N_{M_i}(P_i)/C_{M_i}(P_i)| \) by Lemma 1.2, and \( A_p \) fixes the element \( u^{d_i} \in H^{2d_i}(S_p; \mathbb{Z}) \) since \( N_{M_i}(P_i)/C_{M_i}(P_i) \) fixes the element \( u^{d_i} \). By Lemma 1.4 [9], \( u^{d_i} \) is a stable element for \( S_p \) in \( M_i \), i.e., there exists an element \( v \in H^{2d_i}(M_i; \mathbb{Z}) \) such that \( \text{Res}_{M_i}^{M_i}(v) = \text{Res}_{P_i}^{S_p}(u^{d_i}) = [\text{Res}_{P_i}^{S_p}(u)]^{d_i} \neq 0 \). If we apply the canonical homomorphism \( g^* \) from ordinary cohomology to Farrell cohomology [3, p. 278] we have \( \text{Res}_{P_i}^{M_i}(g^*(v)) = \text{Res}_{P_i}^{S_p}(g^*(u^{d_i})) = \text{Res}_{P_i}^{S_p}(g^*(u))^{d_i} \neq 0 \), i.e., there exists an element \( pr_i^*g^*(v) \in \tilde{H}^{2d_i}(N_{\Gamma}(P_i); \mathbb{Z}) \) such that \( \text{Res}_{P_i}^{N_{\Gamma}(P_i)}(pr_i^*g^*(v)) \neq 0 \) in \( \tilde{H}^{2d_i}(P_i; \mathbb{Z}) \), by Brown-Venkov theorem [2] and the fact that \( N_{\Gamma}(P_i) \) has only one order \( p \) subgroup, the \( p \)-period of
$N_{r}(P)$ divides $2d$. See following diagram.

\[
\begin{array}{ccc}
\hat{H}^{2d_i}(N_{r}(P_i); Z) & \xrightarrow{\text{Res}} & \hat{H}^{2d_i}(P_i; Z) \\
\hat{H}^{2d_i}(M_i; Z) & \xrightarrow{\text{Res}} & \hat{H}^{2d_i}(S_p; Z) & \xrightarrow{\text{Res}} & \hat{H}^{2d_i}(P_i; Z) \\
\hat{H}^{2d_i}(M_i; Z) & \xrightarrow{\text{Res}} & H^{2d_i}(S_p; Z) & \xrightarrow{\text{Res}} & H^{2d_i}(P_i; Z)
\end{array}
\]

Proof of Theorem 2: is basically a similar argument except for the upper bound part. In fact, if $\Gamma$ has a finite $p$-periodic quotient $M$ with torsion free kernel, then $\Gamma$ is $p$-periodic and the $p$-period of $\Gamma$ divides the $p$-period of $M$. This is because the inflation map $\hat{H}^{*}(M) \to \hat{H}^{*}(\Gamma)$ maps an invertible element of $\hat{H}^{*}(M)$ to an invertible element of $\hat{H}^{*}(\Gamma)$. Using Swan Theorem, we obtain that the $p$-period of $N_{r}(P)$ divides the $p$-period of $M$, which is $2|N_{M_i}(Z/p^n)/C_{M_i}(Z/p^n)|$. Also, by Lemma 1.3, the number $2|N_{M_i}(Z/p^n)/C_{M_i}(Z/p^n)|$ divides $2|N_{r}(P_i)/C_{M_i}(P_i)| = 2|N_{r}(P)/C_{M_i}(P_i)|$.

2. An example

Lemma 1.3, Lemma 1.1 and Swan Theorem imply that the equality $|N(S_p)/C(S_p)| = |N(Z/p)/C(Z/p)|$ holds in the case of a finite group $G$ whose a $p$-Sylow subgroup is cyclic, here $Z/p$ is the order $p$ subgroup of $S_p$. Therefore, Theorems 1 and 2 are generalizations of Swan Theorem.

