ON MUCKENHOUPT AND SAWYER CONDITIONS FOR MAXIMAL OPERATORS

Y. RAKOTONDRAITSIMBA

Abstract

Let $M_s(0 \leq s < n)$ be the maximal operator

$$(M_s f)(x) = \sup \left\{ \frac{|Q|^s}{|x|^s} \left\| f \right\|_{L^1(dx)}; \ Q \text{ a cube with } Q \ni x \right\},$$

and $u(x)$ and $v(x)$ be weight functions on $\mathbb{R}^n$. For $1 < p \leq q < \infty$ and $q < q^{-1} \leq (s/n)$, we prove the equivalence of the Sawyer condition

$$\|M_s u^{-1/(p-1)} 1_Q\|_{L^q_u} \leq S \|1_Q\|_{L^p_{v^{-1/(p-1)}}}$$

for all cubes $Q$ to the Muckenhoupt condition

$$|Q|^\frac{d}{n} + \frac{1}{p} - \frac{1}{q} \left( \frac{1}{|Q|} \int_Q u \right)^{1/q} \left( \frac{1}{|Q|} \int_Q v^{-1/(p-1)} \right)^{1-q} \leq A$$

whenever the measure $d\sigma = v^{-1/(p-1)} dx$ satisfies

$$\frac{|Q'|}{|Q|} \leq C \left( \frac{|Q'|}{|Q|} \right)^\nu \text{ for all cubes } Q, Q'$$

with $Q' \subset Q$ and $1 - (s/n) \leq \nu$.

This growth condition is weaker than the $A_{\infty}$ condition usually used to obtain such an equivalence.

0. Introduction

Let $u, v$ weight functions on $\mathbb{R}^n, n \geq 1$ (i.e. nonnegative locally integrable functions). The Hardy-Littlewood maximal operator is given by

$$(M f)(x) = \sup \left\{ \frac{|Q|^s}{|x|^s} \left\| f \right\|_{L^1(dx)}; \ Q \text{ a cube with } Q \ni x \right\}. $$
Throughout this paper $Q$ will denote a cube with sides parallel to the co-ordinate planes. It is fundamental in analysis to characterize the pairs of nonnegative weights $(u, v)$ for which

\[(1) \quad \|Mf\|_{L^p_u} \leq C\|f\|_{L^p_v}
\]

for all functions $f (1 < p < \infty, C = C(n, p, u, v) > 0)$;

here $\|g\|_{L^p_v}$ denotes $\left(\int_{\mathbb{R}^n} |g|^{p} \, dx \right)^{1/p}$, and $dx$ the Lebesgue measure on $\mathbb{R}^n$. Muckenhoupt [Mu] showed that inequality (1) for $u = v$ holds if and only if

\[
\left( \frac{1}{|Q|} \int_Q u \right)^{1/p} \left( \frac{1}{|Q|} \int_Q v^{1/(p-1)} \right)^{1-\frac{1}{p}} \leq A \quad \text{for all cubes } Q.
\]

We write $v \in A_p$. This condition can be viewed as a particular case of $(u, v) \in A(p)$, i.e.

\[
\left( \frac{1}{|Q|} \int_Q u \right)^{1/p} \left( \frac{1}{|Q|} \int_Q v^{1/(p-1)} \right)^{1-\frac{1}{p}} \leq A \quad \text{for all cubes } Q.
\]

It is clear that $(u, v) \in A(p)$ is a necessary condition for (1), but in general it is not a sufficient condition (see [Mu] for a counterexample). A special case of a Sawyer's result [Sa2] shows that (1) is in fact equivalent to $(u, v) \in S(p)$, i.e.

\[
\|(Mv^{-1/(p-1)}1_Q)1_Q\|_{L^p_u} \leq S\|1_Q\|_{L^p_{v^{-1/(p-1)}}} < \infty \quad \text{for all cubes } Q.
\]

However for $u = v$, it is not obvious that $(v, v) \in A(p)$ implies $(v, v) \in S(p)$. This point was solved by Hunt-Kurtz-Neugebauer [Hu-Ku-Ne].

More generally the two weight norm inequality

\[(2) \quad \|M_s f\|_{L^q_s} \leq c\|f\|_{L^p_v} 1 < p \leq q < \infty, 0 < s < n, \left[p^{-1} - q^{-1}\right] \leq (s/n)
\]

for the fractional maximal operator

\[(M_s f)(x) = \sup \left\{ |Q|^{s-1}\|f1_Q\|_{L^1(\partial y)}; \, Q \, \text{a cube with } Q \ni x \right\}
\]

was characterized by Sawyer [Sa2] by the condition $(u, v) \in S(s, p, q)$, i.e.

\[
\|(M_s v^{-1/(p-1)}1_Q)1_Q\|_{L^p_u} \leq S\|1_Q\|_{L^p_{v^{-1/(p-1)}}} < \infty \quad \text{for all cubes } Q.
\]
A necessary condition for (2) is \((u, v) \in A(s, p, q)\), i.e.

\[\frac{|Q|^\frac{n}{p} + \frac{1}{q} - \frac{1}{q}}{\left(\frac{1}{|Q|} \int_Q u^q\right)^{1/q} \left(\frac{1}{|Q|} \int_Q v^{-1/(p-1)}\right)^{1-\frac{1}{p}}} \leq A \text{ for all cubes } Q.\]

Although \((u, v) \in A(s, p, q)\) is not sufficient for (2), it is nevertheless a more easily verifiable condition. So for \(d\sigma = v^{-1/(p-1)} \, dx \in A_\infty\) (i.e. \(d\sigma \in A_r\) for some \(r > 1\)) Perez [Pe] (see also Sawyer [Sa]) proved that \((u, v) \in A(s, p, q)\) implies (2).

