THE FUNDAMENTAL THEOREM OF ALGEBRA BEFORE CARL FRIEDRICH GAUSS

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Abstract

This is a paper about the first attempts of the demonstration of the fundamental theorem of algebra.

Before, we analyze the tie between complex numbers and the number of roots of an equation of n-th degree.

In second paragraph we see the relation between the integration and fundamental theorem.

Finally, we observe the linear differential equation with constant coefficients and the Euler's position about the fundamental theorem and then we consider the d'Alembert's, Euler's and Laplace's demonstrations.

It is a synthesis paper dedicated to Pere Menal a colleague and a friend.

1. Introduction: The Complex Numbers

In the year 1545 Gerolamo Cardano wrote *Ars Magna*\(^1\). In this book, Cardano offers us a process for solving cubic equations, learned from

\(^1\)There are many interesting papers on complex numbers. See, for example, Jones, P. S. [43], Molas, C-Pérez, J. [57] and Remmert, R. [67]. Moreover, in this paper, our interest on complex numbers is limited only in their connexion with algebra and particularly with the *Fundamental Theorem of Algebra*. 


Josèp Vicens FOIX

En la calle mayor de los que han muerto, el deber de vivir iré a gritar

Enrique BADOSA

To be or not to be.

That is the question.

William SHAKESPEARE
Niccolò Tartaglia\textsuperscript{2}. In his book it appears for the first time an special quadratic equation:

\begin{quote}
If some one says you, divide 10 into two parts, one of which multiplied into the other shall produce 30 or 40, it is evident this case or equation is impossible\textsuperscript{3}.
\end{quote}

Cardano says then

\begin{quote}
Putting aside the mental tortures involved, multiply \(5 + \sqrt{-15}\) by \(5 - \sqrt{-15}\,\), making \(25 - (-15)\,\), which is +15. Hence this product is 40 ... This is truly sophisticated ... \textsuperscript{4}
\end{quote}

But, as Remmert remembers us, “it is not clear whether Cardano was led to complex numbers through cubic or quadratic equations”\textsuperscript{5}. The sense of these words is the following: while quadratic equations

\[ x^2 + b = ax, \text{ with } \Delta = \frac{1}{4}a^2 - b < 0, \]

have no real roots \(\text{[and they are therefore impossible equations]},\) cubic equations

\[ x^3 = px + q, \text{ with } \Delta = \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3 < 0, \]

have real roots which are given as sums of imaginary cubic roots\textsuperscript{6}. This question was further developed by Rafael Bombelli in his \textit{L'Algebra}, published in Bologna in 1572. Bombelli worked out the formal algebra of

\begin{quote}
Cardano's rule for cubic equation \(x^3 = px + q\) is

\[ x = \sqrt[3]{\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{q}{2} - \sqrt{\Delta}}, \text{ where } \Delta = \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3. \]
\end{quote}

The history of the process for solving cubic equations is now perfectly known. See, for example, Burton, D. M. [12, 302-313]; Stillwell, J. [76, 59-62]; Vera, F. [80, 47-59] and van der Waerden, B. L. [85, 54-55].

See also Tartaglia, N. [78, 69 and 120].

\textsuperscript{2}Cardano's rule for cubic equation \(x^3 = px + q\) is

\[ x = \sqrt[3]{\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{q}{2} - \sqrt{\Delta}}, \text{ where } \Delta = \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3. \]

\textsuperscript{3}Cardano, G. [16, Ch. 37]. See also Struik, D. J. [77, 67].

The equation \(x^2 - 10x = 40 \text{ or } 30\) has the solutions \(5 \pm \sqrt{13}\) \(\text{or } 5 \pm \sqrt{3}\) \(\text{and both solutions are formally corrects, but in this time they have not any sense.}\)

\textsuperscript{4}Cardano, G. [16, Ch. 37]. See also Struik, D. J. [77, 69 and footnote 7].

The name imaginary is introduced by René Descartes, as we will see soon. But it is debt, perhaps, to following Cardano's words: "... you will nevertheless imagine \(\sqrt{-15}\) to be the difference between ...", completing, in that case, the square.

It is interesting to observe that Cardano accompanied his result over this kind of quadratic equation with the comment: "the result in that case is as subtle as it is useless" [see Cardano, G. [16, Ch. 37, rule 11] and also Struik, D. J. [77, 69]].

\textsuperscript{5}Remmert, R. [67, 57].

\textsuperscript{6}We can see van der Waerden, B. L. [84, 194]: It is not possible solve, by real radicals, an irreductible cubic equation over \(\mathbb{Q}\) whose three roots are all real [casus irreductibilis, following Cardano].
complex numbers. He introduced (in actual notation) the complex unit $i$ and eight fundamental rules of computation:

\[
\begin{align*}
(+1) \cdot i &= +i; 
(+1) \cdot (-i) &= -i; 
(+i) \cdot (+i) &= +1; 
(-1) \cdot i &= -i; 
(-1) \cdot (-i) &= +i; 
(-i) \cdot (+i) &= +1; 
(-i) \cdot (-i) &= -1.
\end{align*}
\]

His principal aim consisted to reduce expressions as $\sqrt{a + bi}$ to the form $c + di$ because then it should be possible to use formally the Cardano's expression by solving the casus irreductibilis $x^3 = 15x + 4$. Bombelli obtains, according the Cardano's expression,

\[x = \sqrt[3]{2} + 11i + \sqrt[3]{2} - 11i.\]

Hence $x = (2 + i) + (2 - i) = 4$.

François Viète wrote in 1591 a higher level paper, which relates algebra to trigonometry. In this paper Viète offers us his solution of the cubic equation by circular functions, which shows that solving the cubic is equivalent to trisecting an arbitrary angle. He starts (in modern notation) from the identity

\[\cos 3\theta = 4\cos^3 \theta - 3\cos \theta\]

[or $x^3 - \frac{3}{4}x - \frac{1}{4}\cos 3\theta = 0$, where $x = \cos \theta$]. Suppose now that the cubic to be solved is given by

\[x^3 - px = q \quad [p, q > 0].\]

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7 It is perhaps interesting to remember that the symbol $i$ for indicate imaginary unit is debt to Euler: "In the following I shall denote the expression $\sqrt{-1}$ by the letter $i$ so that $ii = -1$" [Euler, J. [25, 1301]]. See Kline, M. [44, 410]: "In his earlier work Euler used $i$ (the first letter of infinitesimal) for an infinitely large quantity. After 1777 he used $i$ for $\sqrt{-1}$".

8 Really, Bombelli introduced più di meno [for $+i$] and meno di meno [for $-i$] and rules of calculation such as

\[\text{meno di meno via meno di meno fa meno}\]

which means $(-i) \cdot (-i) = -1$.

See Bombelli, R. [9, 169] or Bertolotti reprint, 133.

9 Bombelli did not through too much on the nature of complex numbers, but he knew, for example, that

\[(2 \pm i)^3 = 2 \pm 11i,\]

so that

\[\sqrt[3]{2} \pm 11i = 2 \pm i.\]

See Bombelli, R. [9, 110] or Bertolotti reprint, 140-141.

10 This paper, "De equatione recognitione et emendatione", written by Viète in 1591, was not published until 1615 by his Scottish friend Alexander Anderson.

