THE PROPERTY \((H_u)\) AND \((\tilde{\Omega})\) WITH THE EXPONENTIAL REPRESENTATION OF HOLOMORPHIC FUNCTIONS

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Abstract

The main aim of this paper is to prove that a nuclear Fréchet space \(E\) has the property \((H_u)\) (resp. \((\tilde{\Omega})\)) if and only if every holomorphic function on \(E\) (resp. on some dense subspace of \(E\)) can be written in the exponential form.

Let \(E\) be a locally convex space. We say that \(E\) has the property \((H_u)\) and write \(E \in (H_u)\) if every holomorphic function \(f\) on \(E\) is of uniform type. This means that there exists a continuous semi-norm \(\rho\) on \(E\) such that \(f\) can be factorized holomorphically through the canonical map \(\omega_\rho : E \to E_\rho\), where \(E_\rho\) denotes the Banach space associated to \(\rho\). On the other hand, we recall that \(E\) is called a space having the property \((\tilde{\Omega})\) if for every neighbourhood \(U\) of \(C \subset E\) there exists a neighbourhood \(V\) of \(O \subset E\) and \(d > 0\) such that for every neighbourhood \(W\) of \(O \subset E\) there exists \(C > 0\) such that

\[
\|u\|_{V}^{*1+d} \leq C\|u\|_{W}^{*d} \]

for \(u \in E^*\), the dual space of \(E\), where

\[
\|u\|_{K}^{*} = \sup\{|u(x)| : x \in K\}
\]

for every subset \(K\) of \(E\).

The properties \((H_u)\) and \((\tilde{\Omega})\) were introduced and investigated by Meise and Vogt in [5]. In the present paper we investigate the property \((H_u)\) and \((\tilde{\Omega})\) by the relation with the exponential representation of entire functions.
1. The property \((H_u)\) and the exponential representation of entire functions

In this section we shall prove the following

**Theorem.** Let \(E\) be a Frechet space. Then \(E\) is nuclear and has the property \((H_u)\) if and only if every entire function on \(E\) with values in a Banach space \(B\) can be written in the form

\[
(\text{Exp})_B f(x) = \sum_{k \geq 1} \xi_k \exp u_k(x)
\]

where the series is absolutely convergent in the space \(H(E, B)\) of holomorphic functions on \(E\) with values in \(B\) equipped with the compact-open topology.

**Proof:** First prove sufficiency of the theorem. Given \(f \in H(E, B)\) with \(B\) is a Banach space. Since \(E\) is a Frechet space we can find a continuous semi-norm \(\rho\) on \(E\) such that

\[
\sum_{k \geq 1} \| \xi_k \| \exp \| u_k \|_\rho^* < \infty,
\]

with

\[
\| u \|_\rho^* = \sup \{|u(x)| : \rho(x) < 1\}.
\]

Indeed, in the converse case let \(\{ \| \cdot \|_\rho \}\) is a fundamental system of semi-norms on \(E\). Then for every \(p\) we have

\[
\sum_{k \geq 1} \| \xi_k \| \exp \| u_k \|_\rho^* = \infty.
\]

Hence for every \(p\) there exists \(k_p\) such that

\[
\sum_{k \leq k_p} \| \xi_k \| \exp \| u_k \|_\rho^* > p.
\]

This inequality implies that for each \(k \leq k_p\) there exists \(x_k^p\) with \(\| x_k^p \|_\rho \leq 1\) such that

\[
\sum_{k \leq k_p} \| \xi_k \| \exp |u_k(x_k^p)| > p.
\]

Put

\[
K = \{ x_1^1, \ldots, x_{k_1}^1, \ldots, x_1^p, \ldots, x_{k_p}^p, \ldots \} \cup \{0\}.
\]
Then $K$ is compact in $E$ and
\[ \sum_{k \geq 1} \|\xi_k\| \exp \|u_k\|_K^* > p \text{ for every } p \geq 1. \]

This is impossible, because
\[ \sum_{k \geq 1} \|\xi_k\| \exp \|u_k\|_K^* < \infty. \]

Thus the form
\[ \sum_{k \geq 1} \xi_k \exp u_k(x) \text{ for } x \in E_\rho, \quad \|x\| < 1 \]
defines a holomorphic function on $U_\rho$, the open unit ball in $E_\rho$ which is Gateaux holomorphic on $E/\text{Ker } \rho$.

