Abstract

Let $G$ be a finitely generated group. We give a new characterization of its Bieri-Neumann-Strebel invariant $\Sigma(G)$, in terms of geometric abelian actions on $\mathbb{R}$-trees. We provide a proof of Brown's characterization of $\Sigma(G)$ by exceptional abelian actions of $G$, using geometric methods.

Introduction.

In a 1987 paper at Inventiones [BNS], Bieri, Neumann and Strebel associated an invariant $\Sigma = \Sigma(G)$ to any finitely generated group $G$. This invariant may be viewed as a positively homogeneous open subset of $\text{Hom}(G, \mathbb{R}) \setminus \{0\}$. It contains information about finitely generated normal subgroups of $G$ with abelian quotient.

In the same issue of Inventiones [Br], Brown introduced HNN-valuations and related $\Sigma$ to actions of $G$ on $\mathbb{R}$-trees. In particular a nonzero homomorphism $\chi : G \to \mathbb{R}$ is in $\Sigma \cap -\Sigma$ if and only if $\mathbb{R}$ is the only $\mathbb{R}$-tree admitting a minimal action of $G$ with length function $|\chi|$ (see Theorem 3.2 below).

A few months earlier, also in Inventiones [Le 1], this author studied singular closed differential one-forms on closed manifolds $M^n$ ($n \geq 3$). We defined complete forms by several equivalent geometric conditions; in the simplest case, a form $\omega$ is complete if and only if every path in $M$ is homotopic to a path $\gamma$ that is either transverse to $\omega$ or tangent to $\omega$ (i.e. $\omega(\gamma'(t))$ never vanishes or is identically 0).

We proved that any form cohomologous to a complete form is also complete, so that completeness defines a subset $U(M)$ in the De Rham cohomology space $H^1_{DR}(M, \mathbb{R}) \simeq \text{Hom}(\pi_1 M, \mathbb{R})$. We also proved that $U(M)$ depends only on the group $G = \pi_1 M$, and in fact $U(M)$ is nothing but $\Sigma(\pi_1 M) \cap -\Sigma(\pi_1 M)$. 


R-TREES AND THE BIERI-NEUMANN-STREBEL INVARIANT

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In this note we use (a generalization of) closed one-forms to give a new characterization of \( \Sigma(G) \), this time in terms of geometric actions of \( G \) on \( \mathbb{R} \)-trees (Theorem 3.1). Assuming for simplicity that \( G \) is finitely presented, we say that an action of \( G \) on an \( \mathbb{R} \)-tree is geometric if it comes from a measured foliation on a finite 2-complex \( K \) with \( \pi_1 K = G \) (see [LP] for a complete discussion). A consequence of Theorem 3.1 is:

**Corollary.** Let \( \chi : G \to \mathbb{R} \) be a nonzero homomorphism, with \( G \) finitely generated.

1. There exists a geometric action of \( G \) on an \( \mathbb{R} \)-tree with length function \( \ell = |\chi| \) if and only if \( \chi \in \Sigma \cup -\Sigma \).
2. The action of \( G \) on \( \mathbb{R} \) by translations associated to \( \chi \) is geometric if and only if \( \chi \in \Sigma \cap -\Sigma \).

We also give a geometric proof of Brown's theorem, by associating a natural \( \mathbb{R} \)-tree \( T^+(f) \) to any real-valued function \( f \) defined on a path-connected space (there is a similar construction in terms of romp-trees in [BS, Chapter II]).

In Part 1 we define closed one-forms relative to a homomorphism \( \chi : G \to \mathbb{R} \), and we reformulate the condition \( \chi \in \Sigma \) in terms of closed one-forms. In Part 2 we recall known facts about abelian actions on \( \mathbb{R} \)-trees (see [CuMo], [Sh]). In Part 3 we prove both characterizations of \( \Sigma \) mentioned above.

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1. **Closed one-forms.**

Let \( \chi : G \to \mathbb{R} \) be a homomorphism. A *closed one-form* relative to \( \chi \) consists of a path-connected space \( X \) equipped with an action of \( G \), together with a continuous function \( f : X \to \mathbb{R} \) such that

\[
f(gx) = f(x) + \chi(g)
\]

for all \( x \in X \) and \( g \in G \).

