ON BOTT-PERIODIC ALGEBRAIC $K$-THEORY

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Abstract

Let $K_*(A; \mathbb{Z}/\ell^n)$ denote the mod-$\ell^n$ algebraic $K$-theory of a $\mathbb{Z}[1/\ell]$-algebra $A$. Snaith [14], [15], [16], has studied Bott-periodic algebraic $K$-theory $K_*(A; \mathbb{Z}/\ell^n)[1/\beta_n]$, a localized version of $K_*(A; \mathbb{Z}/\ell^n)$ obtained by inverting a Bott element $\beta_n$. For $\ell$ an odd prime, Snaith has given a description of $K_*(A; \mathbb{Z}/\ell^n)[1/\beta_n]$ using Adams maps between Moore spectra. These constructions are interesting, in particular, for their connections with the Lichtenbaum-Quillen conjecture [16].

In this paper we obtain a description of $K_*(A; \mathbb{Z}/2^n)[1/\beta_n]$, $n \geq 2$, for an algebra $A$ with $1/2 \in A$ and $\sqrt{-1} \in A$. We approach this problem using low dimensional computations of the stable homotopy groups of $B\mathbb{Z}/4$, and transfer arguments to show that a power of the mod-4 Bott element is induced by an Adams map.

Introduction. Let $\ell$ be a prime number and let $A$ be a commutative ring containing $1/\ell$. For $\ell$ odd or $\ell = 2$ and $n \geq 2$ there exists (see [16]) a “Bott element” $\beta_n \in K_*(A; \mathbb{Z}/\ell^n)$ and Snaith [16] forms $K_*(A; \mathbb{Z}/\ell^n)[1/\beta_n]$, the localization of $K_*(A; \mathbb{Z}/\ell^n)$ obtained by inverting the Bott element. Thus, $K_*(A; \mathbb{Z}/\ell^n)[1/\beta_n]$ is the direct limit of iterated multiplications by $\beta_n$ using the $K$-theory product. The Lichtenbaum-Quillen conjecture [18] has been reformulated as the assertion that for a suitable regular ring $A$, the canonical localization map

$$
\rho : K_*(A; \mathbb{Z}/\ell^n) \to K_i(A; \mathbb{Z}/\ell^n)[1/\beta_n]
$$

is injective for large $i$ (see [17]).

For $\ell$ an odd prime Snaith [16] has obtained a description of $K_*(A; \mathbb{Z}/\ell^n)[1/\beta_n]$ in terms of Adams maps. Recall [2] that an Adams map between mod-$\ell^n$ Moore spectra is a map $A_n : \Sigma^d P(\ell^n) \to P(\ell^n)$ which induces isomorphisms on topological $K$-theory. In [16] Snaith
proved that \( K_i(A; \mathbb{Z}/\ell^n)[1/\beta_n] \) is the direct limit of iterated precompositions of suspensions of mod-\( \ell^n \) Adams maps, i.e.
\[
(1.2) \quad K_i(A; \mathbb{Z}/\ell^n)[1/\beta_n] \cong \lim_{\to} \left( K_i(A; \mathbb{Z}/\ell^n) \xrightarrow{(\Sigma^d A_n)^*} K_{i+d}(A; \mathbb{Z}/\ell^n) \right)
\]
then using (1.2) he obtains a factorization of the localization map (1.1) through the Hurewicz map \( K_i(A; \mathbb{Z}/\ell^n) \to h_i(BGLA^+; \mathbb{Z}/\ell^n) \), where \( h_i \) denotes mod-\( \ell^n \) topological complex K-theory \( KU_i(-, \mathbb{Z}/\ell^n) \) or a suitable defined J-theory \( J_i(-, \mathbb{Z}/\ell^n) \).

In this paper we extended these results to the case \( \ell = 2, n \geq 2 \) assuming that the ring \( A \) contains a fourth root of unity.