Let $x \in E_\rho$. Put
\[ W = \{(1 - t)y + tx : t \in \mathbb{C} \setminus \{0\}, y \in U_\rho\}. \]

Then $W$ is a non-empty open set in $E_\rho$. Hence there exists $z \in W \cap E/\text{Ker } \rho$.

Let
\[ z = (1 - t_0)y_0 + t_0x_0 \]
with $y_0 \in U_\rho$, $t_0 \in \mathbb{C} \setminus \{0\}$.

Then
\[ x = z/t_0 + ((1 - t_0)/t_0)y_0 \]
and hence
\[ \sum_{k \geq 1} \|\xi_k\| \exp |u_k(x)| \leq \]
\[ \leq \sum_{k \geq 1} \|\xi_k\| \exp[(1/t_0)|u_k(x)| + (1 - t_0)/t_0]|u_k(y_0)| \leq \]
\[ \leq \sum_{k \geq 1} \|\xi_k\|[\exp(2/|t_0|)|u_k(x)| + \exp 2|(1 - t_0)/t_0||u_k(y_0)|] < \]
\[ < \infty. \]

Thus
\[ g = \sum_{k \geq 1} \xi_k \exp u_k \]
is a Gateaux holomorphic function on $E_\rho$. Since $g$ is holomorphic on $U_\rho$ by the Zorn Theorem [6], $g$ is holomorphic on $E_\rho$. Obviously $f = g\omega_\rho$ and hence $f$ is of uniform type.

To prove the nuclearity of $E$ for every continuous semi-norm $\rho$ on $E$ write the canonical map $\omega_\rho : E \to E_\rho$ in the form

$$\omega_\rho(x) = \sum_{k \geq 1} \xi_k \exp u_k(x)$$

in which

$$\sum_{k \geq 1} \|\xi_k\|^\ast \|u_k\|^\ast_K < \infty$$

for every compact set $K$ in $E$.

Then

$$\omega_\rho(x) = \sum_{k \geq 1} \xi_k u_k(x) \text{ for } x \in E$$

and

$$\sum_{k \geq 1} \|\xi_k\| \|u_k\|^\ast < \infty \text{ for every compact set } K \subset E.$$

As above there exists a continuous semi-norm $\beta > \rho$ on $E$ such that

$$\sum_{k \geq 1} \|\xi_k\| \|u_k\|_{\beta}^\ast < \infty.$$

This means that the canonical map $\omega_{\beta,\rho}$ from $E_\beta$ to $E_\rho$ is nuclear. Hence $E$ is nuclear.

Assume that $E$ is nuclear and has the property $(H_u)$. Given $f \in H(E, B)$, with $B$ is a Banach space. By hypothesis there exists a continuous semi-norm $\rho$ on $E$ and a holomorphic function $g$ on $E_\rho$ such that $f = g\omega_\rho$. Take a continuous semi-norm $\beta > \rho$ on $E$ such that $T = \omega_{\beta,\rho}$ is nuclear. Write

$$T(x) = \sum_{j \geq 1} t_j u_j(x) e_j$$

with

$$a = \sum_{j \geq 1} |t_j| < \infty \text{ and } \|u_j\| + \|e_j\| \leq 1 \text{ for } j \geq 1.$$

Consider the Taylor expansion of $g$ at $O \in E$,

$$g(x) = \sum_{n \geq 0} P_n g(x)$$
with
\[ P_n g(x) = (1/2\pi i) \int_{|t|=r} (g(tx)/t^{n+1}) \, dt. \]

Choose the two sequences \( \{\xi_k\} \) and \( \{\alpha_k\} \) in \( \mathbb{C} \) such that
\[ z = \sum_{k \geq 1} \xi_k \exp \alpha_k z \text{ for } z \in \mathbb{C} \]
and
\[ C_r = \sum_{k \geq 1} |\xi_k| \exp r|\alpha_k| < \infty \text{ for all } r \geq 0. \]