In the case of a group $\Gamma$ of finite vcd, in general, $|N(S_p)/C(S_p)| \neq |N(Z/p)/C(Z/p)|$ even if all maximal $p$ subgroups $S_p$ of $\Gamma$ are cyclic. For example, let $\Gamma^* = \langle x, y | x^{p^2} = 1, yxy^{-1} = x^{p+1} \rangle$, and $d$ is the minimal positive integer such that $(p+1)^d = 1 \mod (p^2)$. Then $|N((x))/C((x))| = \hat{N}(\langle x \rangle)/C(\langle x \rangle)) = d = p$, but $|N((x^p))/C((x^p))| = 1$. A similar argument to Lemma 1.1 shows the $p$-period of $\Gamma^*$ above equals $2p$. This trivial example shows that the $p$-period of an infinite group $\Gamma$ can not be only described in the form $2LCM\{ |N(Z/p)/C(Z/p)| \}$ in general.

The example $\Gamma^*$ above could lead us to think that the $p$-period of a $p$-periodic group $\Gamma$ equals $2LCM\{ |N(C(p))/C(C(p))| \}$, where $C(p)$ ranges over all conjugacy classes of maximal $p$-cyclic subgroups of $\Gamma$. Recall in the case of a finite group $G$, Swan Theorem can be also stated in the different form: the $p$-period of $G$ equals $2|N(C(p))/C(C(p))|$ (including the case $p = 2$), where $C(p)$ is a maximal $p$-cyclic subgroup of $G$. 
Unfortunately, the next example shows that this is not true.

**Example.** Let \( \Gamma_{n,m} \) denote the congruence subgroup of \( SL(n, Z) \) of level \( m \), i.e., the kernel of the surjective homomorphism \( r_m : SL(n, Z) \rightarrow SL(n, Z/\mathbb{Z})/m \) induced by the reduction mod(\( m \)) (\( m \) may not be prime). It is well-known that the group \( \Gamma_{n,m} \) is always torsion free when \( n \geq 1 \) and \( m \geq 3 \). A result of Charney [4] states that the group \( \Gamma_{n,p} \) is cohomology stable with \( Z/2 \) coefficient for any odd prime \( p \). Define \( \Gamma_p = \lim_{\rightarrow} \Gamma_{n,p} \), then \( H^i(\Gamma_{n,p}, Z/2) = H^i(\Gamma_p, Z/2) \) for \( n \geq 2i + 5 \).

Let \( GL(Z) \) be the infinite general linear group of \( Z \) and \( \omega_i \in H^i(GL(Z); Z/2) \) the \( i \)-th Stiefel-Whitney class of the inclusion \( GL(Z) \rightarrow GL(R) \) for \( i \geq 1 \). We still denote by \( \omega_i \) the image of \( \omega_i \) under the restriction \( H^i(GL(Z); Z/2) \rightarrow H^i(SL(Z); Z/2) \rightarrow H^i(\Gamma_m; Z/2) \).

The calculation in [1] by Arlettaz gives following results: for any odd prime \( p \)

- a) \( \omega_1(\Gamma_p) = 0 \)
- b) \( \omega_2(\Gamma_p) \neq 0 \)
- c) \( \omega_3(\Gamma_p) = 0 \) if and only if \( p = 7 \mod(8) \).

Also, we know from Wu formula for the Steenrod square \( Sq^1(\omega_2) = \omega_1\omega_2 + 2\omega_0\omega_3 = \omega_3 \) in \( H^3(\Gamma_p, Z/2) \). Again, denote by \( \omega_i \) the image of \( \omega_i \) under the restriction \( H^i(\Gamma_{11,5}; Z/2) \rightarrow H^i(\Gamma_{11,11,5}; Z/2) \rightarrow H^i(\Gamma_m; Z/2) \) as long as \( n \geq 11 \).

Let \( \Gamma_0 \) denote the group of the extension \( 1 \rightarrow Z/2 \rightarrow \Gamma_0 \rightarrow \Gamma_{11,5} \rightarrow 1 \) which corresponds to the non-trivial cohomology element \( \omega_2 \in H^2(\Gamma_{11,5}; Z/2) \). Obviously, the group \( \Gamma_0 \) contains only one 2-subgroup \( Z/2 \), and the extension is central. Next, we check that the group \( \Gamma_0 \) is of finite vcd, then show that the 2-period of \( \Gamma_0 \) is greater than 2.