11 See Viète, F. [81, Ch. VI, Th. 3].

12 See Hollingsdale, S. [41, 122-123].
If we introduce an arbitrary constant \( \lambda \), setting \( x = \lambda z \), then

\[
2^3 - \frac{p}{\lambda^2} z - \frac{q}{\lambda^3} = 0.
\]

We can now match coefficients in the two forms

\[
\frac{p}{\lambda^2} = \frac{3}{4} \quad \text{and} \quad \frac{q}{\lambda^3} = \frac{1}{4} \cos 3\theta, \quad \text{so that} \quad \lambda = \sqrt{\frac{4p}{3}}.
\]

With this value of \( \lambda \), we can select a value of \( \theta \) so that

\[
\cos 3\theta = \frac{4q}{\lambda^3} = \frac{q/2}{\sqrt{(p/3)^3}}.
\]

In the casus irreductibilis, we have

\[
\Delta = \left( \frac{q}{2} \right)^2 - \left( \frac{p}{3} \right)^3 < 0 \quad \text{and then} \quad \left| \frac{q/2}{\sqrt{(p/3)^3}} \right| < 1
\]

and thus the condition for three real roots ensures us that \( |\cos 3\theta| < 1 \), which is essential\(^\text{13}\).

In 1637 René Descartes wrote *La Géométrie*\(^\text{14}\). This appendix was his only mathematical work; but a what work! It contains the birth of analytic geometry\(^\text{15}\). In Book III of his *La Géométrie* Descartes gives a brief summary of that was known about equations\(^\text{16}\). Between his

\(^{13}\)Then he proves the equivalence: we have \( \cos 3\theta = \mu \), where \( \mu = \frac{q/2}{\sqrt{(p/3)^3}} \). Given \( \mu \), we can construct a triangle with angle \( 3\theta = \cos^{-1} \mu \). Trisection of this angle gives us the solution \( x = \cos \theta \) of the equation. Conversely, the problem of trisecting an angle with cosine \( \mu \) is equivalent to solve the cubic equation \( 4x^3 - 3x = \mu \).

\(^{14}\)It is, as it is well known, the third appendix of his famous *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences*. The other appendices are *La Dioptrique* and *Les Météores*. For a comment we can see Bos, H. J. M. [10], Milhaud, C. [56, 124–175], Pla, J. [63], or Scott, J. F. [71, 84–157].

\(^{15}\)The analytic geometry was independently discovered by Pierre Fermat, a French amateur mathematician, in his "Ad locos planos et solidos isagoge" [32].

\(^{16}\)John Wallis in his *Algebra* [86] declared that there was little in Descartes which was no to be found in the *Artis Analyticae Praxis* [39] of Harriot' [see Scott, J. F. [71, 138] and Wallis, J. [87, 126]]. But, says Scott [Scott, J. [71, 138]], "this statement is far from true".
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algebraic assertions\textsuperscript{17}, we are interested in the following:

\textit{in every equation there are as many distinct roots as is the number of dimensions of the unknown quantities}\textsuperscript{18}.

This is an important approach to \textit{Fundamental Theorem of Algebra}, but it is not the first and perhaps never the more explicit.

The first writer to assert that "every such equation of the \textit{n}th degree has \textit{n} roots and no more\textsuperscript{19} seems to have been Peter Roth\textsuperscript{19}. The law was next set forth by a more prominent algebraist, Albert Girard, in 1629:

\textit{Every algebraic equation admits as many solutions as the denomination of the highest quantity indicates} \ldots \textsuperscript{20}

Girard gives no proof or any indication of one. He merely explains his proposition by some examples, including that of the equation \(x^4 - 4x + 3 = 0\) whose solutions are 1, 1, -1 + \(i\sqrt{2}\), -1 - \(i\sqrt{2}\).\textsuperscript{21}

\textsuperscript{17}The other important assertions in Book III of \textit{La Géométrie} are:

- A polynomial \(P(x)\) which vanishes at \(c\) is always divisible by the factor \(x - c\) and then

\[ P(x) = (x - c) \cdot Q(x), \text{ where } \deg(Q(x)) = \deg(P(x)) - 1. \]

[This theorem was probably already known by Thomas Harriot, following Remmert, R. \[68, 99 footnote 2\].]

- Descartes' rule of signs: we can determine from this also the number of true and false roots that any equation can have, as follows: Every equation can have as many true roots as it contains changes of signs, from + to - or from - to +, and as many false roots as the number of times two + signs or two - signs are found in succession. (This law was apparently known by Cardano [Cantor, M. \[15, II, 539\], but a satisfactory statement is possibly due to Harriot [Harriot, \[39, 18, 268\]]. See also Smith, E. D. \[73, II, 471\].] [On limitations or mistakes in Descartes' rule see, for example, Scott, J. F. \[71, 140\].]

This rule was formulated in a more precise manner by Isaac Newton in his \textit{Arithmetica Universalis}, composed between 1673 and 1683, perhaps for Newton's lectures at Cambridge, but first published in 1707. Newton's rule counts moreover complex roots.

This Newton's work contains also the formulas, usually known as Newton's identities, for sums of the power of the roots of polynomial equations.

\textsuperscript{19}Descartes, R. \[19, 372\]. English translation in Smith, D.E.-Latham, M. \[75, 159\].

\textsuperscript{19}Peter Roth, who name also appears as Rothe, was a Nürnberg Rechenmeister, died at Nürnberg in 1617. He wrote, in 1600, his \textit{Arithmetica philosophica}, where we can find the quoted statement.

\textsuperscript{20}See Girard, A. \[38\] in Viète and alii \[83, 139\] and in Struik, D. J. \[77, 85\]. See also Tropfke, J. \[79, III(2), 951\] for further details.

\textsuperscript{21}There are opposed opinions about the real content in these formulations. Whilst for Smith [Smith, D. E. \[73, II, 471\]] "this law was more clearly expressed by Descartes
Later another mathematician, named Rahn [or Rohnius], also gave a clear statement of the law in his Teutschen Algebra [66].

The question about these formulations of the Theorem is the following: these algebraists accepted real and complex numbers and only them as solutions of equations? The answer is not easy nor clear. Girard accepts the "impossible solutions" with these words

_Someone could also ask what these impossible solutions are. I would answer that they are good for three things: for the certainty of the general rule, for being sure that there are no other solutions, and for its utility_  

Descartes, by his side, realized the fact that an equation of the nth degree has exactly n roots[23]. But, for Descartes, the _imaginary roots_ do never correspond any real quantity[24].

[19], who not only stated the law but distinguished between real and imaginary roots and between positive and negative real roots in making the total number[2]. For Remmert [Remmert, R. [68, 100]], contrarily, "Descartes takes a rather vague position on the thesis put forward by Girard".

22Girard, A. [38] in Viète and ali [83, 141]. In other side [Viète and ali [83, 142]) he says: "Thus we can give three names to the other solutions, seeing that there are some which are greater than nothing, other less than nothing, and other enveloped, as those which have \(\sqrt{\frac{-1}{3}}\), like \(\sqrt{-3}\) or other similar numbers."

Remmert, R. [68, 99], goes further. He says: "He thus leaves open the possibility of solutions which are not complex". Remmert thinks that, in his ambiguity, Girard leaves open a door to the solutions more complicated than the complex. The problem consists to know the exact sense of the Girard's words "impossible solutions" because, for him, "there are no other solutions". [About this question see also Gilain; C. [37, 93-95].]

23This assert is debt to the Descartes' text. [see Descartes, R. [19, 380]. English translation in Smith, D.E.-Latham, M. [75, 175]]:

> Neither the true nor false roots are real; sometimes they are imaginary; that is, while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation

\[
x^3 - 6x^2 + 13x - 10 = 0
\]

as having three roots, yet there is only one real root, 2, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, remains always imaginary.

In this text there is a rather interesting classification signifying that we may have positive and negative roots that are imaginary.

It seems that for Descartes the roots are always real or imaginary and no other kind of root is possible. [About with this opinion, see Gilain, C. [37, 95-97].]

The use of word _imaginary_ in his actual sense begin here [see Smith, D.E.-Latham, M. [75, 175, footnote 207]].