Such sequence exist by [2]. Formally we have
\[
(gT)(x) = g(Tx) = \sum_{n \geq 0} P_n g(Tx) = \sum_{n \geq 0} P_n g \left( \sum_{j \geq 1} t_j u_j(x)e_j \right) =
\]
\[
= \sum_{n \geq 0} \sum_{j_1, \ldots, j_n \geq 1} t_{j_1} \cdots t_{j_n} u_{j_1}(x) \cdots u_{j_n}(x) P_n g(e_{j_1}, \ldots, e_{j_n}) =
\]
\[
= \sum_{n \geq 0} \sum_{j_1, \ldots, j_n \geq 1} t_{j_1} \cdots t_{j_n} P_n g(e_{j_1}, \ldots, e_{j_n}) \left( \sum_{k \geq 1} \xi_k \exp \alpha_k u_{j_1}(x) \right) \cdots \left( \sum_{k \geq 1} \xi_k \exp \alpha_k u_{j_n}(x) \right) =
\]
\[
= \sum_{n \geq 0} \sum_{j_1, \ldots, j_n \geq 1, k_1, \ldots, k_n \geq 1} t_{j_1} \cdots t_{j_n} \xi_{k_1} \cdots \xi_{k_n} \cdot P_n g(e_{j_1}, \ldots, e_{j_n}) \exp[\alpha_{k_1} u_{j_1}(x) + \cdots + \alpha_{k_n} u_{j_n}(x)].
\]

It remains to check that the right hand side is absolutely convergent in \( H(E, B) \). For each \( r > 0 \) take \( s > C_r \) a.e. Since
\[ \|P_n g(e_{j_1}, \ldots, e_{j_n})\| \leq (n^n/n!s^n)\|g\|_s \]
where
\[ \|g\|_s = \sup\{\|g(x)\| : \|x\| < s\}, \]
and without loss of generality by the nuclearity of \( E \), we may assume that \( g \) is bounded on every bounded set in \( E_p \), we have
\[
\sum_{n \geq 0} \sum_{j_1, \ldots, j_n \geq 1, k_1, \ldots, k_n \geq 1} |t_{j_1}| \cdots |t_{j_n}| |\xi_{k_1}| \cdots |\xi_{k_n}| \cdot \|P_n g(e_{j_1}, \ldots, e_{j_n})\| \exp r|\alpha_{k_1}| + \cdots + |\alpha_{k_n}| \leq
\]
\[
\leq \left( \sum_{n \geq 0} C_r^n a^n n^n / n! s^n \right) \|g\|_s < \infty \text{ for } \|x\| \leq r.
\]
The theorem is completely proved. ■

2. The property (\(\tilde{\Omega}\)) and
the exponential representation of entire functions

The relation between the property (\(\tilde{\Omega}\)) and the exponential representation of entire functions is given by

Theorem 2.1. Let \(E\) be a nuclear Frechet space having the approximation property. Then \(E\) has the property (\(\tilde{\Omega}\)) if and only if there exists a balanced convex compact set \(B\) in \(E\) such that

(i) \(E(B)\) is dense in \(E\), where \(E(B)\) denotes the Banach space spanned by \(B\),

(ii) every holomorphic function on \((E(B), \tau_E)\), where \(\tau_E\) is the topology of \(E(B)\) induced by the topology of \(E\), can be written in the form

\[
(\text{Exp}) : \sum_{k \geq 1} \xi_k \exp u_k
\]

in which the series is absolutely convergent in \(H(E(B), \tau_E)\).

Proof: Since every nuclear Frechet space having the property (\(\tilde{\Omega}\)) has also the property \((H_u)\) \([5]\), and since every holomorphic function on \((E(B), \tau_E)\) can be extended holomorphically to \(E\) \([5]\), where \(B\) is a balanced compact set in \(E\) as in \([5]\), the necessity of the theorem is as in Theorem 1.1.