In this paper we give an analogous result (see Theorem I) for weights \(v\) such that \(d\sigma \in B_v\) with \([1 - (s/n)] < v\), i.e.:

\[\frac{|Q'|}{|Q|} \leq C \left(\frac{|Q'|}{|Q|}\right)^v \text{ for all cubes } Q, Q' \text{ with } Q' \subset Q;\]

here \(|Q|\) denotes \(\int_Q \sigma \, dx\).

If \(d\sigma \in A_\infty\) then \(d\sigma \in B_\delta\) for some \(\delta > 0\) i.e.

\[\frac{|E|}{|Q|} \leq C \left(\frac{|E|}{|Q|}\right)^\delta \text{ for all cubes } Q \text{ and all measurable sets } E \text{ with } E \subset Q.\]

But, as we will see, there are measures \(d\mu\) such that \(d\mu \in B_\delta\) and \(d\mu \notin A_\infty\). First it is known [Ga-Fr] that \(d\sigma \in A_\infty\) implies \(d\sigma \in D_\infty\) i.e.

\[|2Q| \leq D|Q| \text{ for all cubes } Q, D = D(\sigma) > 1;\]

2Q is the cube with the same center as Q but with lengths expanded two times. The condition \(d\sigma \in D_\infty\) is equivalent to \(d\sigma \in D_\varepsilon\) for some \(\varepsilon \geq 1\) (see Proposition VIII below), i.e.

\[|tQ| \leq C|t|^{\varepsilon}|Q| \text{ for all cubes } Q \text{ and all } t \geq 1.\]

Also \(d\sigma \in D_\infty\) implies \(d\sigma \in RD_\nu\) for some \(\nu \in ]0, 1]\) (see Proposition VIII below), i.e.

\[t^{\nu}|Q| \leq C|t|Q| \text{ for all cubes } Q \text{ and all } t \geq 1.\]

The condition \(RD_\nu\) is weaker than the doubling condition \(D_\infty\) (for example if \(w(x) = e^{|x|}\) then \(w \, dx \in RD_\nu\) for some \(\nu \in ]0, 1]\) but \(w \, dx \notin D_\infty\)). Hence if \(d\sigma \in A_\infty\) then \(d\sigma \in D_\infty \cap RD_\nu\) for some \(\nu \in ]0, 1]\). But we can have \(d\sigma \in D_\infty\) with \(d\sigma \notin A_\infty\) (see [Wi] for an example). As we will see below, if \(d\sigma \in B_\nu\) then \(d\sigma \in RD_\nu\) and conversely \(d\sigma \in D_\infty \cap RD_\nu\) implies \(d\sigma \in B_\nu\). The condition \(d\sigma \in D_\infty \cap RD_\nu\) is weaker than \(d\sigma \in A_\infty\).
and it is more verifiable than \( d\sigma \in B_\nu \). So if \( d\sigma \in D_\infty \) then \( d\sigma \in B_\nu \) for \( \nu \) small enough, while \( d\sigma \) does not automatically belong to \( A_\infty \).

Contrary to the Perez's approach [Pe] (which consists to obtain (2) from \( A(s,p,q) \) by exploiting properties of Calderon-Zygmund cubes) our method lies on the same philosophy as the Hunt-Kurtz-Neugebauer [Hu-Ku-Ne] results mentioned above. Using the condition \( d\sigma \in B_\nu \) we directly derive the condition \( S(s,p,q) \) from \( A(s,p,q) \). For applications, the nature of our result leads to the following: "Let \( d\sigma \in D_\infty \). For what reals \( \varepsilon, \nu \) (with \( \varepsilon \geq 1 \) and \( \nu \leq 1 \)) have we \( d\sigma \in D_\varepsilon \) and \( d\sigma \in RD_\nu \)? Can we choose \( \varepsilon \) sufficiently small and \( \nu \) big?"

In Section 1 we begin to state our main result (see Theorem I). Then we give growth conditions (see Proposition II) which are more useful than those used in our result. In Section 2 with the usual weights \( u(x) = |x|^\beta, v(x) = |x|^\alpha \) we recall how to realize the \( A(s,p,q) \) condition (see Proposition IV). In order to answer the above questions we reviewed how \( A_p \Rightarrow D_\infty \) and \( A_p \Rightarrow RD_\nu \) (see Proposition V), \( D_\infty \Rightarrow RD_\nu \) (see Proposition VIII). By these, we bring out precise values of \( \varepsilon \) and \( \nu \) (see Section 4). Proofs of main results are in Section 3.

1. The main result

To include classical maximal functions, we work with the operator

\[
(M_\Phi f)(x) = \sup \{ \Phi(Q)|Q|^{-1}\|f1_Q\|_{L^1(dy)}; Q \text{ a cube with } Q \ni x \}
\]

where \( \Phi \) is a map defined on the set of cubes, taking its values in \( [0, \infty[ \) and satisfying the following growth conditions \( H \):

1) \( \Phi(Q_1) \leq C\Phi(Q_2) \) for all cubes \( Q_1, Q_2 \) with \( Q_1 \subset Q_2, C = C(\Phi) > 0 \).
2) There are \( C_1, C_2 > 0, \lambda, \eta \in [0,1[ \) such that

\[
C_1t^{\eta\lambda}\Phi(Q) \leq \Phi(tQ) \leq C_2t^{\eta\eta}\Phi(Q) \text{ for all cubes } Q \text{ and all } t \geq 1.
\]

When \( \Phi(Q) = 1 \) we obtain the Hardy-Littlewood maximal operator. The fractional maximal operator \( M_\Phi(0 < s < n) \) is given by \( \Phi(Q) = |Q|^{s/n} \). Maximal operators connected to the Bessel potential (see [Ke-Sa]) are defined by \( \Phi(Q) = \int_0^{|Q|^{1/n}} \varphi(s) ds \); and generally \( M_\Phi \) arises in studies of other potential operators (see [Ch-St-Wh]).

