ON A POINTWISE ERGODIC THEOREM FOR MULTIPARAMETER SEMIGROUPS

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Abstract

Let $T_i$ ($i = 1, 2, \ldots, d$) be commuting null preserving transformations on a finite measure space $(X, \mathcal{F}, \mu)$ and let $1 \leq p < \infty$. In this paper we prove that for every $f \in L_p(\mu)$ the averages

$$A_n f(x) = (n + 1)^{-d} \sum_{0 \leq n_i \leq n} f(T_1^{n_1} T_2^{n_2} \ldots T_d^{n_d} x)$$

converge a.e. on $X$ if and only if there exists a finite invariant measure $\nu$ (under the transformations $T_i$) absolutely continuous with respect to $\mu$ and a sequence $\{X_N\}$ of invariant sets with $X_N \uparrow X$ such that $\nu B > 0$ for all nonnull invariant sets $B$ and such that the Radon-Nikodym derivative $\nu = d\nu/d\mu$ satisfies $\nu \in L_q(x_N, \mu)$, $1/p + 1/q = 1$, for each $N \geq 1$.

1. Introduction

We refer to [2] for the basic notation in ergodic theory. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $T_i : X \to X$ ($i = 1, 2, \ldots, d$) be commuting null preserving transformations, where $d \geq 1$ is a fixed integer. Associated with these transformations $T_i$ and for any measurable function $f$ on $X$ we have the averages

$$A_n f(x) = (n + 1)^{-d} \sum_{0 \leq n_i \leq n} f(T_1^{n_1} T_2^{n_2} \ldots T_d^{n_d} x) \quad (n \geq 0)$$

and the maximal operator

$$M f = \sup_{n \geq 0} A_n |f|.$$
Further each $T_i$ defines, by the Radon-Nikodym Theorem, a unique positive linear contraction operator $T_i^*$ on $L_1(\mu)$ by the relation
\[
\int_B T_i^* u \, d\mu = \int_{T_i^{-1}B} u \, d\mu \quad (u \in L_1(\mu), \, B \in \mathcal{F}).
\]

Under the additional hypothesis that all the transformations $T_i$ are invertible, Martín-Reyes [3] has recently proved the equivalence of the following conditions, for $1 \leq p < \infty$.

(a) The sequence $\{A_n f\}$ converges a.e. for all $f$ in $L_p(\mu)$;
(b) There exists a positive measurable function $U$ on $X$ such that
\[
\int_{\{A_n f > t\}} U \, d\mu \leq t^{-p} \int_X |f|^p \, d\mu \quad (t > 0, \, f \in L_p(\mu)).
\]

In this paper, without assuming the invertibility hypothesis on the transformations $T_i$, we intend to characterize those finite measures $\mu$ for which (a) holds.

2. The result

**Theorem.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $T_i : X \to X$ ($i = 1, 2, \ldots, d$) be commuting null preserving transformations. If $1 \leq p < \infty$, then the following are equivalent.

(a) For any $f \in L_p(\mu)$ the sequence $\{A_n f\}$ converges to a finite limit a.e. on $X$.
(b) For any $u \in L_1(\mu)$ the averages
\[
A_n^* u = (n + 1)^{-d} \sum_{0 \leq n_i \leq n} T_1^{n_1} T_2^{n_2} \ldots T_d^{n_d} u \quad (n \geq 0)
\]
converge a.e. on $X$ and also in the norm topology of $L_1(\mu)$; further to every $v \in L_1^+(\mu)$ with $T_i^* v = v$ for all $i = 1, 2, \ldots, d$ there corresponds a sequence $\{X_N\}$ of invariant sets with $X_N \uparrow X$ such that $v \in L_q(X_N, \mu), \, 1/p + 1/q = 1$, for all $N \geq 1$.
(c) There exists $v \in L_1^+(\mu)$ with $T_i^* v = v$ for all $i = 1, 2, \ldots, d$ and a sequence $\{X_N\}$ of invariant sets with $X_N \uparrow X$ such that $\int_B v \, d\mu > 0$ for all nonnull invariant sets $B$ and such that $v \in L_q(X_N, \mu), \, 1/p + 1/q = 1$, for all $N \geq 1$.
(d) $M f < \infty$ for all $f \in L_p(\mu)$. 
(e) There exists a positive measurable function $U$ on $X$ such that

$$\int_{\{M_f > t\}} U \, d\mu \leq t^{-p} \int_X |f|^p \, d\mu \quad (t > 0, f \in L_p(\mu)).$$