Consider the following commutative diagram, where all maps \( R_1, R_2, R_3 \) and \( R_4 \) are restriction maps.

\[
\begin{array}{ccc}
H^2(\Gamma_{11,4}; Z/2) & \xrightarrow{R_3} & H^2(\Gamma_{11,20}; Z/2) \\
\uparrow R_1 & & \uparrow R_2 \\
H^2(SL(11, Z); Z/2) & \xrightarrow{R_4} & H^2(\Gamma_{11,5}; Z/2)
\end{array}
\]

In fact, the map \( R_4 = 0 \) is a special case of the result by Millson [7, p. 85] which states that for any \( n \geq 3 \) the map \( r^* : H^2(SL(n, Z/4); Z/2) \rightarrow H^2(SL(n, Z), Z/2) \) induced by the reduction mod(4) is an isomorphism.
Thus, we obtain the nontrivial second Stiefel-Whitney class $w_2$ in $H^2(\Gamma_{11,5}; Z/2)$, but the restriction of $w_2$ into the cohomology of the finite index subgroup $H^2(\Gamma_{11,20}; Z/2)$ is 0. This actually proves that the group $\Gamma_0$ is finite $vcd$ and the $vcd(\Gamma_0) = vcd(\Gamma_{11,20}) = vcd(SL(11, Z)) = 55$ [3, p. 229].

In order to find a lower bound on the 2-period of $\Gamma_0$, consider two spectral sequences as follows:

1. The Lyndon-Hochschild-Serre spectral sequence of the group extension $1 \to Z/2 \to \Gamma_0 \to \Gamma_{11,5} \to 1$ with $Z/2$ coefficient. This takes the form $E_2^{i,j} = H^i(\Gamma_{11,5}; H^j(Z/2; Z/2)) \Rightarrow H^{i+j}(\Gamma_0; Z/2)$.

2. The Farrell cohomology spectral sequence [2] of the group extension $1 \to Z/2 \to \Gamma_0 \to \Gamma_{11,5} \to 1$ with $Z/2$ coefficient. This takes the form $E_2^{i,j} = H^i(\Gamma_{11,5}; \tilde{H}^j(Z/2; Z/2)) \Rightarrow \tilde{H}^{i+j}(\Gamma_0; Z/2)$.

Let $u \in H^1(Z/2; Z/2)$ be the generator of the cohomology ring $H^*(Z/2; Z/2) = F_2[u]$, and $d_2(u) = w_2 \in H^2(\Gamma_1; Z/2)$ be the second Stiefel-Whitney class corresponding to the extension $1 \to Z/2 \to \Gamma_0 \to \Gamma_{11,5} \to 1$. Then $u$ is transgressive, $d_2(u) = \tau(u) = w_2$, where $\tau$ is the transgression. The element $u^2 = Sq^1(u)$ and $d_2(u^2) = \tau(u^2) = \tau(Sq^1(u)) = Sq^1(\tau(u)) = Sq^1(w_2) = w_3 \neq 0$ in $E_3$ because $H^1(\Gamma_{11,5}; Z/2)$ is trivial.

Consider a commutative diagram involving in both spectral sequences as follows:

$$
\begin{array}{ccc}
H^0(\Gamma_{11,5}; \tilde{H}^2(Z/2; Z/2)) & \longrightarrow & H^3(\Gamma_{11,5}; \tilde{H}^0(Z/2; Z/2)) \\
\uparrow \cong & & \uparrow \cong \\
H^0(\Gamma_{11,5}; H^2(Z/2; Z/2)) & \longrightarrow & H^3(\Gamma_{11,5}; H^0(Z/2; Z/2))
\end{array}
$$

The nontriviality of $d_3$ in the second row implies the nontriviality of $d_3$ in the first row. This shows $Res: \tilde{H}^2(\Gamma_0; Z/2) \to \tilde{H}^2(Z/2; Z/2)$ is trivial since the map $Res$ factors through $E_2^{0,2} = 0$. Therefore, there is no invertible element in $\tilde{H}^2(\Gamma_0; Z/2)$. By the fact that the reduced map $\tilde{H}^2(\Gamma_0; Z)_{(2)} \to \tilde{H}^2(\Gamma_0; Z/2)$ is ring homomorphism, there is no invertible element in $\tilde{H}^2(\Gamma_0; Z)_{(2)}$, i.e., the 2-period of $\Gamma_0$ is greater than 2. We have proved our Theorem 3.