Let \( 1 < p \leq q < \infty \) and \( (u,v) \) be a pairwise of weights. We write \( (u,v) \in S(\Phi,p,q) \) if for some constant \( S > 0 \)

\[
\|(M_\Phi(v)^{1/(p-1)}1_Q)\|_{L^p_u} \leq S\|1_Q\|_{L^{p/(p-1)}_v} < \infty \text{ for all cubes } Q.
\]
Also we write \( (u, v) \in A(\Phi, p, q) \) holds for some \( A > 0 \) if
\[
\Phi|Q|^{\frac{2}{p} - \frac{1}{p'}} \left( \frac{1}{|Q|} \int_Q u \right)^{1/q} \left( \frac{1}{|Q|} \int_Q v^{-1/(p-1)} \right)^{1-\frac{1}{p}} \leq A \text{ for all cubes } Q.
\]
In this paper we always adopt the convention \( 0 \cdot \infty = 0 \). From condition \( A(\Phi, p, q) \) and the Lebesgue theorem whenever \( u \neq 0 \), we see that it is necessary to suppose
\[
H3) \quad \lim_{|Q| \to 0} \left( \Phi(Q)|Q|^{\frac{2}{p} - \frac{1}{p'}} \right) \leq c.
\]
For instance \( H3) \) is satisfied if \( |p^{-1} - q^{-1}| \leq \lambda \). For \( \Phi(Q) = 1 \) \( H3) \) implies \( q \leq p \), and for \( \Phi(Q) = |Q|^{1/n} \) it means \( |p^{-1} - q^{-1}| \leq (s/n) \).

Let \( \rho > 0 \) and \( d\sigma = \sigma dx \) be a weight function. As in Section 0, we write \( d\sigma \in B_\rho \) if there is \( B = B(\sigma) > 0 \) such that
\[
\frac{|Q'|\sigma}{|Q|\sigma} \leq B \left( \frac{|Q'|\sigma}{|Q|\sigma} \right)^\rho \text{ for all cubes } Q, Q' \text{ with } Q' \subset Q.
\]
Also for a weight function \( u \), then \( d\sigma \in B_\rho (u) \) when
\[
\frac{|Q'|\sigma}{|Q|\sigma} \leq B \left( \frac{|Q'|\sigma}{|Q|\sigma} \right)^\rho \text{ for all cubes } Q, Q' \text{ with } Q' \subset Q; B = B(\sigma, u) > 0.
\]
Now we can state our main result:

**Theorem I.**

Let \( 1 < p \leq q < \infty \) and let \( \Phi \) be a function which satisfies \( H1)-2-3. \)

A) If \( (u, v) \in S(\Phi, p, q) \) for a constant \( S > 0 \), then \( (u, v) \in A(\Phi, p, q) \) for the constant \( A = S \).

B) If \( (u, v) \in A(\Phi, p, q) \) for a constant \( A > 0 \), then \( (u, v) \in S(\Phi, p, q) \) whenever one of the following condition is satisfied:

\begin{enumerate}
    \item \( d\sigma = v^{-1/(p-1)} dx \in B_\nu \) with \( 1 - \lambda \leq \nu \)
    \item \( d\sigma = v^{-1/(p-1)} dx \in B_{(p/q)(\nu)}(u) \).
\end{enumerate}

If \( B \) is the constant in the condition on \( d\sigma \) then the constant in \( S(\Phi, p, q) \) takes the form \( S = AB\rho(\Phi, n) \) in case of i), and \( S = AB^{1/p}\rho c(\Phi, n) \) in case of ii), here \( c(\Phi, n) > 0 \) depends only on \( \Phi \) and \( n \).

**Proposition II.**

A) If \( d\sigma \in B_\nu \) for some \( \nu \in [0, \infty[ \), then \( d\sigma \in RD_\nu \). Conversely if \( d\sigma \in D_\infty \cap RD_\nu \) then \( d\sigma \in B_\nu \).

B) If \( d\sigma \in B_{(p/q)}(u) \cap D_\infty \), there are \( \varepsilon \in [1, \infty[ \) and \( \nu \in [0, 1] \) such that \( d\sigma \in RD_\nu \), \( du \in D_\varepsilon \) and \( \nu q \leq \varepsilon p \). Conversely if \( d\sigma \in RD_\nu \) and \( du \in D_\varepsilon \) for some \( \varepsilon \in [1, \infty[ \) and \( \nu \in [0, 1] \) with \( \varepsilon p \leq \nu q \) then \( d\sigma \in B_{(p/q)}(u) \).
Consequently, for the case of the fractional maximal operator, we can state

**Proposition III.**

Let \( 1 < p \leq q < \infty, 0 \leq s < n, \) and \( \left| \frac{1}{p-1} - \frac{1}{q-1} \right| \leq \frac{s}{n} \). Then \((u, v) \in S(s, p, q)\) is equivalent to \((u, v) \in A(s, p, q)\) if one of the following holds:

i) \(du = v^{-1/(p-1)} \, dx \in D_\infty \cap RD_v \) with \( 1 - \frac{s}{n} \leq v \)

ii) \(du = v^{-1/(p-1)} \, dx \in RD_v; \, \, dv \in D_\epsilon \) with \( \epsilon \leq \nu q \).

2. Applications and further results

Assume the condition \( A(s, p, q) \) holds for a constant \( A > 0 \). It is also equivalent to ask

\[
|B|^\frac{s}{n} + \frac{n}{q} - \frac{1}{p} \left( \frac{1}{|B|} \int_B u \right)^{1/q} \left( \frac{1}{|B|} \int_B v^{-1/(p-1)} \right)^{1-\frac{1}{p}} \leq A_1 \text{ for all balls } B
\]

with \( A_1 = Ac(s, n, p, q) \).

