Let \( G \) be an infinite, locally soluble group which is isomorphic to all its nontrivial normal subgroups. If \( G/G' \) has finite \( p \)-rank for \( p = 0 \) and for all primes \( p \), then \( G \) is cyclic.

In paper [2] we considered groups which are isomorphic to all of their nontrivial normal subgroups. The question as to which infinite groups have this property \( P \), say, was first raised by Philip Hall. It was shown in [2] that, if \( G \) is a finitely generated infinite \( P \)-group which contains a proper normal subgroup of finite index, then \( G \) is cyclic, and our conjecture is that \( Z \) is the only finitely generated infinite \( P \)-group which is not simple. It was further remarked in [2] that is should perhaps be possible to deal with the locally soluble case, and this note represents a step in this direction. The following is proved.

**Theorem.** Let \( G \) be an infinite, locally soluble group which is isomorphic to all of its nontrivial normal subgroups. If \( G/G' \) has finite \( p \)-rank for all \( p = 0 \) or a prime then \( G \) is cyclic.

We recall that an abelian group \( A \) has \( p \)-rank \( r \) if the cardinality of a maximal independent subset of elements of \( A \) of order \( p \) is equal to \( r \). In particular, if \( A \) has finite (Prüfer) rank then the \( p \)-ranks of \( A \) are boundedly finite.

There is one aspect of the proof of our theorem which recalls part of the proof from [2], namely the exploitation of "linearity conditions" which are forced by the rank restrictions (in conjunction with property \( P \)). In the case where \( G \) has normal abelian \( p \)-sections of possibly infinite rank, such a technique is bound to fail, and it is not clear how one might approach the case where, for example, \( G \) is an arbitrary locally nilpotent
group with $P$. Clearly such a group is either torsionfree or a $p$-group, but beyond that there is little that we can say at the moment.

**Proof of the theorem:**

Suppose first that $G$ has a nontrivial, torsionfree soluble image $S$ and let $r$ be the 0-rank of $G/G'$. Then, because of property $P$, $S$ has finite Hirsch length (that is, the sum of the 0-ranks of the derived factors of $G$ is finite). Let $F$ denote the Fitting subgroup of $S$. Then $F$ is locally nilpotent and its abelian subgroups have finite 0-rank. Since $F$ is torsionfree, it is nilpotent (see Lemma 6.37 of [3]). Let $A$ be a maximal normal abelian subgroup of $F$. Then $A$ is self-centralizing in $F$ and of rank at most $r$ (again by $P$), and so $F/A$ embeds in the group of (upper) unitriangular $r \times r$ matrices over $\mathbb{Q}$. It follows that $F/A$ and hence $F$ has bounded rank and bounded nilpotency class $c$, say. Let $D = \cap D_i$. Then $S/D$ has bounded derived length. Further, $D$ stabilizes a series of length $c$ in $F$ and so, writing $C$ for the centralizer of $F$ in $S$, we see that $D/C$ is nilpotent ([1, Lemma 3.5]). But $C = Z(F)$ (e.g. Lemma 2.17 of [3]) and so $[C, D] = 1$ and $D$ is nilpotent and hence in $F$. It follows that $S$ has bounded derived length and we can choose $N$ minimal subject to $N < G$ and $G/N$ torsionfree soluble. If $N \neq 1$ then, by property $P$, $N$ has a nontrivial, torsionfree soluble image, contradicting the definition of $N$. Thus $N = 1$ and $G$ is soluble. Clearly $G \cong \mathbb{Z}$ in this case. From now on, we may assume that all soluble images of $G$ are periodic. (If $G$ were to have a nonperiodic soluble image then some abelian normal factor of $G$ would be nontrivial and torsionfree and so, again by $P$, $G/G'$ would have a nontrivial torsionfree image.) Let $H/K$ be an arbitrary chief factor of $G$ - such exists in every nontrivial group. Then $H/K$ is an elementary abelian $p$-group, for some prime $p$, and we see that $G$ therefore has a nontrivial finite $p$-image. Let $P_1 = G'G^p$ and, for $i \geq 1$, let $P_{i+1} = P_i^p P_i^p$. By property $P$, the subgroups $P_i$ form a strictly descending chain of normal subgroups of $G$. Also, each $G/P_i$ is a finite $p$-group. Let $R = \bigcap_{i=1}^{\infty} P_i$ and write $\overline{G} = G/R$, $\overline{P}_i = P_i/R$, $i = 1, 2, \ldots$.

Let $s$ be the rank of $G/P_1$ and let $\overline{A}$ be an arbitrary finitely generated abelian subgroup of $\overline{G}$. The subgroups $\overline{A} \cap \overline{P}_i$ form a descending chain, with trivial intersection, such that each $\overline{A}/\overline{A} \cap \overline{P}_i$ is a finite (abelian) $p$-group of rank at most $s$ (since $\overline{A}_{P_i}/\overline{P}_i$ is subnormal in $\overline{G}/\overline{P}_i$). It
follows that $\bar{A}$ has rank at most $s$, and so $\bar{G}$ is a locally soluble group whose abelian subgroups have bounded rank. By a result of Merzljakov (see p. 89, vol. 2 of [3] for a reference), $\bar{G}$ has finite rank.

Now by Lemma 10.39 of [3], $\bar{G}$ is periodic-by-soluble and hence periodic. Clearly, therefore, $\bar{G}$ is a locally nilpotent $p$-group and hence a Černikov group (Corollary 1 to Theorem 6.36 of [3]). Since $\bar{G}$ is residually finite, it must be finite, contradicting the choice of the subgroups $P_i$. This completes the proof of the theorem.

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Rebut el 14 d'Octubre de 1993