(f) There exists a positive measurable function $U$ on $X$, a constant $r > 0$, and a subsequence $\{n(k)\}$ of $\{n\}$ such that

$$\int_{\{|A_{n(k)}f| > t\}} U \, d\mu \leq t^{-r} \left(\int_X |f|^p \, d\mu\right)^{r/p} \quad (t > 0, f \in L_p(\mu)).$$

We begin by proving the following lemma, which deals with the case $p = \infty$.

**Lemma.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $T_i : X \rightarrow X$ ($i = 1, 2, \ldots, d$) be commuting null preserving transformations. Then the following are equivalent.

(a) The sequence $\{A_n f\}$ converges a.e. on $X$ for all $f \in L_\infty(\mu)$.

(b) The sequence $\{A_n^* u\}$ converges in the norm topology of $L_1(\mu)$ for all $u \in L_1(\mu)$.

(c) For any $u \in L_1^+(\mu)$ with $\|u\|_1 > 0$ the pointwise limit $u_0^*(x) = \lim_n A_n^* (x)$ exists a.e. on $X$ and satisfies $\|u_0^*\|_1 > 0$.

(d) For any $u \in L_1^+(\mu)$ with $\|u\|_1 > 0$ we have

$$0 < \liminf_n A_n^* u_1 < \infty.$$ 

**Proof:**

(a) $\Rightarrow$ (b) follows from a mean ergodic theorem (see e.g. [2, Theorem 2.1.5]).

(b) $\Rightarrow$ (a) and (c). Let $v_0 = \text{strong-} \lim_n A_n^* 1(\in L_1^+(\mu))$. Since $T_i^* v_0 = v_0$ for all $i = 1, 2, \ldots, d$, we have

$$Y \subset T_i^{-1} Y \text{ for all } i = 1, 2, \ldots, d, \text{ where } Y = \{v_0 > 0\}.$$ 

Since the measure $\nu = v_0 \, d\mu$ is invariant under the $T_i$'s, we may regard the transformations $T_i$ as commuting measure preserving transformations on a finite measure space $(Y, v_0 \, d\mu)$. Then, by the classical multi-parameter pointwise ergodic theorem, for any $f \in L_\infty(\mu)$ the sequence
\( \{A_n f\} \) converges a.e. on \( Y \). To prove the a.e. convergence of \( \{A_n f\} \) on \( X \setminus Y \), it is sufficient to show that
\[
(T_1 T_2 \ldots T_d)^{-n} Y \uparrow X.
\]
To do this, let \( B = \lim (T_1 T_2 \ldots T_d)^{-n} Y \). We see easily that \( T_i^{-1} B = B \) for all \( i = 1, 2, \ldots, d \), i.e., \( B \) is an invariant set. Hence
\[
\mu(X \setminus B) = \int_{X \setminus B} A_n^* d\mu = \int_{\{v_0 = 0\}} v_0 d\mu = 0.
\]
To prove (c), let \( u \in L^1_\mu \) and \( \|u\|_1 > 0 \). Since \( \|A_n^* u\|_1 = \|u\|_1 > 0 \) and \( \{A_n^* u\} \) converges in the norm topology of \( L_1(\mu) \), it is sufficient to prove the a.e. convergence of \( \{A_n^* u\} \). Since the transformations \( T_i \) preserve the measure \( \nu = v_0 d\mu \), the classical pointwise ergodic theorem for multiparameter semigroups of Dunford-Schwartz operators and an approximation argument imply that \( \{A_n^* u\} \) converges a.e. on \( Y \).