Conversely, by \([5]\) it suffices to show that every holomorphic function on \((E(B), \tau_E)\) is holomorphic on \(E\). As in Theorem 1.1 there exists a continuous semi-norm \(\rho\) on \(E\) such that

\[
\sum_{k \geq 1} |\xi_k| \exp \|u_k\|_{U_\rho \cap E(B)} < \infty.
\]

Since \(E(B)\) is dense in \(E\), it follows that \(U_\rho \cap E(B)\) is dense in \(U_\rho\), and hence

\[
\sum_{k \geq 1} |\xi_k| \exp \|u_k\|_{U_\rho} < \infty.
\]

Given \(x \in E\). As in Theorem 1.1 put

\[
W = \{(1-t)y - tx : t \in \mathbb{C}\setminus\{0\}, \ y \in U_\rho\}.
\]
Then $W$ is an non-empty open set in $E$ and hence there exists $z \in W \cap E(B)$. Let

$$z = (1 - t_0)y_0 + t_0x \text{ with } t_0 \in \mathbb{C}\{0\}, \quad y_0 \in U_p.$$  

Hence

$$\sum_{k \geq 1} |\xi_k| \exp |u_k(x)| \leq \sum_{k \geq 1} |\xi_k| \exp \left| u_k(z)/t_0 + |(t_0 - 1)/t_0| \cdot u_k(y_0) \right| \leq \sum_{k \geq 1} |\xi_k| \exp 2|u_k(z)/t_0| + \exp 2|t_0 - 1/t_0| \cdot |u_k(y_0)| < \infty.$$  

By the Zorn Theorem [6], it follows that $f$ is holomorphic on $E$. Theorem 2.1 is proved. ■

3. The property $(H_u)$ and $(\tilde{\Omega})$

**Proposition 3.1.** Let $E$ be a Frechet-Schwartz space with the property $(H_u)$. Then every holomorphic function on $E$ with values in a Banach space is of uniform type.

**Proof:** Write $E = \limproj E_n$, where $E_n$ are Banach spaces such that $E$ is dense in $E_n$ for every $n \geq 1$ and the canonical maps $\omega_{n+1,n} : E_{n+1} \to E_n$ are compact. By hypothesis the canonical map

$$S : \limind H_b(E_n) \longrightarrow [H(E)]_{\text{bor}}$$

where $[H(E)]_{\text{bor}}$ denotes the bornological space associated to $H(E)$ and $H_b(E_n)$ for each $n \geq 1$ is the Frechet space of holomorphic functions on $E_n$ which are bounded on every bounded set in $E_n$, is a continuous bijection. Since $H(E)$ is complete, $[H(E)]_{\text{bor}}$ is untrabornological. By the open mapping theorem $S$ is an isomorphism. Given $f : E \to B$ a holomorphic function, where $B$ is a Banach space. Consider the continuous linear map $\hat{f} : B^* \to H(E)$ associated to $f$. Then $\hat{f} : B^* \to [H(E)]_{\text{bor}}$ is continuous. Since $S$ is isomorphic, we can find $n_0$ such that $\text{Im} \hat{f} \subseteq H_b(E_{n_0})$ and $\hat{f} : B^* \to H_b(E_{n_0})$ is continuous. This yields

$$\sup \{|uf(x)| : \|u\| \leq 1, \|x\| \leq r\} = \sup \{|\hat{f}(u)(x)| : \|u\| \leq 1, \|x\| \leq r\} < \infty$$

for all $r \geq 0$.

Thus $f$ induces a holomorphic function $g : E_{n_0} \to B$ such that $g\omega_{n_0} = f$. ■

**Remark.** Proposition 3.1 is a particular case of a recent result of Galindo, Garcia and Maestre [3].
Theorem 3.2. Let $E$ be a nuclear Frechet space with the property $(\tilde{Q})$ and $F$ a Schwartz space with $F \in \mathcal{H}_u$. Then $E \times F \in \mathcal{H}_u$.

We need the following

Lemma 3.3. Let $E$ be a nuclear Frechet space with the property $(\tilde{Q})$ and $F$ a Banach space. Then every holomorphic function on $F \times E$ which is bounded on every bounded set in $F \times E$ is of uniform type.