Let \( B \) be the ball \( B(x_0, R) = \{ y \in \mathbb{R}^n; \ |x - y| < R \} \).

If \( |x_0| \leq 2R \) then \( B \subset B(0, 3R) \) and hence the first member of (3) is majorized by the quantity

\[
c(s, n, p, q)R^{s + \frac{n}{q} - \frac{1}{p}} \left( \frac{1}{R^n} \int_{|y| < R} u \right)^{1/q} \left( \frac{1}{R^n} \int_{|y| < R} v^{-1/(p-1)} \right)^{1-\frac{1}{p}}
\]

which can be easily computed mainly if \( u \) and \( v \) are radial functions.

If \( 2R < |x_0| \) then \((1/2)|x_0| < |y| < (3/2)|x_0| \) for each \( y \in B \) and hence the first member of (3) is now majorized by

\[
c(s, n, p, q)R^{s + \frac{n}{q} - \frac{1}{p}} \left( \sup_{|y| < 2^j R} u(y) \right)^{1/q} \left( \sup_{|y| < 2^j R} v(y)^{-1/(p-1)} \right)^{1-\frac{1}{p}}
\]

where \( j \in \mathbb{N}^* \).

Also if each of functions \( u, v^{-1/(p-1)} \) satisfies a growth condition as:

\[
\left[ \sup_{(1/4)R < |x| \leq 4R} w(x) \right] \leq \frac{c}{R^n} \left( \int_{c_1 R < |y| \leq c_2 R} w(y) \, dy \right)
\]
and if \([p^{-1} - q^{-1} \leq (s/n)]\) then condition \((u, v) \in A(s, p, q)\) is equivalent to

\[
R^{s + n - q - 1}_p \left( \frac{1}{R^n} \int_{|y| < R} u \right)^{1/q} \left( \frac{1}{R^n} \int_{|y| < R} v^{-1/(p-1)} \right)^{1 - \frac{1}{p}} \leq A_2,
\]

\[A_2 = Ac(s, n, p, q).\]

Taking \(u(x) = |x|^\delta, v(x) = |x|^\alpha\) we obtain

**Proposition IV.**

Assume

i) \(1 < p \leq q < \infty, 0 \leq s < n, [p^{-1} - q^{-1}] \leq (s/n)\);

ii) \(-n < \alpha < n(p - 1)\);

iii) \(ps - n < \alpha\);

iv) \(\beta = (q/p)(n + \alpha) - qs - n\);

and define \(u(x) = |x|\beta, v(x) = |x|\alpha\). Then \((u, v) \in A(s, p, q)\).

The condition ii) is equivalent to \(v \in A_p\). Now we recall a known result, yielding \(D_\varepsilon\) or \(RD_\nu\) from the \(A_p\) condition.

**Proposition V.**

A) Let \(1 < p < \infty\), and \(w \in A_p\) for a constant \(A > 0\). Then \(w \in D_p\) i.e.

\[|tQ|_w \leq Dt^np|Q|_w\] for all cubes \(Q\) and all \(t \geq 1\); here \(D = A_p\).

B) Let \(1 < r < \infty\), and \(w \in RH_{r/(r-1)}\) i.e.

\[
\left( \frac{1}{|Q|} \int_Q w^{r/(r-1)} \right)^{1 - \frac{1}{r}} \leq R \left( \frac{1}{|Q|} \int_Q w \right) \text{ for all cubes } Q,
\]

\[R = R(w) > 0\]

then \(w \in RD_{1/r}\) with the constant \(R\).

If \(w \in A_p\) then it is known ([Ga-Fr]) that \(w \in RH_{1+p}\) for some \(\rho > 0\) (which depends on \(n, p, w\)) and so \(w \in RD_\nu\) for some \(\nu \in [0, 1]\). Proposition V can be merely seen by the use of the Hölder inequality.

**Proposition VI.**

Let \(1 < r < \infty, \gamma \in \mathbb{R}\) and \(w(x) = |x|^\gamma\). If \(-n < \min(\gamma, \gamma r)\) then \(w \in RH_r\) and so \(w \in RD_{1-(1/r)}\).

From Propositions III-IV-VI we get
Proposition VII.
Assume
i) $1 < p < q < \infty$, $0 \leq s < n$, \( \{p^{-1} - q^{-1}\} \leq (s/n) \); 
ii) $-n < \alpha < s(p - 1)$; 
iii) $ps - n < \alpha$; 
iv) $\beta = (q/p)(n + \alpha) - qs - n$; 
and define $u(x) = |x|^\beta$, $v(x) = |x|^\alpha$. Then there is $c > 0$ such that 
\[ \|M_s f\|_{L^s} \leq c\|f\|_{L^s} \] 
for all nonnegative functions $f$.

Finally we end with the fact that the $D_{\infty}$ condition implies $D_\epsilon$ or $RD_\nu$ (for some $\epsilon$ and $\nu$).

Proposition VIII.
A) Let $w \in D_{\infty}$: i.e. $|2Q|_w \leq D|Q|_w$ for all cubes $Q$, $D = D(w) > 1$. Then $w \in D_\epsilon$: i.e. $|tQ|_w \leq D^{t\epsilon}|Q|_w$ for all cubes $Q$ and all $t > 1$, with $\epsilon = \frac{\ln D}{2\nu}$.

In particular if $2^n \leq D$ then $\epsilon \geq 1$.

B) Let $w \in D_\epsilon$ with a constant $D > 1$. Then $w \in RD_\nu$: i.e. $t^{\nu}\nu|Q|_w \leq 2^{\nu\epsilon} D|tQ|_w$ for all cubes $Q$ and all $t > 1$, where $\nu = \nu(\epsilon, D, n) = \frac{1}{\ln 2^n} \ln \left[ \frac{12^{n\epsilon} D^2}{2^{n\epsilon} - 1} \right]$.

