EXISTENCE DOMAINS FOR HOLOMORPHIC $L^p$ FUNCTIONS

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Abstract

If $\Omega$ is a domain of holomorphy in $\mathbb{C}^n$, having a compact topological closure into another domain of holomorphy $U \subset \mathbb{C}^n$ such that $(\Omega, U)$ is a Runge pair, we construct a function $F$ holomorphic in $\Omega$ which is singular at every boundary point of $\Omega$ and such that $F$ is in $L^p(\Omega)$, for any $p \in (0, +\infty)$.

1. Statement of the problem

The following notation and terminology will be used without further explanation. The open polydisc in $\mathbb{C}^n$ with center $\alpha$ and radius $r$ is denoted by $\Delta^n(\alpha; r)$; if $n = 1$, then we use the notation $\Delta(\alpha; r)$. For every open set $D$ in $\mathbb{C}^n$, $\theta(D)$ denotes the space of all holomorphic functions in $D$. If $K$ is a compact subset of $D$, we define the $\theta(D)$-hull $\bar{K}_D$ of $K$ by $\bar{K}_D := \{z \in D; |f(z)| \leq \sup_{w \in K} |f(w)|$, for all $f \in \theta(D)\}$. For $p \in (0, +\infty]$, we set $\theta L^p(D) := \theta(D) \cap L^p(D)$. Obviously, $\theta L^\infty(D)$ equals the algebra $H^\infty(D)$ of bounded holomorphic functions in $D$. If $D$ carries a function $F \in \theta L^p(D)$, which cannot be holomorphically extended across the boundary of $D$, then $D$ is said to be an existence domain for $\theta L^p$ or of type $\theta L^p$.

Asking for the conditions under which a bounded domain of holomorphy $\Omega$ is of type $\theta L^p$, we recall the following result: If $\Omega \subset \subset \mathbb{C}^n$ is a domain of holomorphy with $C^\infty$ boundary and $(\alpha_\nu \in \Omega; \nu \in \mathbb{N})$ is a sequence such that $(\lim_{\nu \to \infty} \alpha_\nu) \in \partial \Omega$, then there exists a function $F \in \theta L^2(\Omega)$ satisfying $\lim_{\nu \to \infty} |F(\alpha_\nu)| = +\infty$ ([4]). The question we are interested is the following: Is any bounded domain of holomorphy in $\mathbb{C}^n$ existence domain for $\theta L^p$, for every $p \in (0, +\infty)$? In [1] Catlin showed that any smoothly bounded domain of holomorphy in $\mathbb{C}^n$ is of type $\theta L^\infty$, and consequently of type $\theta L^p$, for every $p \in (0, +\infty)$. However in [6], Sibony showed that there is a bounded Runge complete Hartogs domain of holomorphy $\Omega_S \subset \Delta^2(0; 1)$ ($\Omega_S \neq \Delta^2(0; 1)$) such that all bounded
holomorphic functions in \( \Omega_S \) extend holomorphically to the open unit bidisc, that is \( \Omega_S \) is not of type \( \Theta L_\infty \).

The concern of this note is to give an answer to the above question. Our approach illustrates a partial extension of Catlin's improvement. More precisely, we shall prove that any domain of holomorphy \( \Omega \subset \subset \mathbb{C}^n \), having a compact topological closure into another domain of holomorphy \( U \) such that \( (\Omega, U) \) is a Runge pair, is of type \( \Theta L^p \) for any \( p \in (0, +\infty) \).

### 2. Unbounded holomorphic functions in Runge domains

Let \( \Omega \subset \subset U \) be domains of holomorphy in \( \mathbb{C}^n \). Assume that \( \Omega \) is a bounded Runge domain relative to \( U \).

Let \( \{z_m; m \in \mathbb{N}\} \) be a dense sequence in \( \Omega \), such that every point of the sequence is counted infinitely many times. Let \( r_m \) be the largest number with \( \Delta^n(z_m; r_m) \subset \Omega \). \( \Omega \) can be exhausted by compact sets \( E_j \), so that \( E_j \subset E_{j+1} \). Letting \( K_1 := E_1 \), we find a point \( w_1 \in \Delta^n(z_1; r_1) - \mathring{K}_{1,U} \). Obviously, there exists a \( j_1 > 1 \), with \( w_1 \in E_{j_1} \). Put \( K_2 := E_{j_1} \). Now, there is a point \( w_2 \in \Delta^n(z_2; r_2) - \mathring{K}_{2,U} \). If we set \( K_3 := E_{j_2} (, j_2 > j_1) \), then \( w_2 \in E_{j_2} \). Continuing like this, we find an exhaustive sequence \( \{K_m; m \in \mathbb{N}\} \) of compact subsets of \( \Omega \) and a sequence \( \{w_m; m \in \mathbb{N}\} \) of points of \( \Omega \), with the following properties:

- \( w_m \in K_{m+1} - \mathring{K}_{m,U} (, m \in \mathbb{N}) \),
- whenever \( w \in \partial \Omega \cap \Delta^n(\xi; \rho) \) for a polydisc \( \Delta^n(\xi; \rho) \) and \( V \) is a connected component of \( \Omega \cap \Delta^n(\xi; \rho) \) clustering at \( w \), there exists a subsequence of \( \{w_m; m \in \mathbb{N}\} \) converging to \( w \) in \( V \).

To each \( w_m \) there corresponds a holomorphic function \( f_m \in \Theta(U) \), such that \( |f_m(w_m)| > \sup_{z \in K_m}|f_m(z)| = 1 \). If we let \( 0 < \varepsilon_m < |f_m(w_m)| - 1 \), then \( |f_m(z)| < |f_m(w_m)| - \varepsilon_m | \), whenever \( z \in K_m \). Hence, for suitably chosen numbers \( \nu_m > 0 \), the series

\[
F(z) = \sum_{m=1}^{\infty} \frac{|f_m(z)|^{\nu_m}}{|f_m(w_m)|^{\nu_m} - \varepsilon_m^{\nu_m}}
\]

converges absolutely and compactly on \( \Omega \) and \( |F(w_m)| > m \), for any \( m \). It follows that whenever \( w \in \partial \Omega \), \( \Delta^n(\xi; \rho) \) is a polydisc containing \( w \) and \( V \) is a connected component of \( \Omega \cap \Delta^n(\xi; \rho) \) clustering at \( w \), \( F \) is unbounded in \( V \). So, \( F \) is a function holomorphic on \( \Omega \), which is singular (unbounded) at every boundary point of \( \Omega \) ([3]).
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3. Runge domains of type $\theta L^p$

Let the notations and assumptions be as in Section 2. The principal purpose of this paragraph is to announce the following:

**Theorem 1.** Let $\Omega \subset \subset U$ be domains of holomorphy in $\mathbb{C}^n$ such that $(\Omega, U)$ is a Runge pair. Then, $F \in \theta L^p(\Omega)$, for any $p \in (0, +\infty)$.

**Proof:** The evaluation of more useful choice of $\nu_m$ is our first aim. Let $\delta > 2$. For each $m \in \mathbb{N}$, choose $\nu_m$ so that $|f_m(w_m) - \varepsilon_m| \nu_m \geq \delta^m$. It is easily seen that the power series

$$h(\zeta) = \sum_{m=1}^{\infty} [f_m(w_m) - \varepsilon_m]^{-\nu_m} \cdot \zeta^m$$

converges into the disc $\Delta(0; \delta)$. Define a linear functional

$$\Lambda_h : \mathbb{P}(\mathbb{C}) \to \mathbb{C}; \quad x^m \to \Lambda_h(x^m) := [f_m(w_m) - \varepsilon_m]^{-\nu_m},$$

where $\mathbb{P}(\mathbb{C})$ is the vector space of complex polynomials in $\mathbb{C}$. In order to prove the theorem two lemmas play crucial role:

**Lemma 1.** ([2]) The functional $\Lambda_h$ is continuous and there is a continuous extension of $\Lambda_h$ into $\theta(\Delta(0; \delta_{-1}))$. Further, for each $\zeta \in \Delta(0; \delta)$ there holds $\Lambda_h((1 - x\zeta)^{-1}) = h(\zeta)$ ($x \in \Delta(0; \delta_{-1})$).

**Proof of Lemma 1:** Let $r < \delta$. If $p(x)$ is a polynomial in $x \in \mathbb{C}$, then by Cauchy's integral formula we have

$$|\Lambda_h(p)| \leq M(r) \cdot \sup_{|x| \leq r} |h(x)| \cdot \sup_{|x| \leq r^{-1}} |p(x)|,$$

where the constant $M(r)$ depends only on $r$. Hence, by density, there is a continuous extension of $\Lambda_h$ on $\theta(\Delta(0; \delta_{-1}))$. If now $\zeta \in \Delta(0; \delta)$ and if $\zeta$ is fixed, then the number $\Lambda_h((1 - x\zeta)^{-1})$ is well defined ($\Lambda_h$ acts on the variable $x \in \Delta(0; \delta_{-1})$ and $\zeta$ is regarded as a parameter). By the continuity of $\Lambda_h$, we obtain $\Lambda_h((1 - x\zeta)^{-1}) = h(\zeta)$. ■

The next lemma is a consequence of Lemma 1, but is much more useful since the choice of the functional $\Lambda_h$ is eliminated.
Lemma 2. If \( z \in \Omega \), then there holds

\[
|F(z)| \leq f \left( \frac{1}{\tau} \right) \cdot \sum_{m=1}^{\infty} \frac{|f_m(z)|^m}{\tau^m},
\]

for any \( \tau \in (2, \delta) \) and where the constant \( \mathcal{L} \left( \frac{1}{\tau} \right) \) depends only on \( \tau \) but is independent of \( z \).