**Bott elements and Adams maps.** Let \( A = \mathbb{Z}[1/2, \zeta_4] \) be the ring obtained by adjoining \( \zeta_4 = \sqrt{-1} \) to the ring of integers localized away from 2. Snaith [16, Section 3] considers the following construction:

The inclusion \( \mathbb{Z}/4 \to GL_1A \) given by sending a generator of \( \mathbb{Z}/4 \) to \( \zeta_4 \), and inclusion of permutation matrices induces morphisms
\[
(2.1) \quad \Sigma_r \int \mathbb{Z}/4 \to \Sigma_r \int GL_1A \to GL_rA
\]
where \( \Sigma_r \int G \) is the *wreath product* of the symmetric group \( \Sigma_r \) with the group \( G \). These morphisms induce an infinite loop space map
\[
(2.2) \quad d_1 : (B \sum_{\infty} \int \mathbb{Z}/4)^+ \to BGLA^+
\]

**2.3.** The *Bott element* \( \beta \in K_2(A; \mathbb{Z}/4) \) is defined as the image under the map induced by \( d_1 \) of a generator of order 4,
\[
b \in \pi_2 \left( (B \sum_{\infty} \int \mathbb{Z}/4)^+ ; \mathbb{Z}/4 \right) \approx \pi_2^\beta (B\mathbb{Z}/4; \mathbb{Z}/4)
\]
obtained by stabilization of the generator of \( \pi_2(B\mathbb{Z}/4; \mathbb{Z}/4) \approx \mathbb{Z}/4 \) which maps under the Bockstein morphism \( \pi_2(B\mathbb{Z}/4; \mathbb{Z}/4) \to \pi_1(B\mathbb{Z}/4) \) to the generator of \( \mathbb{Z}/4 \).

The element \( \beta_1 = \beta^4 \in K_8(A; \mathbb{Z}/4) \) is also called a Bott element. The following characterization of the Bott elements is the mod-4 analogue of [6].

**Lemma 2.4.** For \( n > 1 \), the \( 4^{n-1} \) cup power of \( \beta_1 \) in \( K_*(A; \mathbb{Z}/4) \) is the reduction mod-4 of an element \( \beta_n \) in \( K_{8 \cdot 4^{n-1}}(A; \mathbb{Z}/4^n) \).

**Proof:** As in [6, Lemma 2], the proof is by induction on \( n > 1 \) using the fact [12] that the differentials in the mod-4 stable homotopy Bockstein spectral sequence are derivations, and the definition \( K_*(A; \mathbb{Z}/4^n) = \pi_*(K_A; \mathbb{Z}/4^n) \).
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i) $n = 2$: Since $\beta_1^4 = \beta^{4^4}$ and since $d_1 : K_*(A; \mathbb{Z}/4) \to K_{*-1}(A; \mathbb{Z}/4)$ is a derivation, then

$$d_1(\beta_1^4) = d_1(\beta^{4^4}) = 4^2 \cdot 4 \beta_1^4 d_1(\beta) = 0$$

since $K_*(A; \mathbb{Z}/4)$ is a $\mathbb{Z}/4$-module. Thus, $\beta_1^4$ is a $d_1$-cycle and so it survives to $E_2$.

Now, by the description of $E_r$, see [4, Section 5], $\beta_1^4 = \beta^{16} \in E_2$ is represented by the class of a map $P^{8.4}(4) \to KA$ such that there exists a factorization:

$$\begin{array}{ccc}
P^{8.4}(4) & \xrightarrow{i} & KA \\
& \searrow \beta_2 & \\
P^{8.4}(4^2) & \nearrow & \\
\end{array}$$

i.e., $\beta_1^4 = \beta_2 \circ i = i\#(\beta_2)$, i.e. $\beta_1^4$ is the mod-4 reduction of $\beta_2 \in K_{8.4}(A; \mathbb{Z}/4^2) = \pi_{8.4}(KA; \mathbb{Z}/4^2)$ (the mod-4 reduction map is $r_\# = i\#(-)$).

ii) Now, for $n > 2$, inductively we see that the cup powers:

$$\beta_1 = \beta^4, \quad \beta_1^4 = \beta^{4^2}, \ldots, \beta_1^{4^n-1} = \beta^{4^n}$$

are $d_r$-cycles for $1 \leq r \leq n - 1$, and so in particular

$$\beta_1^{4^n-1} = \beta^{4^n} \in E_n^{8.4^n-1}$$

can be represented as

$$P^{8.4^n-1}(4) \xrightarrow{\beta_1^{4^n-1}} KA \xrightarrow{\beta_n} P^{8.4^n-1}(4^n)$$

by the description of $E_n^{8.4^n-1}$ [4, Section 5]. Thus, for $\beta_n \in K_{8.4^n-1}(A; \mathbb{Z}/4^n)$ we have: $\beta_1^{4^n-1} = i\#(\beta_n))$ i.e., the mod-4 reduction of $\beta_n$ is $\beta_1^{4^n-1}$.