In particular if $2 \leq 12^{n\epsilon} D^2$ then $\nu \leq 1$.

Let $\theta > 0$, then $\theta \geq \epsilon$ if and only if $D \leq 2^{n\theta}$ and $\theta \leq \nu$ if and only if $12^{n\epsilon} D^2 \leq \left[ \frac{2^{n\theta}}{2^{3\theta} - 1} \right]$. From this proposition we see that if $w dx \in D_{\infty}$ with a doubling constant $D = D(w) > 1$ then $w \in RD_\nu$ with $\nu = \nu(D, n) = \frac{1}{\ln 2^n} \ln \left[ \frac{D^\epsilon}{D^{\epsilon - 1}} \right]$ where $\epsilon = 4 + \frac{\ln 3}{\ln 2}$.

Part A can be easily obtained by induction. The next part was proved by Strömberg and Torchinsky [St-To], but here we include the proof since we need the precise value of $\nu$.

3. Proofs of the main results

For each cube $Q_0$ we define the local maximal function
\[
(M_{\Phi, Q_0} f)(x) = \sup \{ \Phi(Q)|Q|^{-1}\|f 1_Q\|_{L^1(dy)}; \ Q \ni x, \ Q \subset Q_0 \}.
\]

The proof of Theorem I is based on the following lemmas.
Lemma 1.
There is $C = C(n, \Phi) > 0$ such that for each cube $Q_0$ and for each function $f$ locally integrable whose support is contained in $Q_0$

$$(M_{\Phi, Q_0} f)(x) \leq (M_{\Phi} f)(x) \leq C(M_{\Phi, Q_0} f)(x) \text{ for all } x \in Q_0.$$ 

Lemma 2.
Suppose $(u, v) \in A(\Phi, p, q)$ and $d\sigma$ satisfying one of i)-ii) as in part B of Theorem 1. Let $Q_0$ be a cube with $0 < |Q_0|_\sigma < \infty$. Then

$$\sup_{z \in Q_0} (M_{\Phi, Q_0} 1_{Q_0} \sigma)(z) < A \frac{|Q_0|^{1/p}}{|Q_0|^{1/q}} < \infty.$$ 

Lemma 3.
With the same hypothesis as in Lemma 2, one can find a subcube $Q_1$ of $Q_0$ such that $(M_{\Phi, Q_0} 1_{Q_0} \sigma)(z) < 4 \left( \frac{\Phi(Q_1)}{|Q_1| \sigma} \right)$ for all $z \in Q_0$.

We postpone the proofs below, and we first show how Theorem I is derived from these lemmas.

Proof of Theorem I:
Since

$$\left( \frac{\Phi(Q_0)}{|Q_0| \sigma} \right) 1_{Q_0}(\cdot) \leq (M_{\Phi, Q_0} 1_{Q_0} \sigma)(\cdot) 1_{Q_0}(\cdot)$$

it is clear that if $(u, v) \in S(\Phi, p, q)$ for a constant $S > 0$, then $(u, v) \in A(\Phi, p, q)$ with the constant $A = S$.

Conversely let $(u, v) \in A(\Phi, p, q)$ for a constant $A > 0$, and let $Q_0$ be a cube. If $|Q_0|_\sigma = 0$ then it is trivial to have $(u, v) \in S(\Phi, p, q)$. Also (since $0 \cdot \infty = 0$) if $|Q_0|_\sigma = \infty$ then $(u, v) \in S(\Phi, p, q)$ because in this case $|Q_0|_u = 0$. So we can assume $0 < |Q_0|_\sigma < \infty$. From Lemmas 1 and 3 we first have

$$\|M_{\Phi, Q_0} \sigma\|_{L^p} \leq C \|M_{\Phi, Q_0} 1_{Q_0} \sigma\|_{L^p} C = C(n, \Phi)$$

and

$$\|M_{\Phi, Q_0} 1_{Q_0} \sigma\|_{L^p} \leq 4C \left( \frac{\Phi(Q_1)}{|Q_1| \sigma} \right) |Q_0|^{1/q}.$$

Now suppose $d\sigma = v^{-1/(p-1)} dx \in B_\nu$ with $1 - \lambda \leq \nu$. Then we get

$$\|M_{\Phi, Q_0} \sigma\|_{L^p} \leq C(\Phi, n) \left( \frac{|Q_1|}{|Q_0|} \right)^{\lambda-1} \left( \frac{|Q_1| \sigma}{|Q_0| \sigma} \right) (\Phi(Q_0))^{1/q}$$

$$\leq C(\Phi, n) B \left( \frac{|Q_1|}{|Q_0|} \right)^{\lambda-1+\nu} (\Phi(Q_0))^{1/q} |Q_0|^{1/q}$$

$$\leq C(\Phi, n) BA |Q_0|^{1/p} = C(\Phi, n) BA \|1_{Q_0}\|_{L^p}.$$
Now suppose \( d\sigma = v^{-1/(p-1)} \, dx \in \mathcal{B}(p/q)(u) \). Then we obtain

\[
\|(M_{\Phi}1_{Q_0})1_{Q_0}\|_{L^p} \leq 4C \left( \frac{\Phi(Q_1)}{|Q_1|} \right) |Q_1|^{1/p} \left( \frac{|Q_0|}{|Q_1|} \right)^{1/p} \\
\leq 4CA \left( \frac{|Q_1|}{|Q_0|} \right)^{1/p} \left( \frac{|Q_0|}{|Q_1|} \right)^{1/q} |Q_0|^{1/p} \\
\leq 4CAB^{1/p} \|1_{Q_0}\|_{L^p}. \quad \blacksquare
\]

**Proof of Lemma 1:**

Let \( Q_0 \) be a cube and let \( f \) be a function whose support is contained in \( Q_0 \). Firstly it is clear that

\[(M_{\Phi}Q_0 f)(x) \leq (M_{\Phi}f)(x) \text{ for all } x.\]

For the converse we use the growth properties \( H)1-2 \) of \( \Phi \). Let \( Q \) be a cube which contains \( x \), with \( x \in Q_0 \). We suppose that \( Q_0 \) does not contain \( Q \) (otherwise there is nothing to prove). We distinguish two cases.