Proof: Lemma 3.3 will be proved as in [5] by use Lemmas 3.1 and 3.2 in [5]. Indeed, choose $p$ and $\delta > 0$ such that if $f$ is bounded on $B_\delta \times U_p$, where $f$ is a holomorphic function on $F \times E$ as in the lemma and $B_\delta = \{ z \in F : |z| < \delta \}$. Since $E \in (\tilde{Q})$, by Vogt [8] there exists a balanced convex compact set $K$ in $E$ such that

$$\| f \|_{q+1+d} \leq \| f \|_K \| f \|_p$$

for some $q > p$ and $d > 0$.

We can assume that $E(K)$, $E_q$ and $E_p$ are Hilbert spaces. Write the canonical map $A$ from $E(K)$ to $E_p$ in the form

$$A(x) = \sum_{j \geq 1} \lambda_j (x|e_j) e_j$$

where $\{e_j\}$ is a complete orthonormal system in $E$ and $\{y_j\}$ a orthonormal system in $E_p$ and $\lambda = (\lambda_j) \in s$. Let $\varphi_j$ denote the continuous linear functional on $E_q$ induced by $y_j$. Then

$$\| \varphi_j \|_{q+1+d} \leq |\lambda_j| \text{ for } j \geq 1.$$ 

Take $0 < \varepsilon < \delta$ such that for $\mu = (\varepsilon/j)$ we have

$$\left\{ x \in E : x = \sum_{j \geq 1} \xi_j y_j : |\xi_j| \leq \mu_j \text{ for } j \geq 1 \right\} \subset \{ x \in E_p : \| x \| < 1 \}.$$ 

Put

$$M = \{ m = (m_1, \ldots, m_n, 0, \ldots) \}$$

for each $k \geq 0$ and $m \in M$ put

$$a_{k,m}(z) = (1/2\pi i)^{n+1} \int_{|\tau| = 1} \int_{|\rho_1| = \mu_1} \cdots \int_{|\rho_n| = \mu_n} \frac{g(\tau z, \rho_1 y_1 + \cdots + \rho_n y_n)}{\tau^{k+1} \rho_1^{m_1+1} \cdots \rho_n^{m_n+1}} d\tau d\rho_1 \cdots d\rho_n$$

$$= (1/\lambda^m)(1/2\pi i)^{n+1} \int_{|\tau| = 1} \int_{|w_1| = \rho_1} \cdots \int_{|w_n| = \rho_n} \frac{f(z, w_1 e_1 + \cdots + w_n e_n)}{\tau^{k+1} w_1^{m_1+1} \cdots w_n^{m_n+1}} d\tau dw_1 \cdots dw_n.$$
where $g$ is the holomorphic function on $B_\delta \times \{y \in E_p : \|y\| < 1\}$ is induced by $f$ and

$$\lambda^m = \lambda_1^{m_1} \cdots \lambda_n^{m_n}.$$ 

For $s, t > 0$ put

$$B(s, t) = B_s \times \left\{ x \in E : x = \sum_{j \geq 1} \xi_j e_j, \ |\xi_j| \leq t \mu_j \text{ for } j \geq 1 \right\}.$$ 

By hypothesis

$$N(s, t) = \sup\{|f(w)| : w \in B(s, t)\} < \infty$$

and hence

$$\sup\{|a_{k,m}(z)| : \|z\| < s\} \leq N(s, t) / \lambda^m \mu^m t^{|m|}$$

with

$$|m| = m_1 + \cdots + m_n.$$ 

Let $\eta = 1/1 + d$, $\nu = \gamma = \eta/2$, $\beta = 1 - \gamma$. Given $s > 0$. Take $\sigma > 0$, such that $\sigma^{\gamma} \leq s$.