Proof of Lemma 2: Assuming that \( z \in \Omega \), \( x \in \Delta(0; \delta^{-1}) \) and \( \tau \in (2, \delta) \), we have by Cauchy’s integral formula and by Lemma 1:

\[
|F(z)| = \left| \sum_{m=1}^{\infty} \Lambda_h(x^m) \cdot [f_m(z)]^m \right| = \\
= \left| \frac{1}{2\pi i} \cdot \int_{|\zeta|=\frac{1}{\tau}} \Lambda_h(1/(\zeta - x)) \cdot \left( \sum_{m=1}^{\infty} [f_m(z)]^m \cdot \zeta^m \right) d\zeta \right| \leq \\
\leq L \left( \frac{1}{\tau} \right) \cdot \left( \sup_{|\zeta|=\frac{1}{\tau}} |\Lambda_h(1/(\zeta - x))| \right) \left( \sup_{|\zeta|=\tau} \left\{ \sum_{m=1}^{\infty} \frac{|[f_m(z)]^m|}{\zeta^m} \right\} \right),
\]

that is

\[
|F(z)| \leq \mathcal{L} \left( \frac{1}{\tau} \right) \cdot \sum_{m=1}^{\infty} \frac{|f_m(z)|^m}{\tau^m}. \quad \blacksquare
\]

End of Proof of Theorem 1: Let \( 0 < p \leq +\infty \). By Lemma 2 and by Fatou’s Theorem, it is enough to show that

\[
\sup \left\{ \int_{\Omega} \left( \sum_{m=1}^{v} \frac{|[f_m(z)]^m|}{\tau^m} \right)^p d\lambda(z); \quad v \in \mathbb{N} \right\} < +\infty,
\]

for some \( \tau \in (2, \delta) \). (\( d\lambda(\cdot) \) is the Lebesgue measure in \( \mathbb{C}^n \)).

Suppose \( \tau \in (2, \delta) \). For any \( v \in \mathbb{N} \), choose a positive number \( \frac{2k_v - 1}{k_v} \) (, \( k_v \in \mathbb{N} \)), such that

\[
\int_{\Omega} \left( \sum_{m=1}^{v} \frac{|[f_m(z)]^m|}{\tau^m} \right)^p d\lambda(z) \leq \left( \frac{2k_v - 1}{k_v} \right)^p.
\]

This choice permits us to obtain the following inequalities

\[
\int_{\Omega} \left( \sum_{m=1}^{v} \frac{|[f_m(z)]^m|}{2} \right)^p d\lambda(z) \leq \left( \frac{2k_v - 1}{2k_v} \right)^p \leq 1,
\]
for any \( v \in \mathbb{N} \). Therefore,

\[
\int_{\Omega} \left( \sum_{m=1}^{v} \left| \frac{|f_m(z)|^v}{r^m} \right|^p \right) \, d\lambda(z) \leq 1,
\]

for any \( v \in \mathbb{N} \) and consequently,

\[
\int_{\Omega} \left( \sum_{m=1}^{v} \left| \frac{|f_m(z)|^v}{r^m} \right|^p \right) \, d\lambda(z) \leq 1,
\]

for any \( v \in \mathbb{N} \), which completes the proof. \( \blacksquare \)

We are now in position to formulate the main result of this note, which is an immediate consequence of Theorem 1:

**Theorem 2.** Let \( \Omega \subset U \) be domains of holomorphy in \( \mathbb{C}^n \). Assume that \( \Omega \) is a bounded Runge domain relative to \( U \). Then, \( \Omega \) is an existence domain for \( \Theta L^p \), for any \( p \in (0, +\infty) \). In particular, any bounded Runge domain of holomorphy is of type \( \Theta L^p \), for any \( p \in (0, +\infty) \).

We finally turn to the question whether Sibony's example \( \Omega_S \) in [6] is an existence domain of \( L^p \) holomorphic functions. The answer is a direct consequence of Theorem 2: Since Sibony's example is a bounded Runge domain of holomorphy, it is an existence domain for \( \Theta L^p \), for any \( p \in (0, +\infty) \).

**References**


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