1) For \( |Q_0| \leq |Q| \):

Let \( Q_1 \) be a cube with the same center as \( Q_0 \) but with the lengths \( 3|Q|^{1/n} \). Since \( \eta \leq 1 \) we first have

\[
\frac{\Phi(Q)}{|Q|} \leq \frac{|Q_0| \Phi(Q_1)}{|Q| \, |Q_0|} \\
\leq C(\Phi, n) \left( \frac{|Q_0|}{|Q|} \right)^{1-\eta} \frac{\Phi(Q_0)}{|Q_0|} \\
\leq C(\Phi, n) \frac{\Phi(Q_0)}{|Q_0|}. 
\]

It results that

\[
\frac{\Phi(Q)}{|Q|} \|f1_{Q_0}1_Q\|_{L^1} \leq C(\Phi, n) \frac{\Phi(Q_0)}{|Q_0|} \|f1_{Q_0}\|_{L^1} \\
\leq C(\Phi, n) (M_{\Phi}Q_0 f)(x). 
\]
2) For $|Q| \leq |Q_0|$: 

One can find a cube $Q_2 \subset Q_0$ such that $|Q| = |Q_2|$, $Q \cap Q_0 \subset Q_2$, and $Q \subset 3Q_2$. Hence we get

$$
\frac{\Phi(Q)}{|Q|} \|f1_{Q_0}1_Q\|_L^1 \leq \frac{\Phi(3Q_2)}{|Q_2|} \|f1_{Q_2}\|_L^1
$$

$$
\leq C(\Phi, n) \frac{\Phi(Q_2)}{|Q_2|} \|f1_{Q_2}\|_L^1
$$

$$
\leq C(\Phi, n)(M_{\Phi, Q_0}f)(x). \quad \square
$$

Proof of Lemma 2:

Let $z \in Q_0$ and $Q$ a subcube of $Q_0$ such that $Q \ni z$. Using one of hypothesis in part B of Theorem I we have to show

$$
(\Phi(Q) |Q|_{\sigma}) \leq A \frac{|Q_0|^{1/p}}{|Q_0|^{1/q}}.
$$

This implies: $\sup_{z \in Q_0} (M_{\Phi, Q_0}1_{Q_0\sigma})(z) < \infty$. And so to obtain (7) it suffices to consider $(\Phi(Q) |Q|_{\sigma}) |Q_0|^{1/q}$ and to estimate this with $A|Q_0|^{1/p}$ as we have done in the proof of Theorem I. \hspace{1cm} \square

Proof of Lemma 3:

Since $\sup_{z \in Q_0} (M_{\Phi, Q_0}1_{Q_0\sigma})(z) < \infty$ there is one $y \in Q_0$ such that 

$$(M_{\Phi, Q_0}1_{Q_0\sigma})(x) < 2(M_{\Phi, Q_0}1_{Q_0\sigma})(y) \text{ for all } x \in Q_0.$$ 

Again, there is a subcube $Q_1$ of $Q_0$ which contains $y$ such that 

$$(M_{\Phi, Q_0}1_{Q_0\sigma})(y) < 2 \left( \frac{\Phi(Q_1)}{|Q_1|} |Q_1|_{\sigma} \right)$$

and so

$$
\sup_{z \in Q_0} (M_{\Phi, Q_0}1_{Q_0\sigma})(z) \leq 4 \left( \frac{\Phi(Q_1)}{|Q_1|} |Q_1|_{\sigma} \right). \quad \square
$$

Proof of Proposition II:

Part A

Let $d\sigma \in B_\nu$ for some $\nu \in [0, \infty]$ i.e.

$$
\frac{|Q_1|_\sigma}{|Q_0|_\sigma} \leq B \left( \frac{|Q_1|}{|Q_0|} \right)^\nu \text{ for all cubes } Q_0, Q_1 \text{ with } Q_1 \subset Q_0.
$$
Let $Q$ be a cube and $t \geq 1$. Taking $Q_1 = Q$ and $Q_0 = tQ$ we obtain
\[ t^{nu} |Q|_\sigma \leq R|tQ|_\sigma, \text{ where } R = B \]
which means $d \sigma \in RD_\nu$.

Conversely let $d \sigma \in RD_\nu$ for a constant $R > 0$. Also if $d \sigma \in D_\infty$ then for $Q_1 \subset Q_0$ we have
\[ |Q_1|_\sigma \leq R \left( \frac{|Q_1|}{|Q_0|} \right)^\nu |Q_2|_\sigma \]
where $Q_2$ has the same center as $Q_1$ and $|Q_2| = |Q_0|$
\[ \leq \left( \frac{|Q_1|}{|Q_0|} \right)^\nu |3Q_0|_\sigma \]
\[ \leq RD \left( \frac{|Q_1|}{|Q_0|} \right)^\nu |Q_0|_\sigma \]
where $D$ depends on the constant which is in the doubling condition for $d \sigma$. So it appears that $d \sigma \in B_\nu$ with the constant $B = RD$.

Part B

Let $d \sigma \in B_{(p/q)}(u)$, i.e.
\[ \frac{|Q_1|_\sigma}{|Q_0|_\sigma} \leq B \left( \frac{|Q_1|_u}{|Q_0|_u} \right)^{p/q} \]
for all cubes $Q_0, Q_1$ with $Q_1 \subset Q_0$.