Since $\lambda \in s$, the sequence $(\lambda_j^{\nu}/\mu_j) = (j \lambda_j^{\nu}/\varepsilon) \in l^1$ and hence

$$R = \sup\{|\lambda_k^{\nu}/\mu_k|^{-1} : k \geq 1\} < \infty.$$ 

Put $t = (2Rr)^{1/\gamma}$. Then as in [5] we have

$$\sum_{m \in M} \sum_{k \geq 0} N |m| \sup_{z \in B_s} |a_{k,m}(z)| \prod_{j \geq 1} \|\varphi_j\|^* m_j \leq$$

$$\leq \sum_{m \in M} \sum_{k \geq 0} N |m| (s/\sigma)^k N(\sigma, t)/\mu^m |m| \gamma(|m|) \nu(M(s/\varepsilon)^k/\mu^m)^\beta =$$

$$= N(\sigma, t) \gamma M^\beta \left[ \sum_{k \geq 0} (s/\sigma^{\gamma} \varepsilon^{\beta})^k \right] \prod_{k \geq 1} (1 - \lambda_k^{\nu}/2R \mu_k)^{-1} < \infty$$

where

$$N = \sup\{|f(w)| : w \in B_\delta \times U_p\}.$$ 

As in [5] this implies the series

$$\sum_{m \in M} \sum_{k \geq 0} a_{k,m}(z) \prod_{j \geq 1} \varphi_j(x)^{m_j}$$
converges normally on all sets
\[ B_s \times \{ x \in E_q : \|x\| < r \}, \quad s, r > 0. \]

Hence it defines a holomorphic function \( h \) on \( F \times E_q \) such that
\[ f(z, x) = h(z, \omega_q(x)) \text{ for } (z, x) \in F \times E. \]
The lemma is proved. ■

Now we can prove Theorem 3.2 as follows.

Given \( f \in H(F \times E) \). (i) First show that there exists a neighbourhood \( U \) for \( O \in E \) such that \( f \) is bounded on \( B \times U \) for every bounded set \( B \) in \( F \). In the converse case for each \( p \) there exists a bounded set \( K_p \) in \( F \) such that \( f \) is not bounded on \( K_p \times U_p \). Choose \( \epsilon_j \downarrow 0 \) such that
\[ K = \text{conv} \bigcup_{j \geq 1} \epsilon_j K_j \]
is bounded. Consider the holomorphic function \( g = f|F(K) \times E \). Since every bounded set in \( F(K) \) is bounded in \( F \), it follows that \( g \) is bounded on every bounded set in \( F(K) \times E \). Lemma 3.3 implies there exists a neighbourhood \( U \) of \( O \in E \) such that \( g \) is bounded on \( B \times U \) for every bounded set \( B \) in \( F \). This is impossible.

(ii) Consider the function \( f : E \rightarrow H(F) \) associated to \( f \). Then \( f \) is holomorphic and by (i) it is bounded at \( O \in E \). Then as in [5] or as in Lemma 3.3 we can find \( p \) such that \( f \) can be factorized holomorphically through the canonical map \( \omega_p \) from \( E \) to \( E_p \). Take \( q > p \) such that \( \omega_{q,p} : E_q \rightarrow E_p \) is nuclear. Write
\[ \omega_{q,p}(z) = \sum_{j \geq 1} u_j(z) e_j \]
with
\[ a = \sum_{j \geq 1} \|u_j\| \|e_j\| < \infty. \]
Consider the Taylor expansion of \( f \) at \( O \in E \) in the variable \( z \in E \)
\[ f(z, x) = \sum_{n \geq 0} P_n f(z; x) \]
where
\[ P_n f(z; x) = (1/2\pi i) \int_{|t|=r} (f(tz, x)/t^{n+1}) \, dt \]
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for \((z,x) \in E \times F\).