24Descartes confess that one is quite unable to visualize imaginary quantities [see
This impossibility or difficulty for visualizing imaginary quantities was perhaps the reason which carried the English mathematician John Wallis to give a geometrical interpretation in his *Treatise of Algebra* of 1685.²⁵

He says: "The Geometrical Effection, therefore answering to this Equation

\[ a \cdot a \neq b \cdot a + c = 0 \]

may be this."²⁶

²⁵This representation is quoted in Smith, D. E. [74, 46-54]. See also Stillwell, J. C. [76, 191-192]. In a letter to Collins, May 6, 1673, Wallis suggests a construction a little different from any of the constructions found in his *Algebra* [see Cajori, F. [13]]. We shall see this alternative construction here:

²⁶Wallis, J. [87] in Smith, D. E. [74, 52]. Wallis calls the independent term \( ae \). It is the product of two roots \( a \) and \( e \) of the equation \( a \cdot a \neq b \cdot a + ae = 0 \).

Before this, Wallis offers as the following calculation for solving

\[ a \cdot a - 2a\sqrt{175} + 256 = 0. \]
On $AC\alpha = b$ bisected in $C$, erect a Perpendicular $CP = \sqrt{c}$. And taking $PB = \frac{1}{2}b$ make a Rectangular Triangle [figure 3a].

If $PC = \sqrt{c} < \frac{1}{2}b = PB$, then the solutions are Real and are precisely $AB$ and $B\alpha$.

Figure 3a

[In this case $AB = \frac{1}{2}b - \sqrt{\frac{1}{4}b^2 - c}$ and $\alpha B = \frac{1}{2}b + \sqrt{\frac{1}{4}b^2 - c}$ [see Smith, D. E. [74, 53]].]

But if $PC > PB$, "the above construction fails" and "the Right Angle will be at $B$". Then the solutions are Imaginary and are $AB$ and $\alpha B$ [see figure 3b]. [Now $AB = \frac{1}{2}b - i\sqrt{c - \frac{1}{4}b^2}$, $\alpha B = \frac{1}{2}b + i\sqrt{c - \frac{1}{4}b^2}$. Wallis uses the later $BC$ to obtain the imaginary part of the solution.]

This geometrically representation was not accepted by the mathematicians\(^{27}\) and would be still necessary to wait a hundred years to obtaining

The solutions are $a = \sqrt{175} + \sqrt{-81}$ and $e = \sqrt{175} - \sqrt{-81}$. The geometrical representation is [following Wallis, J. [87] in Smith, D. E. [74, 50-51]]:

Figure 2

\(^{27}\)In Stillwell, J. [76, 192], we can see the Wallis' figures and the modern
the correct and acceptable representation. We shall not comment this work.

2. The technique of integration and complex quantities

The eighteenth century use of the integral concept was limited. Newton represented the transcendental functions as series and integrated these functions term by term. Gottfried Wilhelm Leibniz and Johann Bernoulli treated the integral as the inverse of the differential.

In this context the decomposition of rational fractions [or functions] into partial [or simple] fractions made possible a decisive step in integral calculus.

The problem was calculate the integral

$$\int \frac{P}{Q} \, dx,$$

where $P$ and $Q$ are polynomials and $\deg(P) < \deg Q$ and, for getting it, Gottfried Wilhelm Leibniz and Johann Bernoulli, together other mathematicians of his time, saw the necessity to express every real polynomial as product of real factors of first and second degree. This fact shows us that they had very much confidence in the Fundamental Theorem of Algebra.

representation.

28 The satisfactory geometrically representation of complex quantities was carried by the Norwegian mathematician Caspar Wessel in 1797 and independently by the Swiss Jean Robert Argand in 1806. This last work, despite its considerable merit, remained unnoticed until a French translation appeared in 1897.

29 See Kline, M. [44, 406]: "If $dy = f'(x) \cdot dx$, then $y = f(x)$. That is, a Newtonian antiderivative was chosen as the integral, but differentials were used in place of Newton's derivatives".

30 The existence of an integral was never questioned.

31 The Arithmetica Universalis of Isaac Newton contains, as we have said before, the substance of Newton's lectures from 1673 to 1683 at Cambridge. In it are found many important results in equations theory, such as the fact that the imaginary roots of a real polynomial "must occur in conjugate pairs". This fact is a very important result and it was naturally accepted by the mathematicians of the end of seventeenth century. But, following Leibniz, this fact presents difficulties, as we shall see next.

32 See Leibniz, G. W. [49], [51] and Bernoulli, Jh. [6].

The chance did that in 1702, July 10, Johann Bernoulli, thinking to enumerate him a new result, wrote to Leibniz that had found the integral of differential quantities $\frac{p}{q} \, dx$, where $p$ and $q$ are polynomials. But Leibniz responded: "No only I have already the solution of this problem, but moreover I have it from the first years in which I practiced the higher geometry. In this result I have seen an essential component of
The exact Bernoulli’s text is:

Let the differential be $p dx : q$ which $p$ and $q$ express rational quantities composed arbitrarily of a single variable $x$ and constants; one seeks the integral or the algebraic sum or the means of reducing it to the quadrature of the hyperbola or the circle, the one or the other always being possible.\[34\]

And next he says that $\frac{dx}{x^2}$ is the differential of logarithm of $x \pm a$. Therefore

$$\int \frac{a \, dx}{x + f} + \int \frac{b \, dx}{x + g} + \int \frac{c \, dx}{x + h} + \cdots = \log \left\{ (x + f)^a \cdot (x + g)^b \cdot (x + h)^c \cdot \cdots \right\}.$$

But the remarkable question is that “complex numbers made their entry to the theory of circular functions”. The process is the following: he observes that “one transforms the differential $\frac{adz}{b^2 - z^2}$ into a logarithmic differential $\frac{adz}{b^2 + z^2}$ and reciprocally”\[35\] and, as a corollary, “one transforms the differential $\frac{adz}{b^2 - z^2}$ in the same way into $-\frac{adz}{2bzt}$, an imaginary logarithmic differential and reciprocally”\[36\]. But then he observes that $\frac{adz}{b^2 + z^2}$ can

my science of the infinite, and hence of the integral analysis ...” [Leibniz, G. W. [49, 703]].

“When, in 1746, Jean le Rond d’Alombert drew people’s attention to the need to prove that theorem, he was to cite Bernoulli’s paper as a particularly important use of it” [see Fauvel, J.-Gray, J. [31, 455]].

\[34\] Bernoulli, Jh. [6] in Opera Omnia, 1, 393 or Fauvel, J.-Gray, J. [31, 439].

\[35\] He uses the change of variable $z = \frac{t^2 - 1}{t^2 + 1}$ and observes that $\frac{adz}{b^2 - z^2}$ goes over into $\frac{adt}{2bt}$ [see Fauvel, J.-Gray, J. [31, 438]]. How does Johann Bernoulli obtain this result? It is clear that Johann Bernoulli knows the integral of rational functions as $\frac{a}{b^2 - z^2}$, because he knows the decomposition of the rational functions into simple functions:

$$\frac{a}{b^2 - z^2} = \frac{a}{2b} \cdot \frac{1}{b - z} + \frac{a}{2b} \cdot \frac{1}{b + z}.$$

And then he applies his technique and obtains

$$\int \frac{a \, dx}{b^2 - z^2} = \frac{a}{2b} \cdot \log \frac{b + z}{b - z} = \frac{a}{2b} \cdot \int \frac{dt}{t},$$

where $t = \frac{b + z}{b - z}$.