2.5. Definition. Let $X$ be an algebra over $A = \mathbb{Z}[1/2, \zeta_4]$, define [16]:

$$K_i(X; \mathbb{Z}/4^n)[1/\beta_n] := \lim (K_i(X; \mathbb{Z}/4^n) \xrightarrow{\beta_n} K_{i+d}(X; \mathbb{Z}/4^n) \to \cdots)$$

where $d = \deg(\beta_n) = 8 \cdot 4^{n-1}$. Notice that $K_*(X; \mathbb{Z}/4^n)[1/\beta_n]$ is periodic of period $d$, i.e., $K_i(X; \mathbb{Z}/4^n)[1/\beta_n] \approx K_{i+d}(X; \mathbb{Z}/4^n)[1/\beta_n]$. These
groups are called the mod-4\(^n\) Bott-periodic algebraic K-theory groups of \(X\).

In this section we prove that an appropriate choice for the 2-primary Bott elements is given by an Adams map between mod-4\(^n\) Moore spectra. First, we recall some properties of these 2-primary Adams maps, see [5] for details on these maps.

2.6. 2-Primary Adams maps. Let \(u \in KU_0(S^0) = \pi_2(BU) = \mathbb{Z}\) be a (Bott) generator. Then, \(\bar{u} = u^{2r} \in KU_{2r}(S^0) = \pi_{2r}(BU) = \mathbb{Z}\) is independent of the choice of \(u\). This \(\bar{u}\) will be called a Bott class.

Now, consider real K-theory \(KO_*\) and the complexification map
\[c : KO_*(-) \to KU_*(-)\]

Choose a generator \(v \in KO_{8r}(S^0) = \pi_{8r}(BO) = \mathbb{Z}\) such that \(c(v) = \bar{u}\) is the Bott class in \(KU_{8r}(S^0)\).

Now, let \(n \geq 1\) and consider the Moore spectrum \(P(2^n) = S^0 \cup 2^n\ e^1\).

Using this spectrum to introduce coefficients in \(KO\)-theory, write:
\[KO_*(X; \mathbb{Z}/2^n) = [P(2^n), X \wedge KO_*]\]
for \(X\) any spectrum and \(KO\) the spectrum representing \(KO_*\)-theory (see [1, Part 3]).

Now, for \(v \in KO_{8r}(S^0) = [S^0; KO]_{8r}\) we have that:
\[
\bar{v} = 1 \wedge v \in [P(2^n) \wedge S^0, P(2^n) \wedge KO]_{8r} = [P(2^n), P(2^n) \wedge KO]_{8r} = KO_{8r}(P(2^n); \mathbb{Z}/2^n)
\]
is a generator, called the mod-2\(^n\) Bott class.

Now, let \(h_{KO} : \pi_*(X; \mathbb{Z}/2^n) \to KO_*(X; \mathbb{Z}/2^n)\) be the \(KO\)-Hurewicz map defined as follows: If \([f] \in \pi_*(X; \mathbb{Z}/2^n) = [P(2^n), X]_*\) is represented by a map \(f : P(2^n) \to X\) of degree \(r\), then \(f\) induces
\[f_* : KO_*(P(2^n); \mathbb{Z}/2^n) \to KO_*(X; \mathbb{Z}/2^n)\]
and we define:
\[h_{KO}(f) = f_*(e) \in KO_*(X; \mathbb{Z}/2^n)\] where \(e \in KO_0(P(2^n); \mathbb{Z}/2^n) \approx \mathbb{Z}/2^n\) is a generator.

2.7. Definition. A map \(A_n : \Sigma^n P(2^n) \to P(2^n)\), representing an element \(A_n \in \pi_d^n(P(2^n); \mathbb{Z}/2^n)\), is called an Adams map iff
\[h_{KO}(A_n) = \bar{v} = \text{Bott class} \in KO_d(P(2^n); \mathbb{Z}/2^n)\]

2.8. Remark. Observe that if \(A_n\) is an Adams map then \((A_n)_*\) is a \(KO_*\)-isomorphism. M. C. Crabb and K. Knapp, [5], have proved the following:
2.9. Proposition. [5, 3.2]. Let $d = d(n) = \max(8, 2^{n-1}), n \geq 1$. Then, there exists a family of maps $A_n \in \pi_8^s(P(2^n); \mathbb{Z}/2^n) = [\Sigma^d P(2^n), P(2^n)]_0$ such that:

(1) $A_n$ is an Adams map.