We have

\[
f(\omega_{q,p}(z), x) = \sum_{n \geq 0} P_n f \left( \sum_{j \geq 1} u_j(z) e_j; x \right) = \sum_{n \geq 0, j_1, \ldots, j_n \geq 1} P_n f(e_{j_1}, \ldots, e_{j_n}; x) u_{j_1}(z) \ldots u_{j_n}(z).
\]

Moreover

\[
\sum_{n \geq 0} s^n \sum_{j_1, \ldots, j_n \geq 1} \|u_{j_1}\| \ldots \|u_{j_n}\| \|P_n f(e_{j_1}, \ldots, e_{j_n}, \ldots)\|_K = \sum_{n \geq 0} s^n \sum_{j_1, \ldots, j_n \geq 1} \|u_{j_1}\| \|e_{j_1}\| \ldots \|u_{j_n}\| \|e_{j_n}\| \|P_n f(e_{j_1}/\|e_{j_1}\|, \ldots, e_{j_n}/\|e_{j_n}\|, \ldots)\|_K \leq \left( \sum_{n \geq 0} s^n a^n n^n / \rho^n n! \right) \|f\|_{B_\rho \times K} < \infty
\]

for all \(\rho > \text{aes}\) and all compact set \(K\) in \(F\), where

\[
\|f\|_{B_\rho \times K} = \sup\{|f(z,x)|: \|z\| < \rho, x \in K\}
\]

and

\[
\|P_n f(e_{j_1}, \ldots, e_{j_n}, \ldots)\|_K = \sup\{|P_n f(e_{j_1}, \ldots, e_{j_n}, x)|: x \in K\}.
\]

Let \(B = \{z \in E_q : \|z\| = 1\}\). Consider the function

\[
f : \mathbb{C} \times \tilde{F} \to 1^\infty(B)\) with \(F \cong \mathbb{C} \times \tilde{F}\),
\]

given by

\[
\tilde{f}(t, x) = \left\{ \sum_{n \geq 0} t^n \sum_{j_1, \ldots, j_n \geq 1} P_n f(e_{j_1}, \ldots, e_{j_n}, x) u_{j_1}(z) \ldots u_{j_n}(z) \right\}_{z \in B}.
\]

For each \(N \in \mathbb{N}\) put

\[
S_N(t, x) = \left\{ \sum_{n \leq N} t^n \sum_{j_1, \ldots, j_n \geq 1} P_n f(e_{j_1}, \ldots, e_{j_n}, x) u_{j_1}(z) \ldots u_{j_n}(z) \right\}_{z \in B}.
\]
Since for every $k \geq 1$, the functions

$$S_{n,k}(t, x) = \sum_{n \leq N} t^n \sum_{j_1 + \ldots + j_n \leq k} P_n f(e_{j_1}, \ldots, e_{j_n}, x) u_{j_1}(z) \ldots u_{j_n}(z)$$

are holomorphic on $F$ with values in $1^\infty(B)$ and

$$S_{n,k} \rightarrow S_N$$

as $k \rightarrow \infty$ uniformly on every compact set in $F$, we infer that $S_N$ is holomorphic for $N \geq 1$. On the other hand, since $S_N \rightarrow \tilde{f}$ uniformly on compact set in $F$, it follows that $\tilde{f}$ is holomorphic. By Proposition 3.1 there exists a continuous semi-norm $\rho$ on $F$ and a holomorphic function $\tilde{g}$ on $F$ with values in $1^\infty(B)$ such that

$$\tilde{f}(t, x) = \tilde{g}(t, \omega_\rho(x)) \text{ for } (t, x) \in \mathbb{C} \times \tilde{F}.$$ 

We may assume that $\tilde{g}$ is bounded on every bounded set in $\mathbb{C} \times \tilde{F}$, because $F$ is Schwartz. Then

$$= \sup \left\{ \sum_{n \geq 0} t^n \sum_{j_1, \ldots, j_n \geq 1} P_n f(e_{j_1}, \ldots, e_{j_n}, x) \cdot u_{j_1}(z) \ldots u_{j_n}(z) : |t| \leq s, \ z \in B, \ \rho(x) \leq s \right\}$$

$$= \sup\{ ||\tilde{f}(t, x)|| : |t| \leq s, \ \rho(x) \leq s \}$$

$$= \sup\{ ||\tilde{g}(t, x)|| : |t| \leq s, \ \rho(x) \leq s \}$$

$$< \infty$$

for all $s \geq 0$.

Consequently $f$ is of uniform type.

The theorem is proved. ■

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