And then $x = b \cdot \frac{t - 1}{t + 1}$.

\[36\] Similarly, the differential $\frac{adz}{b^2 - z^2}$ goes over, by the substitution $z = \frac{b^2 - 1}{b^2 - t}$, into $-\frac{adt}{2bt}$ [see Fauvel, J.-Gray, J. [31, 438]].

Remember that, in 1699, Jakob Bernoulli had evaluated [see Bernoulli, Jk. [1699], 868–870] the integral of $\frac{a^2 \, dx}{b^2 - z^2}$, using the change of variable $x = \frac{b^2 z}{b^2 - 1}$; this converts the integrand $\frac{a^2 \, dz}{b^2 - z^2}$ into $\frac{dt}{2bt}$ [See Kline, M. [44, 407].]
be transformed also [using now \( z = \sqrt{\frac{1}{a} - b^2} \)] into the differential of "a sector or circular arc \( \frac{adz}{2\sqrt{1-b^2}} \) and reciprocally". Finally he observes that the integral of

\[
\frac{adz}{b^2 + z^2}
\]

depends on the quadrature of the circle, and moreover

\[
\frac{adz}{b^2 + z^2} = \frac{1}{2b} \cdot \frac{adz}{b + iz} + \frac{1}{2b} \cdot \frac{adz}{b - iz}
\]

which are two differentials of imaginary logarithms: one sees that imaginary logarithms can be taken for real circular sectors because the compensation which imaginary quantities makes on being added together of destroying themselves in such a way that their sums is always real\(^{37}\).

We have observed there the introduction of imaginary logarithmic differential into the integration of rational functions\(^{38}\).

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\[
\tan^{-1} z = \frac{1}{2i} \cdot \log \frac{i - z}{i + z}.
\]

In this sense it is interesting to note that, several years later, in 1712, Johann Bernoulli carried out the integration to obtain an algebraic relation between \( \tan n\theta \) and \( \tan \theta \). His argument is as follows. Given

\[ y = \tan n\theta, \quad x = \tan \theta, \]

we have

\[ n\theta = \tan^{-1} y = n \cdot \tan^{-1} x; \]

hence, taking differentials,

\[ n\, d\theta = \frac{dy}{1 + y^2} = n \cdot \frac{dx}{1 + x^2} \]

and then

\[ \left[ \frac{1}{y + i} - \frac{1}{y - i} \right] \cdot dy = n \left[ \frac{1}{x + i} - \frac{1}{x - i} \right] \cdot dx. \]

Integration gives

\[ \log \frac{y + i}{y - i} = \log \left[ \frac{x + i}{x - i} \right]^n \]

and whence

\[ (x - i)^n \cdot (y + i) = (x + i)^n \cdot (y - i). \]

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\(^{38}\)We do not explain the history of imaginary logarithms. But there are many papers on complex logarithms as, for example, Cajori, F. [14], Kline, M. [44, 407–408]; Naux, I. [58] and Stillwell, J. [76, 220–222].
But this situation is not easier than it seems. In his presentation about the integral of rational functions, Leibniz shows us a difficulty, a limitation or merely a question. It is always possible decompose a real polynomial into a product of real linear factors or real quadratic factors? or, every polynomial has always a real and complex root and, with every complex root, has also the conjugate complex root? Although always Leibniz is clear and rotund when he says

As soon as I had found my Arithmetic Quadrature, reducing the quadrature of circle into a rational quadrature and observing that the sum

\[ \int \frac{dx}{1 + x^2} \]

depends of the quadrature of the circle, I immediately observed that a time reduced to the summation of a rational expression, all quadrature can be converted in many kinds of summation of the more simple. And I will show, by a decomposition proceeding of a new genus because it must be in this manner. This proceeding consists to convert a product of factors into a sum; this is, to transform a fraction with a denominator of higher degree, equal to product of roots, into a sum of fractions with simple denominators,

when he must integrate \( \int \frac{dx}{x^4 + a^4} \) he finds a problem. It is possible obtain \( \frac{1}{x^4 + a^4} \) to multiply \( \frac{1}{x^2 + ia^2} \) by \( \frac{1}{x^2 - ia^2} \), but they are not real. And it is not possible to obtain a real decomposition, because

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39 This assertion is absolutely clear in Newton, I. [59], as we have seen in the footnote 32.

40 Leibniz, G. W. [50, 351-352].

In this work Leibniz obtains naturally the integration of rational functions, as for example

\[ \int \frac{dx}{x^2 - 1} = \frac{1}{2} \int \frac{dx}{x - 1} - \frac{1}{2} \int \frac{dx}{x + 1}, \]

although "\( \int \frac{dx}{y} \) is the quadrature of the hyperbola".

Next year Leibniz studies the case in which the roots are not simple and therefore the sum is transformed into the sum of fractions with multiple denominators [see Leibniz, G. W. [51]].
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\[ x^4 + a^4 = \left[ x + a\sqrt{i} \right] \cdot \left[ x - a\sqrt{i} \right] \cdot \left[ x + a\sqrt{-i} \right] \cdot \left[ x - a\sqrt{-i} \right] \]

and therefore it is not possible to reduce \( \int \frac{dx}{x^4 + a^4} \) to the quadrature of the circle nor to the quadrature of the hyperbola. It would be necessary to introduce the quadrature of \( \int \frac{dx}{x^4 + a^4} \) as a new function.

There is neither hesitation about the importance which Leibniz granted the complex numbers and his contributions, "when they were almost forgotten", were remarkable. Between these it is interesting to observe that he obtained an imaginary decomposition of a positive real number which surprised his contemporaries and enriched the theory of imaginaries:

\[ \sqrt{6} = \sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}} \]

41 Leibniz does not observe that

\[ x^4 + a^4 = [x^2 + a\sqrt{2} + a^2] \cdot [x^2 - a\sqrt{2} + a^2]. \]

The possible mistake is due to have begun by the complex conjugate decomposition

\[ x^4 + a^4 = [x^2 + ia^2] \cdot [x^2 - ia^2]. \]

42 We have already introduced the quadrature of the hyperbola \( \int \frac{dx}{x^2 + a^2} \) and the quadrature of the circle \( \int \frac{dx}{x^2 + a^2} \). Then, says Leibniz, "I wait that we will be able to follow this progression and we will find the problems related with \( \int \frac{dx}{x^4 + a^4}, \int \frac{dx}{x^8 + a^8}, \ldots \)" [see Leibniz, G. W. [50, 360]].

43 Moreover, for Leibniz, complex numbers are the natural consequence of have accepted real numbers: "From the irrationals are born the impossible or imaginary quantities whose nature is very strange but whose usefulness is not to be despised" [see Leibniz, G. W. [50, 51]].

44 See a letter from Leibniz to Huygens, written in 1674 or 1675 [Gerhardt, C. I. [36, 563] and see also Hofmann, J. E. [1972], 147 and McClemon, R. B. [55]: "I once came upon two equations of this kind \( x^2 + y^2 = b, x \cdot y = c \). He obtains then

\[ y = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}} \quad \text{and} \quad x^2 - \frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2} = 0 \quad \text{or} \quad x = \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}}. \]

Then

\[ d = x + y = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}} + \sqrt{\frac{b^2}{2} - \sqrt{\frac{b^2}{4} - c^2}} \quad \text{or} \quad d^2 = b + 2c. \]
Moreover, as says Boyer, "Leibniz did not write the square roots of complex numbers in standard complex form, nor was he able to prove his conjecture that

\[ f(x + iy) + f(x - iy) \text{ is real,} \]

if \( f(x) \) is a real polynomial."\(^{45}\)

Finally in an unpublished Leibniz's paper\(^{46}\) appears the so-called de Moivre's formula. He does not explain how he found it, but it is comprehensible to us as

\[ 2y = \sqrt{x + \sqrt{x^2 - 1}} + \sqrt{x - \sqrt{x^2 - 1}}, \]

where \( x = \cos \theta, y = \cos \frac{\theta}{n} \).\(^{47}\) But these important mathematical contributions did not enough to clarify the nature and reality of the complex numbers.