To prove that \( \lim A_n^* u(x) = 0 \) a.e. on \( X \setminus Y \), we use Brunel's Theorem (see e.g. [2, Theorem 6.3.4]) concerning an ergodic inequality for commuting linear contraction operators on \( L_1(\mu) \): there exists a constant \( K_d > 0 \) and a positive linear operator \( Q \) on \( L_\infty(\mu) \) of the form
\[
Qf(x) = \sum_{n_i \geq 0} a(n_1, n_2, \ldots, n_d) f(T_1^{n_1} T_2^{n_2} \ldots T_d^{n_d} x),
\]
where \( a(n_1, n_2, \ldots, n_d) > 0 \) and \( \sum_{n_i \geq 0} a(n_1, n_2, \ldots, n_d) = 1 \), such that if \( Q^* \) denotes the positive linear operator on \( L_1(\mu) \) associated with \( Q \), then
\[
\limsup_{n} A_n^* u \leq K_d \cdot \limsup_{n} (n + 1)^{-1} \sum_{i=0}^{n} Q^{*i} u \quad (u \in L^1_\mu(\mu)).
\]
Let \( C = \{x : \sum_{i=0}^{\infty} Q^{*i} u = \infty\} \setminus Y \). Since \( \|Q^*\|_1 = 1 \), it follows that \( Q_{1_C} \geq 1_C \), where \( 1_C \) being the indicator function of \( C \). Thus we have \( C \subset T_i^{-1} C \) for all \( i = 1, 2, \ldots, d \), and hence
\[
\mu C \leq \int_{X} A_n 1_C d\mu = \int_{C} A_n^* 1 d\mu = \int_{C} v_0 d\mu = \int_{\{v_0 = 0\}} v_0 d\mu = 0.
\]
This proves that \( \lim A_n^* u(x) = 0 \) a.e. on \( X \setminus Y \).

(c) \( \Rightarrow \) (d). Obvious.
(d) ⇒ (b). There exists \( v_0 \in L_1^+(\mu) \) with \( T_i^*v_0 = v_0 \) for all \( i = 1, 2, \ldots, d \) such that if \( v \in L_1^+(\mu) \) satisfies \( T_i^*v = v \) for all \( i = 1, 2, \ldots, d \) then \( \{v > 0\} \subseteq \{v_0 > 0\} \). Let \( Y = \{v_0 > 0\} \) and

\[
B = \lim_{n} (T_1T_2\ldots T_d)^{-n}Y.
\]

Since \( B \) and \( X \setminus B \) are invariant sets, it follows that if \( u \in L_1^+(X \setminus B, \mu) \) and \( \|u\|_1 > 0 \) then the function

\[
\tilde{u}_0 = \liminf_{n} A_n^*u \quad \text{satisfies} \quad \{\tilde{u}_0 > 0\} \cap \{v_0 > 0\} = \emptyset.
\]

But this is impossible, since \( T_i^*\tilde{u}_0 = \tilde{u}_0 \in L_1^+(\mu) \) for all \( i = 1, 2, \ldots, d \) and (d) implies that \( \mu\{\tilde{u}_0 > 0\} > 0 \). We conclude that

\[
(T_1T_2\ldots T_d)^{-n}Y \uparrow X.
\]

Hence by an approximation argument we see that \( \{A_n^*u\} \) converges in the norm topology of \( L_1(\mu) \) for all \( u \in L_1(\mu) \), completing the proof. ■

**Proof of the Theorem:** (a) ⇒ (b). The first part of (b) follows from the lemma. To prove the second part, let \( v \in L_1^+(\mu) \) be such that \( T_i^*v = v \) for all \( i = 1, 2, \ldots, d \). Putting \( Y = \{v > 0\} \), we see that the transformations \( T_i \) can be regarded as commuting null preserving transformations on the measure space \((Y, \mu)\). Since the measure \( \nu = v d\mu \) is invariant under the transformations \( T_i \), it follows that these \( T_i \) are conservative on \((Y, \mu)\).

By this and the fact that for each \( f \) in \( L_p(Y, \mu) \) the sequence \( \{A_n f\} \) converges to a finite limit a.e. on \( Y \), we can apply Theorem 3.1 in [4] to infer that there exists a sequence \( \{Y_N\} \) of sets in \( \mathcal{I}_Y \), where

\[
\mathcal{I}_Y = \{B \in \mathcal{F} : B \subset Y, B = Y \cap T_i^{-1}B \text{ for all } i = 1, 2, \ldots, d\},
\]

such that \( Y_N \uparrow Y \) and \( v \in L_q(Y_N, \mu) \) for all \( N \geq 1 \). Then, letting

\[
X_N = \left[ \lim_{n} (T_1T_2\ldots T_d)^{-n}Y_N \right] \cup \left[ X \setminus \lim_{n} (T_1T_2\ldots T_d)^{-n}Y \right],
\]

we have \( v \in L_q(X_N, \mu) \) for all \( N \geq 1 \), \( X_N \uparrow X \), and \( X_N \in \mathcal{I} \) where

\[
\mathcal{I} = \{B \in \mathcal{F} : B = T_i^{-1}B \text{ for all } i = 1, 2, \ldots, d\}.
\]