Suppose also $d \sigma \in D_\infty$. Let $Q$ be a cube and $t \geq 1$. Taking $Q_1 = Q$ and $Q_0 = tQ$ and using the fact that $d \sigma \in D_{\epsilon'}$ for some $\epsilon' \geq 1$ (see Proposition VIII) we obtain
\[ \frac{1}{t^{nu'}} D^{-1} \leq \frac{|Q|_\sigma}{|tQ|_\sigma} \leq B \left( \frac{|Q|_u}{|tQ|_u} \right)^{p/q} \]
that is
\[ |tQ|_u \leq (DB)^{q/p} t^{nu'q/p} |Q|_u \]
which means $du \in D_\epsilon$ with $\epsilon = \epsilon'(q/p) \geq 1$. Also since $u dx \in RD_\nu$, for some $\nu' \in [0,1]$ (see Proposition VIII) we get
\[ \frac{|Q|_\sigma}{|tQ|_\sigma} \leq B \left( R \frac{1}{t^{nu'}} \right)^{p/q} \]
that is
\[ t^{n\nu'q/p} |Q|_\sigma \leq B(R)^{p/q} |tQ|_\sigma \]
which means $d\sigma \in RD_{\nu}$ with $\nu = \nu'(p/q) \leq 1$. On other hand we must have for all $t \geq 1$

$$1 \leq DB(R)^{p/q} t^{n[\epsilon'-\nu'(p/q)]}$$

hence $0 \leq \epsilon' - \nu'(p/q)$, or $\nu q \leq \epsilon p$.

Conversely let $d\sigma \in RD_{\nu}$, $du \in D_{\epsilon}$ for some $\epsilon \in [1, \infty[\text{ and } \nu \in [0, 1]$ with $\epsilon p \leq \nu q$. For all cubes $Q_1, Q_0$ with $Q_1 \subset Q_0$ we have

$$\frac{|Q_1|}{|Q_0|} \left( \frac{|Q_0|}{|Q_1|} \right)^{p/q} \leq RD \left( \frac{|Q_1|}{|Q_0|} \right)^{\nu - \epsilon(p/q)}$$

$$\leq RD$$

which implies $d\sigma \in B(p/q)(u)$ for constant $B = (RD)^{q/p}$.

4. Proofs of further results

Proof of Proposition IV:
Let $R > 0$. The condition ii) implies that $v$ and $v^{-1/(p-1)}$ are locally integrable functions and

$$\int_{|y| < R} v^{-1/(p-1)} \, dy = \int_{|y| < R} |y|^{-\alpha/(p-1)} \, dy \sim R^{n-\alpha/(p-1)}.$$

From iii) and iv) we have $\beta = (q/p)(n + \alpha) - qs - n > -n$, and so

$$\int_{|y| < R} u \, dy = \int_{|y| < R} |y|^\beta \, dy \sim R^{(n+\alpha)(q/p)-qs} (\mathbb{1}[n + \alpha](q/p) - qs > 0).$$

Since $[p^{-1} - q^{-1}] \leq (s/n)$ we only have to estimate

$$R^{s+\frac{n}{p}-\frac{n}{p}} \left( \frac{1}{R^n} \int_{|y| < R} u \right)^{1/q} \left( \frac{1}{R^n} \int_{|y| < R} v^{-1/(p-1)} \right)^{1-\frac{1}{q}} \text{ (see Section 2).}$$

Using the two equivalences above this last quantity is equivalent to

$$R^{s+(n/q)-(n/p)}(R^{(n+\alpha)(q/p)-qs-n})^{1/q}(R^{-\alpha/(p-1)})^{1-\frac{1}{q}} = R^{s+(n/q)-(n/p)+(\alpha/p)+s-(n/q)-(\alpha/p)} = 1.$$

Proof of Proposition VI:
Let $-n < \min(\gamma, \gamma\tau)$, and let $B$ be the ball $B(x_0, R)$. 

...
1) If $|x_0| \leq 2R$ then $B \subset B(0,3R)$ and $B(0,R) \subset 3B$. Hence

$$\left( \frac{1}{|B|} \int_B w^r \right) \leq \left( \frac{c(n)}{R^n} \int_{|y|<3R} |y|^\gamma \, dy \right) \sim R^{\gamma r}$$

and since $-n < \gamma$ then, by Propositions IV-V, $w \, dx \in D_\infty$ and it follows

$$\left( \frac{1}{|B|} \int_B w \right) \geq D(\gamma) \left( \frac{1}{|B|} \int_{3B} w \right) \geq D'(\gamma) \left( \frac{c(n)}{R^n} \int_{|y|<R} |y|^\gamma \, dy \right) \sim R^\gamma.$$

2) If $2R < |x_0|$ then $(1/2)|x_0| < |y| < (3/2)|x_0|$ for each $y \in B$ and it results

$$\left( \frac{1}{|B|} \int_B w^r \right) \sim (2^j R)^\gamma r \quad \text{with} \quad j \in \mathbb{N}^*,$n and \( \left( \frac{1}{|B|} \int_B w \right) \sim (2^j R)^\gamma.$

In all cases, since $w \, dx \in D_\infty$, we get

$$\left( \frac{1}{|Q|} \int_Q w^r \right) \leq D(n,\gamma) \left( \frac{1}{|Q|} \int_Q w \right) \text{ for all cubes } Q$$

and hence $w \, dx \in RH_r$. \( \blacksquare \)

**Proof of Proposition VII:**

Let $\sigma(x) = v^{-1/(p-1)}(x) = |x|^{-\alpha/(p-1)}$. Note that $d\sigma \in A_p$ and so $d\sigma \in D_\infty$ (see Proposition V). If $\alpha \leq 0$, then $-n < \gamma = -\alpha/(p-1) < \gamma r$ for all $r > 1$. Choose $r > 1$ with $(n/s) \leq r$ then from Proposition VI: $d\sigma \in D_\infty \cap RD_{1-(1/r)}$ with $[1-(s/n)] \leq \nu = [1-(1/r)]$. To obtain the same conclusion for $\alpha > 0$, we choose $r > 1$ such that $(n/s) \leq r < [n(p-1)/\alpha]$, and so $-n < \gamma r \leq \gamma$.