Finally

\[ \sqrt{b + 2c} = \sqrt{\frac{b}{2} + \sqrt{\frac{b}{4} - c^2}} + \sqrt{\frac{b^2}{2} - \sqrt{\frac{b^2}{4} - c^2}} \]

If we put \( b = 2 \) and \( c = 2 \), there results \( \sqrt{5} = \sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}} \).

But that what results really surprising is the use of Cardano's rule by obtaining this kind of results. Taking Albert Girard's equation

\[ x^3 - 13x - 12 = 0,\]

whose true root is 4 and the sum of roots is zero, Leibniz obtains

\[ 4 = 2 + \sqrt{-\frac{1}{3}} + 2 - \sqrt{-\frac{1}{3}} = 3 \sqrt[3]{6 + \sqrt{\frac{-1225}{27}}} + 3 \sqrt[3]{6 - \sqrt{\frac{-1225}{27}}}. \]

By using the equation

\[ x^3 - 48x - 72 = 0, \]

he shows finally that

\[ -6 = 3 \sqrt[3]{36 + \sqrt{-2800}} + 3 \sqrt[3]{36 - \sqrt{-2800}} \]

Hofmann says us "the identity (*) is implicit in Euclide's book X, 47–54 [if \( 4c^2 < b^2 \)], but "nobody noticed it at the time".

\(^{45}\)Boyer, C. B. [11, 444]. This conjecture is done by Leibniz in Gerhardt, C. I. [36, 550].

\(^{46}\)Leibniz, G. W. [49].

\(^{47}\)See Hofmann, J.E. [1972], 145–146; Schneider, I. [72, 224–229] and Stillwell, J.
Leibniz adventures his mistic nature, saying: “The nature, mother of the eternal diversities, or the divine spirit, are zealous of her variety by accepting one and only one pattern for all things. By these reasons she has invented this elegant and admirable proceeding. This wonder of Analysis, prodigy of the universe of ideas, a kind of hermaphrodite between existence and non-existence, which we have named imaginary roots”.

This mysterious character stood during several centuries, may be until the Euler’s time with the contributions of the own Euler and d’Alembert.

Kline is absolutely clear in this sense:

Complex numbers were more of a bone to the eighteenth-century mathematicians. These numbers were practically ignored from their introduction by Cardan until about 1700. Then complex numbers were used to integrate by the methods of partial fractions, which was followed by the lengthy controversy about complex numbers and the logarithms of negative and complex numbers. Despite his correct resolution of the problem of the logarithms of complex numbers, neither Euler nor the other mathematicians were clear about those numbers.

Kline tried to understand what complex numbers really are, and in his “Vollständige Anleitung zur Algebra”, which first appeared in Russian in 1768–69 and in Germany in 1770 and is the best algebra text of the eighteenth century, says,

Because all conceivable numbers are either greater than zero or less than 0 or equal to 0, then it is clear that the square roots of negative numbers cannot be included among the possible numbers [real numbers]. Consequently we must say that these are impossible numbers. And this circumstance leads us to the concept of such numbers, which by their nature are impossible, and ordinarily are called imaginary or fancied numbers, because they exist only in the imagination.

Euler made mistakes with complex numbers. In this Algebra he writes \( \sqrt{-1} \cdot \sqrt{-4} = \sqrt{4} = 2 \), because \( \sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b} \). He also gives \( i = 0.2078795763 \), but misses other values of this quantity.


48 Leibniz, G. W. [49, 387].

49 Kline, M. [44, 554].
3. The three first attempts to prove the Fundamental Theorem of Algebra

One possible enunciation of the Fundamental Theorem of Algebra\(^{50}\) is:

*Every polynomial \( P(x) \) with real coefficients has a complex root.*

Before 1799, year in which Karl Friedrich Gauss gave his first rigorous proof of Fundamental Theorem of Algebra\(^{51} \), three important mathematicians had already made three attempts to prove the Theorem. The first is debt to a French mathematician and philosopher, Jean le Rond d'Alembert, and was published in 1748, but elaborated in 1746. Three years later, in 1749, Leonhard Euler gave an algebraic demonstration, very different of the d'Alembert's demonstration. This demonstration was completed by Joseph Louis Lagrange in 1772\(^{52} \). Several years later another French mathematician, Pierre Simon Laplace, tried to prove the Theorem. It was the year 1795\(^{53} \).

There are excellent papers about the Fundamental Theorem of Algebra. See, for example, Bashmakova, I. [4], Dieudonné, J. et alii [20, 68–71], Gilain, C. [37], Houzel, C. [42], Petrova, S. S. [61], Remmert, R. [67] and van der Waerden, B. L. [1980], 94–102.

The Gilain's text offers us a distinction between the Fundamental Theorem of Algebra —sometimes known as the d'Alembert's Theorem— and the Theorem of linear factorization —sometimes known as the Kronecker Theorem— very clever for understand posterior developments and clarify the different kinds of demonstrations [see Gilain, C. [37, 92]].

But I think that, historically, this distinction is not clear. The former mathematicians to Gauss was not conscious of that fact.

Gauss considered the Theorem so important that he gave four proofs; the principles on which the first is based was discovered by Gauss in October 1797, but the proof was not published until 1799. In this proof, similar to d'Alembert's attempt of proof, he does not introduce complex numbers. He proves the Theorem in the form:

*Every polynomial \( P(x) \) with real coefficients can be factored into linear or quadratic factors.*

The second and third proofs of Theorem were published in 1816. The second proof is purely algebraic, following perhaps the Euler's intention. The forth proof is based in the same principle of the first and was published in 1849. In this proof Gauss uses already complex numbers more freely because, he says, "they are now common knowledge". In the third proof he used, in fact, that what we today know as the Cauchy integral theorem.

A half century dedicated by Gauss to prove the Theorem. Following these different demonstrations we can find precisely the differences noted by Gilain.

The Euler and Lagrange attempts were published, respectively, in 1751 and 1774.

Pierre Simon Laplace made an attempt to prove the Theorem, quite different from the Euler-Lagrange attempt but also algebraic, in his *Leçons de mathématiques donnés à l'Ecole Normal*, published in 1812.
Really therefore was Euler the first of these three mathematicians which asserted the true of the Theorem. So in a letter to Nikolaus Bernoulli, Euler enunciates the factorization theorem for real polynomials, closing the question posed by Leibniz.

We have already seen that "does not seem to have occurred to Leibniz that \(\sqrt{i}\) could be of the form \(a + bi\), because if he had seen that

\[
\sqrt{i} = \frac{1}{2} \sqrt{2} \cdot [1 + i] \quad \text{and} \quad \sqrt{-i} = \frac{1}{2} \sqrt{2} \cdot [1 - i],
\]

he would have noticed that the product of the factors

\[
[X + a\sqrt{i}] \cdot [X + a\sqrt{-i}] \quad \text{and} \quad [X - a\sqrt{i}] \cdot [X - a\sqrt{-i}]
\]

are both reals and then he would have obtained

\[
X^4 + a^4 = [X^2 + a\sqrt{2}X + a^2] \cdot [X^2 - a\sqrt{2}X + a^2].
\]

So he would have avoid his mistake. It is remarkable that he should not have been led to this factorization by the simple advice for writing \(X^4 + a^4 = [X^2 + a^2]^2 - 2a^2 X^2\) (see Remmert, R. [67, 100]).