(2) In the homotopy commutative diagram:

\[ \begin{array}{ccc}
\Sigma^d P(2^n) & \xrightarrow{A_n} & P(2^n) \\
\downarrow i & & \downarrow j \\
\Sigma^d S^0 & \xrightarrow{\alpha_n} & \Sigma S^0
\end{array} \]

where $i$ and $j$ are inclusion into the bottom cell and projection onto the top cell respectively, and $\alpha_n$ is defined by the composite $\alpha_n \simeq j \circ A_n \circ i$, we have that $[\alpha_n] \in \pi_{8-1}^s(S^0)$ generates the 2-primary component of the image of $J$ if $n \geq 4$ (a subgroup of order $2^n$ if $1 \leq n < 4$).

2.10. Remark. Recall, see e.g. [19], that $2\pi_8^s(S^0) = \mathbb{Z}/16$ generated by the Hopf map $\sigma$. From the coefficient sequence

\[ \cdots \xrightarrow{2\pi_8^s(S^0) \rightarrow 2\pi_8^s(S^0; \mathbb{Z}/4) \rightarrow 2\pi_8^s(S^0; \mathbb{Z}/2) \rightarrow \cdots} \]

and since $2\pi_8^s(S^0) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, it follows that $r$ is injective.

Also, from the Universal Coefficient Sequence [11] we see that

\[ 2\pi_8^s(S^0; \mathbb{Z}/4) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \]

Let $\hat{\sigma}$ a generator of order 4 in $2\pi_8^s(S^0; \mathbb{Z}/4)$. Observe that $\partial(\hat{\sigma}) = 4\sigma$.

Consider now the following diagram for $q$ sufficiently large:

\[ \begin{array}{ccc}
P^{q+8}(4) & \xrightarrow{A_1} & P^q(4) \\
\downarrow i & & \downarrow j \\
S^{q+7} & \xrightarrow{\alpha_1} & S^0
\end{array} \]

where $\alpha_1$ is a map that represents $\hat{\sigma}$, $i$ and $j$ are inclusion into the bottom cell and projection onto the top cell respectively, and $\alpha_1 \simeq a_1 \circ i = i^#(a_1)$ represents $\partial(\hat{\sigma}) = 4\sigma$ (recall that the Bockstein morphism $\partial$ is given by $i^#$). Then, $\alpha_1 \in \pi_8^s(S^0)$ has order 4 since $\alpha_1 = \partial(\hat{\sigma}) = 4\sigma$ and $\sigma$ is of order 16.

The Toda bracket $\{4, \alpha_1, 4\} = 0$ by [19, 3.7].
The Adams $e$-invariant [2, Section 3] of $\alpha_1$ is: $e(\alpha_1) = 1/4 (\text{mod } 1)$ since $\alpha_1 = \partial(\hat{\sigma}) = 4\sigma$ and $e(\sigma) = 1/16 (\text{mod } 1)$ by [1].

It follows, from [2, 12.5] that there exists a map $A_1$ making the previous diagram homotopy commutative, and moreover $A_1$ is an Adams map.

2.11. Transfer maps. Let $H \subseteq G$ be finite groups, and let $n = [G : H]$ be the index of $H$ in $G$. As usual, let $\Sigma_r$ denote the $r$-th symmetric group for $1 \leq r \leq \infty$. The natural morphisms:

$$G \longrightarrow \Sigma_n \int H \longrightarrow \Sigma_\infty \int H$$

induce, upon applying the classifying space functor $B(-)$ and the plus construction $(-)^+$ (Section 1.1.), maps:

$$BG \longrightarrow B(\Sigma_n \int H) \longrightarrow (B \Sigma_\infty \int H)^+ \simeq Q_0(BH_+)$$

where the equivalence is that of [7]. The natural extension of the map $BG \longrightarrow Q_0(BH_+)$ to $Q_0(BG_+)$ is called the (stable) transfer map, and we will denote it by:

$$t : Q_0(BG_+) \longrightarrow Q_0(BH_+)$$

2.12. Theorem. Let $b \in \pi_2^s(BZ/4; Z/4) = Z/4 \oplus Z/2$ be the generator of order 4. Let $t : Q_0(BZ/4)_+ \longrightarrow Q_0(S^0)$ be the transfer map associated to the inclusion $1 \hookrightarrow Z/4$. Consider $b^4 \in \pi_8^s(BZ/4; Z/4)$ and let $\hat{\sigma} \in \pi_8^s(S^0; Z/4)$ be as in (2.10). Then

$$t_\# : \pi_8^s(BZ/4; Z/4) \longrightarrow \pi_8^s(S^0; Z/4)$$

sends $b^4$ to $\hat{\sigma}$.

Proof: The transfer map $t_\#$ can be factored as:

$$t_\# : \pi_8^s(BZ/4; Z/4) \overset{t_2}{\longrightarrow} \pi_8^s(RP^\infty; Z/4) \overset{t_1}{\longrightarrow} \pi_8^s(S^0; Z/4)$$

where $RP^\infty = BZ/2$, $t_1$ is the transfer map associated to $1 \hookrightarrow Z/2$ and $t_2$ is the transfer associated to $Z/2 \hookrightarrow Z/4$.

Consider now the following commutative diagram:

$$\begin{array}{ccc}
\pi_8^s(BZ/4; Z/4) & \overset{t_2}{\longrightarrow} & \pi_8^s(RP^\infty; Z/4) & \overset{t_1}{\longrightarrow} & \pi_8^s(S^0; Z/4) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
\pi_7^s(BZ/4) & \overset{f_2}{\longrightarrow} & \pi_7^s(RP^\infty) & \overset{f_1}{\longrightarrow} & \pi_7^s(S^0)
\end{array}$$
where \( f_i, i = 1, 2 \) are the morphisms induced by the group inclusions, \( t_i \) are the corresponding transfer maps, and \( \partial \) the Bockstein morphisms.

We know that \( \partial(\hat{\sigma}) = 4 \cdot \sigma \).

Similarly, if \( \hat{a} \) is a generator of order 4 of \( \pi^*_7(RP^\infty; \mathbb{Z}/4) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), then: \( \partial(\hat{a}) = 4 \cdot a \).

Also, if \( \hat{b} \) is a generator of order 8 of \( \pi^*_7(B\mathbb{Z}/4) = \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), then: \( \partial(b^4) = 2 \cdot \hat{b} \).

By Kahn-Priddy [8], see also [7, Remark 4, p. 26], \( t_1 \) is split surjective on the 2-primary components. Thus: \( t_1(a) = \sigma \in 2\pi^*_7(S^0) \), and so, by the commutativity of the right-hand side square of the diagram above we have: \( t_1(\hat{a}) = \hat{\sigma} \).

Now, observe that \( f_2 \cdot t_2 \) = multiplication by 2 on \( \pi^*_7(B\mathbb{Z}/4) \) so that \( t_2(\hat{b}) = 2 \cdot a \in \pi^*_7(RP^\infty) \). Therefore \( t_2 \cdot \partial(b^4) = 4 \cdot a = \partial(\hat{a}) \). Hence \( t_2(b^4) = \hat{a} \). \( \blacksquare \)

2.13.

i) Recall that for \( A = \mathbb{Z}[1/2, \zeta_4] \) we defined (2.3)

\[
\beta = (d_1)_\#(b) \in K_2(A; \mathbb{Z}/4)
\]

where \( d_1 : (B \bigoplus_{n=1}^{\infty} \mathbb{Z}/4)^+ \rightarrow BGLA^+ \).

