A COUNTEREXAMPLE IN OPERATOR THEORY

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Abstract

The purpose of this note is to give an explicit construction of a bounded operator $T$ acting on the space $L^2[0,1]$ such that $|Tf(x)| \leq \int_0^1 |f(y)| \, dy$ for a.e. $x \in [0,1]$ and, nevertheless, $\|T\|_{S_p} = \infty$ for every $p < 2$. Here $\| \cdot \|_{S_p}$ denotes the norm associated to the Schatten-von Neumann classes.

A. Definitions and statement of the problem

The purpose of this note is to give an alternative and direct construction to the one presented in reference [1] and we shall follow closely the lines of introduction contained in that paper.

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $S, T$ be two bounded linear operators on $L^2(X, \mathcal{F}, \mu)$. The operator $S$ dominates pointwise $T$ if it happens that $|Tf(x)| \leq S(|f|)(x)$ a.e. $x$, for every function $f \in L^2$. For example: if $T = T_K$ is an integral operator associated to a $\mu \otimes \mu$-measurable kernel $K$ then obviously $T_{|K|}$ dominates $T_K$. For an operator $T$ on a Hilbert space $H$ we have the singular numbers

$$S_n(T) := \inf\{\|T - T_n\|; \text{rank}(T_n) < n\}.$$ 

If $T$ is compact then it is known that $S_n(T) = \lambda_n(\sqrt{T^*T})$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots$ denotes the sequence of eigenvalues of $\sqrt{T^*T}$ arranged in non-increasing order and repeated according to their multiplicities.

The Schatten-von Neumann classes $S_p = S_p(H)$ are defined by

$$S_p = \{T \text{ bounded } \mid \sum_{n=1}^{\infty} |S_n(T)|^p < \infty\} \text{ if } 0 < p < \infty.$$
and

\[ S_\infty = \{ T \text{ bounded} \mid \lim_{n \to \infty} S_n(T) = 0 \}. \]

Among them the more important ones are \( S_1, S_2, S_\infty \) which correspond, respectively, to nuclear, Hilbert-Schmidt and compact operators.

Suppose that we have the information that the operator \( S \in Sp(H) \) pointwise dominates \( T \). Does it follow necessarily that \( T \in Sp \)?

This is a natural question whose answer is known to be YES when \( p = 2n \) is an even natural number. In the paper quoted above a construction of a probabilistic nature is introduced to observe that the answer to our question is NO when \( 0 < p < 2 \). Here we present an explicit example where such situation occurs.

**B. The counterexample**

In the following we shall consider \( X = [0,1] \) and \( \mu = dx \) is Lebesgue measure. Let us define the operator \( T \) by the formula:

\[ T f(x) = \int_0^1 e^{2\pi i (x - y) \cdot \nu(y)} f(y) dy \]

where \( \nu(t) = \left\lfloor e^t \right\rfloor \) and \( \lfloor x \rfloor \) denotes the integer part of the real number \( x \) i.e.

\[ \nu(x) = n \text{ if } x \in I_n = \left( \frac{1}{\log(n+1)}, \frac{1}{\log(n)} \right) \]

\[ n = 3, 4, \ldots, I_2 = \left( \frac{1}{\log 3}, 1 \right). \]

Then we have:

\[ T^* f(x) = \int_0^1 e^{-2\pi i (y - x) \cdot \nu(x)} f(y) dy \]

and

\[ T^* T f(x) = \left[ \int_{I_{\nu(x)}} e^{-2\pi i z \cdot \nu(x)} f(x) dz \right] e^{2\pi i \nu(x)}. \]

Let us consider the family of functions

\[ f_k(x) = e^{2\pi i k \cdot x} \chi_{I_k}(x) \]

where \( \chi_{I_k} \) is the indicator function of the interval \( I_k \) i.e. \( \chi_{I_k}(x) = 1 \) if \( x \in I_k \) and \( \chi_{I_k}(x) = 0 \) otherwise.
Then we have:

$$T^*T f_k(x) = \mu(I_k)f_k(x)$$

i.e., $f_k$ is an eigenfunction corresponding to the eigenvalue $\mu(I_k) = \frac{1}{\log(k+1)}$. 

Therefore $\sqrt{T^*T}$ has eigenvalues $\sqrt{\mu(I_k)} \sim \frac{1}{k^{1/2}\log k}$ and, by well-known results, the decreasing sequence of singular values of $T$ must satisfy

$$S_n(T) \geq \frac{1}{n^{1/2}\log n}$$

which implies $T \notin S_p$, if $p < 2$. On the other hand it is clear that

$$|Tf(x)| \leq \int_0^1 |f(y)| \, dy = S(|f|)$$

and $\text{rank}(S) = 1$ which yields $S \in S_p$ for every $0 < p$.

Remark. There is nothing particularly special about the division points $\frac{1}{\log n}$ and the reader may consider the more general operator

$$T f(x) = \sum_{n=1}^{\infty} \int_{x_{n-1}}^{x_n} e^{2\pi i (x-y)n} f(y) \, dy$$

where $x_n$ is any increasing sequence with $x_0 = 0$ and $x_n$ tending to 1 as $n \to \infty$. It has the kernel $\sum_{n=1}^{\infty} e^{2\pi i (x-y)} \chi_n(y)$ where $\chi_n$ denotes the characteristic function of the interval $(x_{n-1}, x_n)$ while the adjoint kernel is

$$\sum_{n=1}^{\infty} e^{-2\pi i (x-y)} \chi_n(x).$$

This yields

$$T^*T f(x) = \sum_{n=1}^{\infty} e^{-2\pi i x} \chi_n(x) \int_{x_{n-1}}^{x_n} e^{-2\pi i y} \chi_n(y) \, dy.$$

It follows then that the functions $f_n(x) = e^{-2\pi i x} \chi_n(x)$ are eigenfunctions with corresponding eigenvalues given by $\lambda_n = x_n - x_{n-1}$. And this yields the estimate

$$\|T\|_{S_p} \geq \left( \sum_{n=1}^{\infty} \lambda_n^{p/2} \right)^{1/p}.$$
References


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