See also Kline, M. [44, 597-598]: "... Leibniz did not believe that every polynomial with real coefficients could be decomposed into linear and quadratic factors. Euler took the correct position. In a letter to Nikolaus Bernoulli of October 1, 1742, Euler affirmed without proof that a polynomial of arbitrary degree with real coefficients could be so expressed [see Euler, L. [1862], 1, 525]. Nikolaus did not believe the assertion to be correct and gave the example of

\[
x^6 - 4x^3 + 2x^2 + 4x + 4
\]

with the imaginary roots \(1 + \sqrt{2 + \sqrt{-3}}, 1 - \sqrt{2 + \sqrt{-3}}, 1 + \sqrt{2 - \sqrt{-3}}, 1 - \sqrt{2 - \sqrt{-3}}, \) which he said contradicts Euler's assertion [see Euler, L. [27, II, 695]].

On December 15, 1742, Euler into a letter to Goldbach [see Euler, L. [27, I, 170-171]], after assert that he doubted once when he saw this example, did it doubt once seen the example, "pointed out the complex roots occur in conjugate pairs, so the product of \(x - [a + bi]\) and \(x - [a - bi]\), wherein \(a + bi\) and \(a - bi\) are a conjugate pair, gives a quadratic expression with real coefficients. Euler then showed that his was true for Bernoulli's example. But Goldbach, too, rejected the idea that every polynomial with real coefficients can be factored into real factors and gave the example \(x^4 + 72x - 20\) [see the letter from Goldbach to Euler of February 5, 1743 in Euler, L. [27, I, 193]].

Euler then showed Goldbach that the later had made a mistake and that he [Euler] had proved this theorem for polynomials up to the sixth degree. However, Goldbach was not convinced, because Euler did not succeed in giving a general proof of this assertion.

The reader interested to follow the succession of these letters can see, for example, Gilain, C. [37, 106-108].
Next year, in a very important paper, Euler thinks about the homogeneous $n$th-order differential equation with constant coefficients

$$0 = Ay + B \frac{dy}{dx} + C \frac{d^2y}{dx^2} + D \frac{d^3y}{dx^3} + \cdots + L \frac{d^ny}{dx^n},$$

where $A, B, C, D, \ldots, L$ are constants. He points out that the general solution of [1] must contain $n$ arbitrary constants and the solution will be a sum of $n$ particular solutions $y_j$, every one multiplied by an arbitrary constant. So the general solution of $y$ has the form

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n.$$ 

Then he makes in [1] the substitution

$$y = e^{\int r \, dx},$$

with $r$ constant, and obtains the polynomial equation in $r$,

$$A + Br + Cr^2 + \cdots + L r^n = 0.$$ 

In fact, the general solution depends of the factorization of the polynomial [3] and of the nature of its roots —reals or complex; simple or multiple—, and indirectly his result depends essentially of the Fundamental Theorem.

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55Euler, L. [22].

56Each root $r_j$ of the polynomial equation [3] furnishes a partial solution into the sum [2] in accordance with the nature of each root $r_j$, $j = 1, \ldots, n$:

- if $r_j$ is a real simple root of [3], then it furnishes into the sum [2] the summand

$$z_j = D_j e^{r_j x};$$

- if $r_j$ is a multiple real root of multiplicity $k$, the $k$ equal roots $r_j$ furnish into the sum [2] the summand

$$z_{j,k} = e^{r_j x} \left[ D_0 + D_1 x + \cdots + D_{k-1} x^{k-1} \right];$$

- if $r_j = \alpha_j + i \beta_j$ is a simple complex root of [3], then it and its conjugate $\overline{r_j} = \alpha_j - i \beta_j$ furnish into the sum [2] the summand

$$z_j^* = e^{\alpha_j x} \left[ D_1^* \cos \beta_j x + D_2^* \sin \beta_j x \right];$$

and finally,

- if $r_j = \alpha_j + i \beta_j$ is a multiple complex root of multiplicity $k$, then the $k$ equal roots $r_j = \alpha_j + i \beta_j$ and their $k$ conjugate roots furnish into the sum [2] the
But, as we have already said, the first attempt of demonstration of the Fundamental Theorem of Algebra is debt to d'Alembert\textsuperscript{57}.

3.1. The d'Alembert's attempt.

Really d'Alembert proves the existence of the root of $P(x)$ in two steps\textsuperscript{58}:

1. There is the \textit{minimum} $x_0$ of the module $|P(x)|$;
2. The d'Alembert's lemma: if $P(x_0) \neq 0$, then any neighborhood of

\begin{equation}
x^*_{j,k} = \sum_{\ell=0}^{k-1} e^{2\pi i j \ell} \{ D_1^* \cos \beta_2 x + D_2^* \sin \beta_3 x \}
\end{equation}

Somewhat later [Euler, L. \textsuperscript{[24]}] he treated the nonhomogeneous nth-order linear differential equation

\[X(x) = Ay + B \frac{dy}{dx} + C \frac{d^2 y}{d^2 x}\]

\textsuperscript{57}D'Alembert remembers the Johann Bernoulli's text and then he says: "Nobody, what I know, have went more far [in the question of the decomposition of polynomials], if we exclude mister Euler, which in the tome VII of Miscellanea Berolinensia declares that he has demonstrated the proposition in the general case. But I seem me that Euler never has published yet on this theorem [d'Alembert, J. le Rond \textsuperscript{[2, 183]}].

\textsuperscript{58}See d'Alembert, J. le Rond \textsuperscript{[2]} and Petrova, S. S. \textsuperscript{[62]}. In the d'Alembert's words:

\textit{In order to reduce in general a differential rational function to the quadrature of the hyperbola or to that of the circle, it is necessary, according to the method of M. Bernoulli [Mem. Acad. Paris, 1702], to show that every rational polynomial, without a divisor composed of a variable $x$ and of constants, can always be divided, when it is of even degree, into trinomial factors $xx + fx + g$, $xx + hx + i$, etc., of which all coefficients $f, g, h, i, \ldots$ are real. It is clear that this difficulty affects only the polynomial that cannot be divided by any binomial $x + a, x + b$, etc., because we can always by division reduce to zero all the real binomials, if two are any, and it can easily be seen that the products of these binomials will give real factors $xx + fx + g$ [see Struik, D. J. \textsuperscript{[77, 89, footnote 1]}].}
$x_0$ contains a point $x_1$ such that $|P(x_1)| < |P(x_0)|$\textsuperscript{59}.

Then, if 1 and 2 are true and $x_0$ is the point in which $|P(x)|$ attains the minimum, then $|P(x_0)| = 0$. This is the sketch of the d'Alembert's proof\textsuperscript{60}.

The second step is, for d'Alembert, the more important and the proof offered by d'Alembert depends essentially on the Newton's method.

\textsuperscript{59}D'Alembert accepts without demonstration the step 1 and the Newton's method. A simple elementary proof of d'Alembert lemma was given by Argand in 1806. This mathematician was one of the co-discoverers of the geometric representation of complex numbers. He represents the complex numbers as vectors into the plane. Then

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a vector $OA_{n+1}$. The demonstration consists to see that it is possible to choose $x$ such that the point $A_{n+1}$ coincides with $O$. By seeing this, he explains

$$P(x_0 + \Delta x) = P(x_0) + A \Delta x + \text{terms in } (\Delta x)^2, (\Delta x)^3, \cdots = P(x_0) + A \Delta x + \epsilon$$

where $A$ is constant and $|\epsilon|$ is small compared to $|\Delta x|$ when $|\Delta x|$ is small.