ii) We also defined (2.3)

\[
\beta_1 = \beta^4 = ((d_1)_\#(b))^4 = (d_1)_\#(b^4) \in K_8(A; \mathbb{Z}/4)
\]

iii) Now, in order to have a similar description for the higher Bott elements \( \beta_n \in K_*(A; \mathbb{Z}/4^n) \) of (2.4) for \( n > 1 \), we proceed as in [16, Section 3], but first we point out, as communicated to me by V. Snaith in a corrigendum of his paper [16], that diagram (3.7) of [16] should be replaced by the following commutative (up to an inner automorphism) diagram, where in the case considered by Snaith we replace 4 by and odd prime \( \ell \):

\[
\begin{array}{ccc}
\Sigma_n \int \mathbb{Z}/4 & \xrightarrow{d_1 \times d_2 \eta} & GL_n \mathbb{Z}[1/2, \zeta_4] \times GL_n \mathbb{Z}[1/2, \zeta_4] \\
\downarrow \quad t & & \downarrow \oplus \\
\Sigma_{4n} & \xrightarrow{d_2} & GL_{4n} \mathbb{Z}[1/2, \zeta_4] \quad \leftarrow \quad GL_{2n} \mathbb{Z}[1/2, \zeta_4]
\end{array}
\]
where \( s : GL_r \to GL_m, m \geq r \), is the stabilization map,
\[
d_1 : \Sigma r \int \mathbb{Z}/4 \to \Sigma r \int GL_1 \mathbb{Z}[1/2, \zeta_4] \to GL_r \mathbb{Z}[1/2, \zeta_4]
\]
is induced by the inclusion \( \mathbb{Z}/4 \cong \mu_4 \to GL_1 \mathbb{Z}[1/2, \zeta_4] \),
\[
d_2 : \Sigma m \to GL_m \mathbb{Z} \to GL_m \mathbb{Z}[1/2, \zeta_4]
\]
is induced by inclusion of permutation matrices,
\[
\eta : \Sigma n \int \mathbb{Z}/4 \to \Sigma n
\]
is induced by the morphism \( \mathbb{Z}/4 \to 1 \),
\[
t : \Sigma m \int \mathbb{Z}/4 \to \Sigma 4m
\]
is the transfer morphism induced by sending a generator of \( \mathbb{Z}/4 \) to the cycle \((1, 2, 3, 4) \in \Sigma 4\).

Snaith's proof of the commutativity (up to an inner automorphism) of this diagram runs as follows:

The right hand route corresponds to the module
\[
(\mathbb{Z}[1/4]^n \otimes_{\mathbb{Z}[1/4]} \mathbb{Z}[1/4]^3) \oplus (\mathbb{Z}[1/4]^n)
\]
where \( \Sigma n \int \mathbb{Z}/4 \) acts on the second summand by the permutation representation \( \eta \), and on the first summand by the tensor product action on the first factor of \( \eta \) with the translation action of \( \mathbb{Z}/4 \) on \( \mathbb{Z}[\zeta_4][1/4] \) as \( \mathbb{Z}[1/4] \)-module. As in the proof of Lemma (3.8) of [16], this is conjugate to the tensor product action of \( \eta \otimes \iota \) where
\[
\iota : \mathbb{Z}/4 \to \Sigma 4
\]
is the natural inclusion.

iv) From this (modified) diagram, applying the classifying space functor, the plus construction and taking \( n \to \infty \), we obtain a commutative diagram, which replaces Corollary (3.10) of [16],
\[
\pi_*(B\Sigma_{\infty} \int Z/4^+ ; Z/4) \xrightarrow{(d_1)^\# \times (d_2^\eta)^\#} K_*(Z[1/2, \zeta_4] ; Z/4) \times K_*(Z[1/2, \zeta_4] ; Z/4)
\]
\[
\pi_*(B\Sigma_{\infty}^+ ; Z/4) \xrightarrow{(d_2)^\#} K_*(Z[1/2, \zeta_4] ; Z/4)
\]
and since \((B \sum_\infty ^1 Z/4)^+ \simeq Q_0(BZ/4_+)\) and \(b \in \pi_2(BZ/4_+; Z/4)\) originates in \(\pi_2(BZ/4; Z/4)\) then \(\eta_\#(b) = 0\) and hence \(\eta_\#(b^4) = 0\) also.

Therefore, we have the formula:

\[
(d_2)_\#t_\#(b^4) = (d_1)_\#(b^4),
\]

which is essentially (3.12) in [16] and is used to derive Lemma (3.13) of [16] and its consequences.