Finally using Propositions IV-III and the Sawyer theorem [Sa2] then

$$\| M_\sigma f \|_{L^q} \leq c \| f \|_{L^p} \text{ for all functions } f. \quad \blacksquare$$

**Proof of part B of Proposition VIII:**

We need the following lemmas whose proofs will be given below.

**Lemma 4.**

Let $w \, dx \in D_\varepsilon$ for some $\varepsilon \in [1,\infty]$ and with a constant $D = D(w) > 1$. Then $\| M_{\varepsilon} f \|_{L^q} \leq 6^m c \| f \|_{L^p}$ for each cube $Q$.\( \blacksquare \)
Lemma 5.

Let \( w \, dx \in D_\varepsilon \) for some \( \varepsilon \in [1, \infty] \) and with a constant \( D = D(w) > 1 \). Then \( \left| \frac{1}{2} Q \right|_w \leq \beta |Q|_w \) for each cube \( Q \), with \( \beta = \frac{12^{n^2} D^2 - 1}{12^{n^2} D^2} \), and so \( \beta \in [0, 1] \).

The part B can be derived from Lemma 5. Indeed if \( Q \) is a cube then

\[
|Q|_w \leq \beta^m |2^m Q|_w \quad \text{for each } m \in \mathbb{N}^*.
\]

Let \( t > 1 \). There is \( k = k(t) \in \mathbb{N}^* \) such that \( 2^{k-1} < t \leq 2^k \) (so \( [(\ln t)/(\ln 2)] \leq k \)). It results

\[
|Q|_w \leq \beta^k |2^k Q|_w \\
\leq 2^{n^2} D \beta^k |tQ|_w \\
= 2^{n^2} D \left( \frac{\ln \beta}{\ln 2} \right) \ln t |tQ|_w \\
\leq 2^{n^2} D \left( \frac{\ln \beta}{\ln 2} \right) \ln t |tQ|_w \\
= 2^{n^2} D \left[ \frac{1}{t} \right] \frac{\ln \beta}{\ln 2} |tQ|_w
\]

and

\[
t^{n^2} |Q|_w \leq 2^{n^2} D |tQ|_w \quad \text{with} \quad \nu = \frac{\ln \beta}{\ln 2^n}.
\]

If \( 2 \leq 12^{n^2} D^2 \) we get

\[
(12^{n^2} D^2 + 2^n) \leq (2^{n^2} 12^{n^2} D^2 + 2^{n^2} 12^{n^2} D^2) = 2^n 12^{n^2} D^2
\]

or \( 12^{n^2} D^2 \leq 2^n (12^{n^2} D^2 - 1) \) which implies \( \frac{1}{\beta} \leq 2^n \) and so \( \nu = \frac{\ln \frac{1}{\beta}}{\ln 2^n} \leq 1. \)

Proof of Lemma 4:

In the proof of the Theorem I we have already used the following geometric argument:

"Let \( Q_1, Q_2 \) two cubes such that \( Q_1 \cap Q_2 \neq \emptyset \) and \( |Q_1|^{1/n} \leq |Q_2|^{1/n} \); then \( Q_1 \subset 3Q_2. \)" Let \( Q \) be a cube and \( Q_0 \) a subcube of \( (Q \setminus (2^{-1} Q)) \) with lengths \( \left( \frac{1}{4} \right) |Q|^{1/n} \) and let \( Q_1 = (2^{-1} Q) \). Then \( (2^{-1} Q) \cap 2Q_0 \neq \emptyset \) and \( |2^{-1} Q|^{1/n} \leq |2Q_0|^{1/n} \). Using this argument we obtain \( (2^{-1} Q) \subset 3(2Q_0) = 6Q_0 \) and then

\[
\left| \frac{1}{2} Q \right|_w \leq 6Q_0|_w \\
\leq 6^{n^2} D |Q_0|_w \\
\leq 6^{n^2} D \left| Q \setminus \left( \frac{1}{2} Q \right) \right|_w.
\]
Proof of Lemma 5:
Let $Q$ be a cube. By hypothesis

$$2^{-n\varepsilon}|Q|_w \leq D \left| \frac{1}{2} Q \right|_w, \quad D = D(w) > 1.$$ 

So using Lemma 4 we get

$$2^{-n\varepsilon}|Q|_w \leq 6^{n\varepsilon} D^2 \left|Q \setminus \left( \frac{1}{2} Q \right) \right|_w$$

$$\leq 6^{n\varepsilon} D^2 \left[ |Q|_w - \left| \frac{1}{2} Q \right|_w \right].$$

It results that

$$\left| \frac{1}{2} Q \right|_w \leq \beta |Q|_w$$

with

$$\beta = \frac{6^{n\varepsilon} D^2 - 2^{-n\varepsilon}}{6^{n\varepsilon} D^2} = \frac{12^{n\varepsilon} D^2 - 1}{12^{n\varepsilon} D^2}, \text{ and so } \beta \in [0, 1[. \quad \blacksquare$$

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References


Centre Polytechnique Saint Louis
Ecole de Physique et de Mathématiques industrielles
13, Boulevard de l’Hautil
95 092 Cergy-Pontoise cedex
FRANCE

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