3. The $p$-period of the mapping class group $\Gamma_g$

The $p$-periodicity of the mapping class group is studied in a different paper of the author [11]. As an application of the theorem 1, we obtain the $p$-period of a $p$-periodic mapping class group $\Gamma_g$ when $g < p(p-1)/2$. 
Recall that the mapping class group $\Gamma_g$ is defined to be the group of path components of orientation preserving diffeomorphisms of the closed orientable surface $S_g$ of genus $g > 1$. Next, we define a set $B_{g,p}$ for surface $S_g$ and a prime $p$.

**Definition.** For $p$ odd, let $2g - 2 = mp - i, 0 \leq i \leq p - 1$.

\[
B_{g,p} = \{i, i + p, i + 2p, \ldots, i + \left(\frac{2g}{p-1}\right) - m\}
\]

if $i \neq 1$.

\[
B_{g,p} = \{1 + p, 1 + 2p, \ldots, 1 + \left(\frac{2g}{p-1}\right) - m\}
\]

if $i = 1$.

And for $p = 2$,

\[
B_{g,2} = \{0, 4, 8, \ldots, 2g + 2\}
\]

if $g$ is odd.

\[
B_{g,2} = \{2, 6, 10, \ldots, 2g + 2\}
\]

if $g$ is even.

**Remarks.**

1. The notation $\lfloor \cdot \rfloor$ here means the integer part.

In case $i \neq 1$, $2g/(p-1) < m$, define $B_{g,p} = \emptyset$.

In case $i = 1$, $2g/(p-1) < m + 1$, define $B_{g,p} = \emptyset$.

2. It is proved in [11] that the set $B_{g,p}$ is exactly the set of all possible number of fixed points when an order $p$ diffeomorphism acts on the surface $S_g$.

**Lemma 3.1.** For the mapping class group $\Gamma_g$, there is a formula

\[
\text{LCM}\left\{\frac{N(x)}{C(x)}\right\} = \text{LCM}\left\{\gcd(p-1, b_i)\right\},
\]

where $(x)$ ranges over all conjugacy classes of $Z/p$ in $\Gamma_g$, $b_i$ ranges over all $b_i \in B_{g,p}$.

**Proof:** 1) Assume $\lfloor N(x)/C(x) \rfloor = d$. Then there exists an integer $r$ such that $x \approx x^r \approx \cdots \approx x^{r^{d-1}}$ ($\approx$ means "is conjugate to" in $\Gamma_g$) so that $d$ is the minimal positive integer satisfying $r^d = 1 \mod(p)$. The $d$ divides $p - 1$ obviously. Let $b$ be the number of fixed points of the $x$ action on $S_g$, $\sigma(x) = (\beta_1, \beta_2, \ldots, \beta_b)$ the fixed point datum, where $\beta_i \in Z/p - \{0\}$ (cf. [10]).

Let us define a permutation $r^*$ on the ordered $b$-tuple $(\beta_1, \beta_2, \ldots, \beta_b)$. Set $r^*(\beta_1, \beta_2, \ldots, \beta_b) = (r\beta_1, r\beta_2, \ldots, r\beta_b)$, $(r^*)^2 = (r^2)^* \cdots \cdots (r^*)^{d-1} = (r^{d-1})^*$. It is well-defined since $\sigma(x) = \sigma(x^{r^2}) = \cdots = \sigma(x^{r^{d-1}})$ as an unordered $b$-tuples [12]. We can decompose $r^* = (\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_s})(\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_t}) \cdots (\beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_u})$, a product of cyclic permutations. Notice that permutations $r^*$, $(r^*)^2$, \ldots $(r^*)^{d-1}$ do not have fixed points. Otherwise, there exists $\beta_i$ such that $r_j \beta_i = \beta_i \mod(p), 1 \leq j \leq d - 1$. This forces $r_j = 1 \mod(p)$, a contradiction. But, of course, $(r^*)^d = (r^d)^* = \text{Id}$. These imply
The $p$-period of an infinite group