(b) ⇒ (c). It is enough to put \( v = \text{strong-lim}_{n} A_n^*1 \).

(c) ⇒ (a). Put \( Y = \{v > 0\} \). It follows (cf. the proof of the lemma) that

\[
(T_1T_2\ldots T_d)^{-n}Y \uparrow X.
\]
Hence it is sufficient to prove that for each $f$ in $L_p(Y, \mu)$ the sequence \{${A_n f}$\} converges to a finite limit a.e. on $Y$; this follows from the equivalence of (a) and (f) of Theorem 3.1 in [4], since the transformations $T_i$ may be regarded as commuting conservative null preserving transformations on the measure space $(Y, \mu)$.

(a) $\Rightarrow$ (d). Obvious.

(d) $\Rightarrow$ (e). This follows from Nikishin's Theorem (see e.g. [1, p. 536]).

(e) $\Rightarrow$ (f). Obvious.

(f) $\Rightarrow$ (a). We may suppose that $0 < U \leq 1$ on $X$. Using an approximation argument we see (cf. the proof of (d) $\Rightarrow$ (a) of Theorem 3.1 in [4]) that

$$\lim \sup_{\mu B \to 0} \sup_{k \geq 1} \int_B A_{n(k)}^* f d\mu = 0.$$ 

Hence by a mean ergodic theorem we see that the sequence \{${A_n^* 1}$\} converges in the norm topology of $L_1(\mu)$. Write $v = \text{strong-lim} A_n^* 1$ and $Y = \{v > 0\}$. Since $T_i^* v = v$ for all $i = 1, 2, \ldots , d$ and $(T_1 T_2 \ldots T_d)^{-n} Y \uparrow X$, it follows from the classical multiparameter pointwise ergodic theorem that for any $f \in L_p^+(\mu)$ the limit

$$f^*(x) = \lim_n A_n f(x)$$

exists a.e. on $X$ (but may be equal to infinity on some subset of $X$).

To prove that $f^* < \infty$ a.e. on $X$, we observe that \{${f^* = \infty}$\} $\subset \liminf_k \{A_n(k) f > t\}$ for all $t > 0$; hence by Fatou's Lemma and (f)

$$\int_{\{f^* = \infty\}} U d\mu \leq \liminf_k \int_{\{A_n(k) f > t\}} U d\mu \leq t^{-r} \left( \int_X f^p d\mu \right)^{r/p}.$$ 

Letting $t \uparrow \infty$, we have $\int_{\{f^* = \infty\}} U d\mu = 0$ and $\mu \{f^* = \infty\} = 0$. The proof is complete. ■

Since the above proofs of the implications (a) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a) hold for $0 < p < \infty$, we have the

**Corollary.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $T_i : X \to X$ ($i = 1, 2, \ldots , d$) be commuting null preserving transformations. Let $0 < p, r < \infty$. If there exists a subsequence $\{n(k)\}$ of $\{n\}$ such that the operators $A_{n(k)}$ are equicontinuous mappings from $L_p(\mu)$ to $L_r(\mu)$ then
for any $f \in L_p(\mu)$ the sequence $\{A_nf\}$ converges to a finite limit a.e. on $X$.

Proof: This follows from the equivalence of (a) and (f) of the theorem. ■

References

4. R. Sato, Multiparameter pointwise ergodic theorems for Markov operators on $L_\infty$, submitted for publication.

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Added in proof: An obvious argument shows that condition (c) of the theorem may be sharpened as follows. (c') There exists $v \in L^+_\infty(\mu)$ with $T_i^*v = v$ for all $1 \leq i \leq d$ such that $\int_B v \, d\mu > 0$ for all nonnull invariant sets $B$. 