The closed one-form is *geometric* if \( G \) acts as a group of covering transformations and the base \( X/G \) is (homeomorphic to) a finite CW-complex. Note that this forces \( G \) to be finitely generated.

**Example 1.** Let \( G \) be the trivial group. Then any function on a path-connected space defines a closed one-form.
Example 2. Let $\omega$ be a closed differential one-form on a closed manifold $M$. Let $\chi : \pi_1M \to \mathbb{R}$ be the homomorphism given by integrating $\omega$ along loops. Let $p : X \to M$ be the universal covering. Then any $f : X \to \mathbb{R}$ such that $df = p^*\omega$ defines a geometric closed one-form relative to $\chi$.

Example 3. Let $\Gamma$ be the Cayley graph of $G$ relative to some fixed generating system. Given a homomorphism $\chi : G \to \mathbb{R}$, view it as a function on the set of vertices of $\Gamma$, and extend it affinely and $G$-equivariantly to a function $f$ defined on the whole of $\Gamma$. This defines a closed one-form relative to $\chi$. It is geometric if and only if the generating system is finite.

Example 4. Any abelian action of $G$ on an $\mathbb{R}$-tree $T$ defines a closed one-form $f : T \to \mathbb{R}$ (see Part 2).

If $f$ is a closed one-form on $X$, we denote $X_{>c} = f^{-1}(c, +\infty)$ for $c \in \mathbb{R}$.

**Proposition 1.1.** Let $f : X \to \mathbb{R}$ be a geometric closed one-form relative to a nonzero homomorphism $\chi : G \to \mathbb{R}$. For any $c \in \mathbb{R}$, the set $X_{>c}$ has at least one component on which $f$ is unbounded. This component is unique if and only if $\chi \in \Sigma(G)$.

**Proof:** Since $f$ is geometric, the group $G$ acts on $X$ as a group of covering transformations. Let $X' = X/G'$, where $G'$ is the commutator subgroup of $G$. The function $f$ induces $f' : X' \to \mathbb{R}$. Let $X'_{>c} = f'^{-1}(c, +\infty)$. By [BNS, Part 5], there exists a unique component $A'$ of $X'_{>c}$ on which $f'$ is unbounded, and $\chi \in \Sigma(G)$ if and only if the natural map from $\pi_1 A'$ to $G'$ is onto (compare [Le 1, Parts IV and V] and [Si]). The proposition follows. ■

2. Abelian actions on $\mathbb{R}$-trees.

Suppose a finitely generated group $G$ acts by isometries on an $\mathbb{R}$-tree $T$.

The *length function* $\ell : G \to \mathbb{R}^+$ is defined as $\ell(g) = \inf_{x \in T} d(x, gx)$. The action is *trivial* if there is a global fixed point (equivalently if $\ell \equiv 0$), *minimal* if there is no proper invariant subtree. The action (or the length function) is called *abelian* if $\ell$ is the absolute value of a nonzero homomorphism $\chi : G \to \mathbb{R}$. Two minimal actions of $G$ with the same length function $\ell$ are equivariantly isometric, except maybe if $\ell$ is abelian. Brown's theorem (Theorem 3.2 below) is concerned with this "maybe".

A nontrivial action is abelian if and only if there is a fixed end $e$. We can then define a closed one-form on $T$, as follows. Given $x \in T$, there is a unique isometric embedding $i_x : (-\infty, 0] \to T$ such that $i_x(-\infty) = e$.
and \( i_x(0) = x \). Fixing a basepoint \( m \in T \), we define \( f(x) \) as the only real number such that \( i_x(t) = i_m(t + f(x)) \) for \( |t| \) large enough ("Busemann function"). Then \( f \) is a closed one-form on \( T \), relative to some nonzero homomorphism \( \chi : G \to \mathbb{R} \) satisfying \( \ell = |\chi| \). This homomorphism measures how much elements of \( G \) push away from \( e \).

An abelian action is called *exceptional* if there is only one fixed end \( e \). We can then define \( \chi \) unambiguously, and we say that the action is *associated* to \( \chi \). If there are two fixed ends (i.e. if there is an invariant line), we say that the action is associated to both \( \chi \) and \( -\chi \).