Then, choosing the adequate direction of vector $\Delta x$, it is possible obtain that $A \Delta x$ was opposite in direction to $P(x_0)$. Then

$$|P(x_0 + \Delta x)| < |P(x_0)|.$$

\textsuperscript{60}By seeing a complete proof of this kind, see, for example, Aleksandrov et alii [1]; Dörrie, H. [21, 108-112], or Stillwell, J. [76, 197-200].

\textsuperscript{61}The first step was naturally accepted in the eighteenth century. The rigorous demonstration can be seen into Cauchy, A. [1821], Ch. X: "For every polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x],$$

there is a $x_0 \in \mathbb{C}$ such that $|P(x_0)| = \inf |P(C)|$.}
of polygon. Applying this, d'Alembert obtains

\[ x - x_0 = \sum_{k \geq 0} c_k \cdot |y - y_0|^k. \]

The equation (*) shows that, if \( y \) is a real point very close to \( y_0 \), it is the image of any \( x \) which appears into the form \( p + q \sqrt{-1} \). Then the demonstration of the Theorem is founded if we can prove that \( y_0 = 0 \) is the image of any \( x \) which will be naturally real or imaginary.

D'Alembert examines the set of real images \( y \) and takes the minimum \( y_0 \) which associate \( x \) is of complex form. But following the development (*), all real number \( y \) very close to \( y_0 \) must be also an image of the complex numbers \( x \). Then, if \( y_0 \neq 0 \), there is an image closer to zero than \( y_0 \). Contradiction. This contradiction establishes the Theorem.

It is interesting to note two important facts which were observed by d'Alembert in his work. The first are corollaries I and II and proposition III and says: "if a complex number \( a + b \sqrt{-1} \) is a root of the polynomial \( P(x) \), then \( a - b \sqrt{-1} \) is another root of \( P(x) \) and then \( P(x) \) can always be decomposed into quadratic factors of the kind \( x^2 + mx + n \)."

The second fact, contained in the demonstration but not mentioned explicitly, is: "if \( P(x) \) is a real polynomial and we substitute \( x \) by a complex number \( z = z_1 + iz_2 \), where \( z_1, z_2 \) are real numbers, then we obtain \( P(z) = Q_1(z_1) - i Q_2(z_2) \), where \( Q_1(x) \) and \( Q_2(y) \) are real polynomials. Then \( P(z) = 0 \) if \( Q_1(z) = 0 \) and \( Q_2(z) = 0 \)."

3.2. The Euler-Lagrange's attempt.

The idea of Euler's demonstration was to decompose every monic polynomial with real coefficients \( P(x) \) of degree \( 2^n \geq 4 \) into a product

\[ x - x_0 = \sum_{k \geq 0} c_k \cdot |y - y_0|^k \]
in a neighborhood of \( y_0 \). This theorem was proved rigorously by Puisieux in 1850.

It is possible to avoid this theorem like we can see, for example, in Dörrie, H. [21, 108-112].

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52 See Newton, I. [59] and Stillwell, J. [76, 125-126]. The sense of this theorem is the following: "To every pair \( (x_0, y_0) \) of complex numbers with \( y_0 - P(x_0) = 0 \), there correspond an increasing series \( \{q_k\} \) of rational numbers such that

53 See d'Alembert, J. le Rond [2, 189].

54 See d'Alembert, J. le Rond [2, 190-191].

55 See d'Alembert, J. le Rond [2, 186-187].

56 This fact is essential in the first Gauss' demonstration [see, for example, Hollingdale, S. [41, 319-322]].

57 See Euler, L. [23] and Lagrange, J.-L. [45].
$P_1(x) \cdot P_2(x)$ of two monic polynomials with real coefficients of degree $m = 2^n - 1$.

Thus, if $P(x)$ is a polynomial of the form

$$P(x) = x^{2m} + B x^{2m-2} + C x^{2m-3} + \ldots,$$

the polynomials $P_1(x), P_2(x)$ now take the form

$$x^m + u x^{m-1} + \alpha x^{m-2} + \beta x^{m-3} + \ldots$$

$$x^m - u x^{m-1} + \lambda x^{m-2} + \mu x^{m-3} + \ldots$$

Then Euler asserts that $\alpha, \beta, \ldots, \lambda, \mu, \ldots$ are real functions in $B, C, \ldots, u,$ and that, by elimination of $\alpha, \beta, \ldots, \lambda, \mu, \ldots,$ is obtained a monic real polynomial in $u$ of degree $\left(\frac{2m}{m}\right)$ whose constant term is negative. Now this polynomial in $u$ has a zero $u$ by the intermediate value theorem as Euler clearly knew\textsuperscript{68}. Now we can follow quickly the Euler's steps\textsuperscript{69}:

1. If the equation has a root of the form $x + y \sqrt{-1}$, then there is also another of the form $x - y \sqrt{-1}$\textsuperscript{70};
2. Every equation of odd degree has a least one root;
3. Every equation of even degree with negative absolute term has at least one positive and one negative root\textsuperscript{71}.

But it is the forth theorem which gives us the key of his ideas:

Every equation of the forth degree, as

$$x^4 + A x^3 + B x^2 + C x + D = 0$$

can always be decomposed into two real factors in the second degree.

First, setting $x = y - \frac{1}{2} A$, he obtains that every equation of the forth degree can be of the form $x^4 + M x^2 + N x + P = 0$. If we decompose this equation in two equations of the second degree, we have

$$[x^2 + u x + \alpha] \cdot [x^2 - u x + \beta] = 0.$$
If we compare this product with the proposed equation, we shall find

\[ M = \alpha + \beta - \mu^2, \quad N = [\beta - \alpha] \mu, \quad P = \alpha \beta \]

from which we derive

\[ \mu^6 + 2M \mu^4 + [M^2 - 4P] \mu^2 - N^2 = 0, \]

"from which the value of \( \mu \) must be found. And since the absolute term \(-N \cdot N\) is essentially negative, we have hope that this equation has at least two real values\(^2\)."

Among the corollaries to Theorem 4 there is the statement that the resolution into real factors is now also proved for the fifth degree, and Scholiwm II points out that, if the roots of the given fourth-degree equation are \( x_1, x_2, x_3, x_4 \), then the sixth-degree equation in \( \mu, \mu \) being the sum of two roots of the given equation, will have the six roots \( x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_3 + x_4 \). Since \( x_1 + x_2 + x_3 + x_4 = 0 \), we can write for \( \mu \) the values \( \mu_1, \mu_2, \mu_3, -\mu_1, -\mu_2, -\mu_3 \), and the equation in \( \mu \) becomes

\[ (\mu_1^2 - \mu_1^2) \cdot (\mu_2^2 - \mu_2^2) \cdot (\mu_3^2 - \mu_3^2) = 0^3. \]

\(^2\)When we take one of them as \( \mu_1 \), then the values of \( \alpha \) and \( \beta \) will also be real, seeing that

\[ 2\beta - \mu_1 + M - \frac{N}{\mu}, \quad 2\alpha = \mu_1 + M - \frac{N}{\mu}. \]

\(^3\)We can observe that the fourth roots \( x_1, x_2, x_3, x_4 \) of the equation

\[ x^4 + M x^2 + N x + P = 0 \]

satisfies

\[ x_1 + x_2 + x_3 + x_4 = 0. \]

Then \( \mu \) can have \( \binom{4}{2} = 6 \) different values. Then \( \mu \) satisfies an equation of the sixth degree whose coefficients are reals

\[ F_6(\mu) = 0. \]

We have \( u_1 = x_1 + x_2, u_2 = x_1 + x_3, u_3 = x_1 + x_4, u_4 = x_2 + x_3, u_5 = x_2 + x_4, u_6 = x_3 + x_4 \) and then

\[ u_1 = -u_6, u_2 = -u_5, u_3 = -u_4 \]

and then the equation \([3]\) has the form

\[ F_6(\mu) = [\mu^2 - \mu_1^2] \cdot [\mu^2 - \mu_2^2] \cdot [\mu^2 - \mu_3^2]. \]
Next to, into the theorem 5, he establishes

*Every equation of degree 8 can always be resolved into two real factors of the fourth degree.*

The problem consists to see that not only $u$, but also the other coefficients $\alpha, \beta, \gamma, \delta, \epsilon, \psi$ are reals, a reasoning which Lagrange and, more later, Gauss objected.