$s = t = \cdots = u = d$, i.e., the number $|N((x))/C((x))|$ divides the number $b_i$ of fixed points of the $x$ action on the surface $S_g$. We have showed that $\text{LCM}\{|N((x))/C((x))|\}$ divides $\text{LCM}\{\gcd(p-1,b_i)\}$, where $\langle x \rangle$ ranges over all conjugacy classes of $Z/p$ in $\Gamma_g$, $b_i$ ranges over all $b_i \in B_{g,p}$.

2) Conversely, assume $\gcd(p-1,b_i) = d$. Then there is a mod($p$) integer $r$ so that $d$ is a minimal positive integer satisfying $r^d = 1 \pmod{p}$.

**Case 1.** $b_i \neq 0$. If $d \neq 1$, then $r \neq 1$. Consider the unordered $b_i$-tuples $\sigma = (1, r, r^2, \ldots, r^{d-1}, 1, r, r^2, \ldots, r^{d-1}, \ldots, 1, r, r^2, \ldots, r^{d-1})$. Since $(b_i/d)(1+r+r^2+\cdots+r^{d-1}) = 0 \pmod{p}$. There exists an element $x \in \Gamma_g$, $x^p = 1$, and the it's representative fixed point datum $\sigma(x)$ is $\sigma$, i.e., the unordered $b_i$-tuples $\sigma$ can be realized as a fixed point datum of an order $p$ element in $\Gamma_g$ [6]. Obviously, $\sigma(x) = \sigma(x^r) = \sigma(x^{r^2}) = \cdots = \sigma(x^{r^{d-1}})$ or $x \approx x^r \approx x^{r^2} \approx \cdots \approx x^{r^{d-1}}$ in $\Gamma_g$. This implies that the number $d$ divides the order $|N((x))/C((x))|$. If $\gcd(p-1,b_i) = d = 1$, for any order $p$ element $x$ in $\Gamma_g$ with the number of fixed points $b_i$, obviously 1 divides $|N((x))/C((x))|$.

**Case 2.** $b_i = 0$. On the one hand, we have $\gcd(p-1,b_i) = p - 1$. On the other hand, the $x$ acts on $S_g$ freely. All order $p$ free actions are conjugate by [5], this implies $|N((x))/C((x))| = p - 1$.

So, $\text{LCM}\{\gcd(p-1,b_i)\}$ divides $\text{LCM}\{|N((x))/C((x))|\}$. ■

**Proof of Theorem 4:** Let $\mu : \Gamma_g \to Sp(2g,Z)$ be the canonical homology representation and $p : Sp(2g,Z) \to Sp(2g,F_q)$ be the reduction map. Here $q$ can be chosen a primitive root of mod($p$) such that $q \geq 3$, and $q^{p-1}$ is not congruent to 1 mod($p^2$) (by the Dirichlet theorem).

Now Ker$(p\mu) = N$ is a torsion free, normal, finite index subgroup of $\Gamma_g$ and a $p$-Sylow subgroup of the finite quotient $\Gamma_g/N = Sp(2g,F_q)$ is elementary abelian if $2g < p(p-1)$. Then we can use Theorem 1 and Lemma 3.1 to finish the proof. ■

A list of the $p$-period of a $p$-periodic mapping class group $\Gamma_g$ can be also found in the Appendix C of the author's thesis [12].

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References

1. Artlettaz, D., Sur les classes de Stiefel-Whitney des sous-groupes


12. Xia, Y., Farrell-Tate cohomology of the mapping class group, Thesis, The Ohio State University.

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