### 3. Characterizations of \( \Sigma \).

Let \( f : X \to \mathbb{R} \) be continuous, with \( X \) path-connected. Assume \( f \) has bounded variation in the following sense: given \( x, y \in X \), there exists a path \( \gamma : [0, 1] \to X \) from \( x \) to \( y \) such that \( f \circ \gamma \) has bounded variation. The infimum of the total variation of \( f \circ \gamma \) over all paths \( \gamma \) from \( x \) to \( y \) then defines a pseudometric \( d(x, y) \) on \( X \).

We let \( T(f) \) be the associated metric space: points of \( T(f) \) are equivalence classes for the relation \( d(x, y) = 0 \). Denote \( \pi : X \to T(f) \) the natural projection and \( \lambda : T(f) \to \mathbb{R} \) the map such that \( \lambda \circ \pi = f \).

If \( f \) is a closed one-form relative to some \( \chi : G \to \mathbb{R} \), there is an induced isometric action of \( G \) on \( T(f) \) with \( \lambda(gx) = \lambda(x) + \chi(g) \). When \( T(f) \) is an \( \mathbb{R} \)-tree, the length function \( \ell \) of this action satisfies \( \ell \geq |\chi| \) (since \( \lambda \) does not increase distances).

**Definition.** Consider an abelian action of a finitely generated group \( G \) on an \( \mathbb{R} \)-tree \( T \), associated to \( \chi : G \to \mathbb{R} \). The action is geometric if and only if there exists a geometric closed one-form \( f : X \to \mathbb{R} \) relative to \( \chi \) such that \( T(f) \) is \( G \)-equivariantly isometric to \( T \).

**Theorem 3.1.** Let \( \chi : G \to \mathbb{R} \) be a nontrivial homomorphism, with \( G \) a finitely generated group. There exists a geometric abelian action of \( G \) on an \( \mathbb{R} \)-tree associated to \( \chi \) if and only if \( \chi \in -\Sigma \).

**Proof:**

Let \( f : X \to \mathbb{R} \) be a geometric closed one-form relative to \( \chi \). Assume that \( T(f) \) is an \( \mathbb{R} \)-tree and the action of \( G \) on \( T(f) \) is abelian, associated to \( \chi \). We show \( \chi \in -\Sigma \).

Fix \( g \in G \) with \( \chi(g) < 0 \), and fix \( x \in X \) with, say, \( f(x) = 0 \). For \( A \) large enough, the path component \( U \) of \( f^{-1}(-A, A) \) containing \( x \) meets every orbit for the action of \( G \) on \( X \): this is because \( X/G \) is a finite complex. We may also assume that \( A \) has been chosen so that \( gx \in U \).
We then claim that any \( y \in X \) with \( f(y) \leq -A \) belongs to the same component of \( f^{-1}(-\infty, A) \) as \( x \). This will imply \( \chi \in -\Sigma \) by Proposition 1.1.

Choose an infinite path \( \gamma : [0, +\infty) \to X \) such that \( \gamma|_{[0,1]} \) is a path from \( x \) to \( gx \) in \( U \) and \( \gamma(t + n) = g^n\gamma(t) \) for \( n \in \mathbb{N} \) and \( t \in [0, 1] \). Since \( \chi(g) < 0 \) this path is contained in \( f^{-1}(-\infty, A) \).

Given \( y \in X \) with \( f(y) \leq -A \), fix \( h \in G \) such that \( hy \in U \), and choose a path \( \delta \) from \( hy \) to \( x \) in \( U \). Consider the infinite path \( \rho \) obtained by applying \( h^{-1} \) to \( \delta \gamma \): it starts at \( y \) and passes through \( h^{-1}x \), \( h^{-1}gx \), \( h^{-1}g^2x \), \ldots. It is contained in \( f^{-1}(-\infty, A) \) since \( f(y) \leq -A \).

The image of \( \gamma \) in \( T(f) \) contains all points \( g^n\pi(x) \) \((n \in \mathbb{N})\), while the image of \( \rho \) contains all points \( h^{-1}g^n\pi(x) = (h^{-1}gh)^n\pi(h^{-1}x) \). Now the translation axes of \( g \) and \( h^{-1}gh \) intersect in a half-line containing the fixed end \( e \) (unless they are equal). Furthermore \( g \) and \( h^{-1}gh \) both push towards \( e \) since \( \chi(g) < 0 \). It follows that \( \gamma \) and \( \rho \) are contained in the same component of \( f^{-1}(-\infty, A) \), so that \( \chi \in \Sigma \).