Lagrange takes this equation but he observes that when $u$ takes the value 0 into the rational expressions of the other coefficients of $P_1(x)$ and $P_2(x)$ as function of $u$, it is possible obtain undefined coefficients of the form $\frac{0}{0}$. For avoid this, he takes as unknown $v = 2n + a_{n-1}$ and then observes that the “imaginary roots” of the

His constant term is $-u_1^2 u_2^2 u_3^2$. The product $u_1^2 u_2^2 u_3^2$ is real? There is. Euler does not explain this with detail. He says only that this product is real because the fundamental theorem of the theory of symmetric functions.

We can reasoning this: Despite this product was not a symmetric function of the symbols $x_1, x_2, x_3, x_4$, it is unvariable when we do all possible permutations of the roots of the equation [1], under the condition [2], between the roots of the equation [1]. Really this product can be obtained of the following:

$$u_1^2 u_2^2 u_3^2 = \frac{1}{4} \left\{ (x_1 + x_2) \cdot (x_1 + x_3) \cdot (x_1 + x_4) + (x_1 + x_2) \cdot (x_2 + x_3) \cdot (x_2 + x_4) + \right.$$

Remember that the fundamental theorem of the theory of symmetric functions says:

*Every rational function of roots of an algebraic equation

$$\phi(x_1, x_2, \ldots, x_n)$$

which takes $k$ different values when it makes all possible permutations of roots, satisfies an algebraic equation of degree $k$ whose coefficients are rational functions of the coefficients of the given equation.*

Then, if $k = 1$, the function $\phi(x)$ satisfies a rational expression of the coefficients of the given equation.

Euler uses largely this fundamental theorem, but he only develop, with a sufficient rigour, for the general case of the second degree equations, but the theorem in his general form was proved firstly by Lagrange in his transcendental paper *Réflexions sur la résolution algébrique des équations* [1771]. So it will be necessary hope the Lagrange's apports by obtaining the general result.

74 First, the term $x^7$ is eliminated, so that the two supposed factors can be written $x^4 - u x^3 + \alpha x^2 + \beta x + \gamma$ and $x^4 + u x^3 + \delta x^2 + \epsilon x + \psi$. Since $u$ expresses the sum of four roots of the eight-degree equation, it can have $\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 70$ values, and it will satisfy an equation of the form

$$0 = [u^2 - p^2] \cdot [u^2 - q^2] \cdot [u^2 - r^2] \cdot [u^2 - s^2] \cdots$$

with 35 factors. The absolute term is negative, and the reasoning continues as before.
equation in the unknown $v$ are the expressions

$$v_{\sigma} = \sum_{k=1}^{r} z_{\sigma(k)} - \sum_{k=1}^{r} z_{\sigma(k+r)}$$

where $\sigma$ runs over the set $S_n$ of all permutations of set $\{1, 2, \ldots, n\}$. It is easy to see that the product of $v_{\sigma}$ is always $\leq 0$. Next he avoids the case in which the product is zero, substituting $v_{\sigma}$ for a useful combination of the coefficients of $P_1$ with real coefficients and then using his results contained in a paper of 1770-1771 on permutations of an equation, finishes rightly the demonstration.

3.3. The Laplace's attempt.

In the year 1795, Pierre Simon Laplace made an attempt to prove the Fundamental Theorem. This attempt was completely algebraic, but quite different from the Euler-Lagrange attempt. This mathematician and politician assumes, as his predecessors, that the roots of polynomials "exist".

Laplace says:

"Of this it results a demonstration very simple of this general theorem which we have announced before and which says that every equation of even degree can be solved into real factors of second degree."

His prove is the following: Let be $x_1, x_2, \ldots, x_n$, where $n = 2^k q, k \leq 1, q \in 2\mathbb{N} + 1$, the roots of the polynomial

$$P(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} + \cdots + (-1)^n b_n \in \mathbb{R}[x], n \leq 1.$$
The equation \( Q_t(x) \) which roots are \( x_i + x_j + t(x_i x_j) \), where \( t \in \mathbb{R} \) arbitrary and \( i < j \), has a degree of the form \( 2^{k-1} q' \), where \( q' \in 2\mathbb{N} + 1 \).

Then Laplace proceeds by induction on \( k \):

- If \( k = 1 \), the new polynomial \( Q_t(x) \) will have an odd degree and then it will be a least a real root \( x_i + x_j + t(x_i x_j) \).

It is clear that there is infinitely many real values \( t \) such that, for a same \( x_i \) and \( x_j \),

\[
x_i + x_j + t(x_i x_j) \in \mathbb{R}.
\]

Then there are \( t_1 \neq t_2, t_1, t_2 \in \mathbb{R} \), such that \( x_i + x_j + t_1(x_i x_j), x_i + x_j + t_2(x_i x_j) \in \mathbb{R} \). Then the quantities

\[
[t_1 - t_2](x_i x_j), \ x_i x_j \text{ and } x_i + x_j
\]

are all real. So the factor \( x^2 - [x_i + x_j]x + x_i x_j \) will be a real factor of second degree of \( P(x) \);

- If \( k > 1 \), then \( P(x) \) will have a real factor of second degree if every equation of degree \( 2^{k-1} q' \) has a factor of second degree, because infinitely many

\[
x_i + x_j + t(x_i x_j), i < j, t \in \mathbb{R}
\]

will be complex numbers [that is: they are of the form \( \alpha + i\beta, \alpha, \beta \in \mathbb{R} \)] and then, following the precedent reasoning, there are two roots \( x_i, x_j \) of \( P(x) \) such that \( x_i + x_j, x_i x_j \in \mathbb{C} \). Therefore the factor

\[
x^2 - [x_i + x_j]x + x_i x_j \in \mathbb{C}[x]
\]

and it divides exactly \( P(x) \). Then

\[
x^2 - [x_i + x_j]x + x_i x_j \in \mathbb{C}[x]
\]

divides also \( P(x) \). Thus the following real polynomial of forth degree

\[
[x^2 - [x_i + x_j]x + x_i x_j] \cdot [x^2 - (\overline{x_i} + \overline{x_j})x + \overline{x_i} \overline{x_j}] =
\]

\[
= [x^2 - \text{Re}(x_i + x_j)x + \text{Re}(x_i x_j)]^2 + [\text{Im}(x_i x_j) - \text{Im}(x_i + x_j)]^2.
\]

This quantity, "as we have seen", can be solved in two real factors of second degree.

\[\text{Laplace applies the following corollary of the Intermediate value Theorem: "Every polynomial of odd degree has at least one real root".}\]

\[\text{See Laplace, P.-S. \cite{47, 60-63}.}\]

\[\text{Laplace considers the case in which the two factors}\]

\[
[x^2 - [x_i + x_j]x + x_i x_j], [x^2 - (\overline{x_i} + \overline{x_j})x + \overline{x_i} \overline{x_j}]
\]
Then the problem is finished because \( P(x) \) has a real factor of second degree iff every real equation of degree \( 2^{k-1} q', q' \in 2N+1 \) has a similar factor, and then [for the same reason] iff every equation of \( 2^{k-2} q'', q'' \in 2N+1 \) has a similar factor and following we establish the proof.

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