Conversely, suppose \( \chi \in -\Sigma \). First assume that \( G \) is finitely presented. Let \( M \) be a closed manifold with \( \pi_1 M = G \). Consider a geometric closed one-form \( f : X \to \mathbb{R} \) as in Example 2 of Part 1. To fix ideas we may assume that \( f \) is a Morse function.

Since \( X \) is simply connected (it is the universal covering of \( M \)), it is known [GS] that \( T(f) \) is an \( \mathbb{R} \)-tree (see [Le 2, Corollary 111.5]). Since \( \chi \in -\Sigma \) the function \( \lambda : T(f) \to \mathbb{R} \) is bounded on every component of \( \lambda^{-1}(-\infty, c) \) but one. It follows that the action of \( G \) on \( T(f) \) is abelian, associated to \( \chi \): letting \( \lambda \) go to \(-\infty \) defines an end \( e \) of \( T(f) \) which is invariant under the action.

Now let \( G \) be any finitely generated group. Using (i) \( \iff \) (iii) in [BNS, Proposition 2.1] we can find an epimorphism \( q : H \to G \), with \( H \) finitely presented, such that \( \chi' = \chi \circ q \) belongs to \(-\Sigma(H)\). Apply the previous construction to \( H \) and \( \chi' \). Let \( Y \) be the normal covering of \( M \) with group \( G \) and \( g : Y \to \mathbb{R} \) the map induced by \( f \).

The length function of the action of \( H \) on the \( \mathbb{R} \)-tree \( T(f) \) is \( \ell = |\chi'| \). It vanishes on the kernel \( K \) of \( q \). It follows from [Le 2, corollary of Theorem 2] that \( T(g) = T(f)/K \) is an \( \mathbb{R} \)-tree. The action of \( G \) on this \( \mathbb{R} \)-tree is abelian, associated to \( \chi \).

**Theorem 3.2 (Brown).** Let \( \chi : G \to \mathbb{R} \) be a nontrivial homomorphism, with \( G \) finitely generated. Then \( \chi \in \Sigma \) if and only if there exists no exceptional abelian action associated to \( \chi \).

We start the proof with a general construction. Let \( f : X \to \mathbb{R} \) be continuous, with \( X \) path connected. We construct an \( \mathbb{R} \)-tree \( T^+(f) \) as
follows. Given \( x, y \in X \), define
\[
\delta(x, y) = f(x) + f(y) - 2\sup_{\gamma \in [0,1]} \min_{t \in [0,1]} f(\gamma(t)),
\]
the supremum being over all paths \( \gamma : [0,1] \to X \) from \( x \) to \( y \). This is a pseudodistance on \( X \) and we let \( T^+(f) \) be the associated metric space.

**Proposition 3.3.** The space \( T^+(f) \) is an \( \mathbb{R} \)-tree. If \( \mu : T^+(f) \to \mathbb{R} \) is the map induced by \( f \), all sets \( \mu^{-1}(-\infty, c) \) are path-connected, so that \( T^+(f) \) has a preferred end \( e = \mu^{-1}(-\infty) \).

**Proof:**

We first prove that \( T^+(f) \) is an \( \mathbb{R} \)-tree. By [AB, Theorem 3.17] it suffices to show that \( \delta \) satisfies the 0-hyperbolicity inequality
\[
\delta(x, y) + \delta(z, t) \leq \max\{\delta(x, z) + \delta(y, t), \delta(x, t) + \delta(y, z)\}.
\]
By linearity we need only worry about the terms \( \delta' = \sup \min f \circ \gamma \). They satisfy inequalities such as
\[
\delta'(x, y) \leq \min\{\max(\delta'(x, z), \delta'(z, y)), \max(\delta'(x, t), \delta'(t, y))\}
\]
and we conclude by applying the inequality
\[
\min\{\max(a, c), \max(b, d)\} + \min\{\max(a, b), \max(c, d)\} \leq \max(a+d, b+c),
\]
valid for any four real numbers \( a, b, c, d \).

Let \( \pi^+ \) be the projection from \( X \) to \( T^+(f) \). Given \( x, y \in X \) in \( f^{-1}(-\infty, c) \) with, say, \( f(y) \leq f(x) \), choose a path \( \gamma : [0,1] \to X \) from \( x \) to \( y \). If \( (p, q) \) is a maximal interval in \( (f \circ \gamma)^{-1}(f(x), +\infty) \), we have \( (\pi^+ \circ \gamma)(p) = (\pi^+ \circ \gamma)(q) \) and we can change \( \pi^+ \circ \gamma \) on \( (p, q) \) so that it becomes constant on \( [p, q] \). Doing this for all intervals \( (p, q) \) yields a path from \( \pi^+(x) \) to \( \pi^+(y) \) in \( \mu^{-1}(-\infty, c) \).

If \( f \) is a closed one-form relative to \( \chi : G \to \mathbb{R} \), the natural action of \( G \) on \( T^+(f) \) fixes \( e \). It is abelian, associated to \( \chi \) (note that this action is nongeometric whenever \( \chi \notin -\Sigma \), by Theorem 3.1).

To prove Theorem 3.2, we fix a finite generating system \( S \) for \( G \) with \( \chi(s) > 0 \) for every \( s \in S \) and we consider the corresponding Cayley graph \( \Gamma \).

First assume \( \chi \notin \Sigma \). Let \( f : \Gamma \to \mathbb{R} \) be as in Example 3 of Part 1. We claim that the abelian action of \( G \) on \( T^+(f) \) is exceptional.
Let $u_1$ and $u_2$ be vertices of $\Gamma$ belonging to distinct components $U_1, U_2$ of some $f^{-1}(c, +\infty)$. Fix $s \in S$. The whole ray $u_i, u_is, u_is^2, \ldots, u_is^n, \ldots$ is contained in $U_i$. Writing $u_is^n = (u_isu_i^{-1})^nu_i$ we see that $u_isu_i^{-1}$ and $u_isu_2^{-1}$ do not have the same translation axis in $T^+(f)$. This means that the action is exceptional.

Now assume $\chi \in \Sigma$. Let $T$ be an $\mathbb{R}$-tree with a minimal abelian action associated to $\chi$. We show that the action is not exceptional.

Choose $x \in T$ belonging to the translation axis of every $s \in S$. Consider a $G$-equivariant map $\varphi : \Gamma \to T$, affine on each edge, sending 1 to $x$. It is surjective because the action is minimal.

Define $f : T \to \mathbb{R}$ as in Part 2. The choice of $x$ implies that $g = f \circ \varphi$ is monotonous on each edge of $\Gamma$. It follows that $g$ is unbounded on every component of $g^{-1}(c, +\infty)$, so that $g^{-1}(c, +\infty)$ is connected for every $c \in \mathbb{R}$ by Proposition 1.1. Projecting to $T$ we see that every $f^{-1}(c, +\infty)$ is connected: the action is not exceptional.

Combining Theorems 3.1 and 3.2 we get:

**Corollary.** Let $\chi : G \to \mathbb{R}$ be a nonzero homomorphism, with $G$ finitely generated.

1. If $\chi \in \Sigma \cap -\Sigma$, the action of $G$ on $\mathbb{R}$ by translations is the only minimal action with length function $\ell = |\chi|$. It is geometric.
2. If $\chi \in \Sigma$ but $\chi \notin -\Sigma$, there exist geometric exceptional abelian actions associated to $-\chi$. The only minimal action associated to $\chi$ is the action on $\mathbb{R}$, it is not geometric.
3. If $\chi \notin \Sigma \cup -\Sigma$, there exist both exceptional abelian actions associated to $\chi$ and exceptional actions associated to $-\chi$. No action with length function $|\chi|$ is geometric.

Combining with Theorem B.1 from [BNS] we obtain:

**Corollary.** Let $G$ be finitely generated. The following conditions are equivalent:

1. Every nontrivial action of $G$ on $\mathbb{R}$ by translations is geometric.
2. The commutator subgroup $G'$ is finitely generated.

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