I thank professor Snaith for communicating the above results to me.

v) Now, using the formula in (iv) we have:

\[
\beta_1 = (d_1)_\#(b^4) = (d_2)_\#t_\#(b^4) = (d_2)_\#(\hat{\sigma})
\]

by (2.12) where \(\hat{\sigma} = j \circ A_1\), with \(A_1\) an Adams map (2.10). vi) Now, \(d_2 : B \sum_\infty ^+ \to BGLZ[1/2, \zeta_4]^+\) is the base-point component of the 0-th spaces of the unit

\[
D : S^0 \to KZ[1/2, \zeta_4]
\]

of the algebraic \(K\)-theory ring spectrum of \(A = Z[1/2, \zeta_4]\). Therefore,

\[
\beta_1 = (d_2)_\#(\hat{\sigma}) = D_\#(\hat{\sigma})
\]

2.14. Now, to have the desired description for the higher Bott elements \(\beta_n \in K_*(A; Z/4^n)\) of (2.4) for \(n > 1\), we proceed as in [16, Section 3] as follows: We want \(\beta_n \in D_\#(\pi_8^s 4n-1(S^0; Z/4^n))\) where

\[
D_\# : \pi_8^s(S^0; Z/4^n) \to K_*(A; Z/4^n).
\]

By induction on \(n\) suppose \(\beta_n \in D_\#(\pi_8^s 4n-1(S^0; Z/4^n))\) and consider \(\beta_{n+1} \in K_{8*4n}(A; Z/4^{n+1})\).

Let \(r_\# : \pi_*(-; Z/4^{n+1}) \to \pi_*(-; Z/4^n)\) be the reduction map.

Let \(x_n \in \pi_{8*4n-1}(S^0; Z/4^n)\) be such that \(D_\#(x_n) = \beta_n\), and consider \(x_{n+1} \in \pi_{8*4n}(S^0; Z/4^n)\). Since the differentials in the homotopy Bockstein spectral sequence are derivations [12] then \(\partial_n(x_{4n}) = 0\) since \(\pi_8^s(S^0; Z/4)\) is a \(Z/4\) module. Thus, there exists \(x_{n+1} \in \pi_{8*4n}(S^0; Z/4^{n+1})\) such that \(r_\#(x_{n+1}) = x_{n+1}\). Now, since \(D_\#\) is a ring map we have \(D_\#(x_{4n}) = \beta_{n+1}\).

Therefore, by naturality we have:

\[
\begin{array}{c}
x_{n+1} \\
D_\# \\
D_\#(x_{n+1}) \end{array} \quad \xrightarrow{r_\#} \quad \begin{array}{c}
x_{4n} \\
D_\# \\
\beta_{n+1}\end{array}
\]
i.e., \( D_\#(x_{n+1}) \) is an element of \( K_{8.4^n}(A; \mathbb{Z}/4^{n+1}) \) that reduces mod-4 to \( \beta_n^4 \).

Therefore, we may choose \( \beta_{n+1} = D_\#(x_{n+1}) \) since this element reduces to \( \beta_n^4 \) which itself reduces to \( (\beta_1^{4n-1})^4 = \beta_1^{4n} \) by (2.4).

2.15. Remark. Analogously to [16, Section 3], we can see that for \( n \geq 1 \), a suitable choice for \( x_n \in \pi_i^s(S^0; \mathbb{Z}/4^n) \) is given by an Adams map, i.e. by \( a_n = j \circ A_n \) where \( j \) and \( A_n \) are maps in the diagram:

\[
\begin{array}{ccc}
P(4^n) & \xrightarrow{\beta_n^i} & \mathcal{P}(4^n) \\
\downarrow & & \downarrow \\
\mathcal{S}(4^n) & \xrightarrow{\alpha_n} & \mathcal{S}(4^n)
\end{array}
\]

where \( d_n = \max(8, 4^{n-1}) = \deg(A_n) \), and \( A_n \) an Adams map.

2.16. Now, let \( X \) be a commutative \( A \)-algebra, \( A = \mathbb{Z}[1/2, \zeta_4] \). Then \( KX \) is a \( KA \)-module. We denote this action by

\[ \mu : KA \wedge KX \to KX. \]

Let \([g] \in K_i(X; \mathbb{Z}/4^n) = \pi_i(KX; \mathbb{Z}/4^n)\) be represented by a map of spectra \( g : P(4^n) \to KX \) of degree \( i \). Consider a representative \( \beta_n : P(4^n) \to KA \) of the Bott element \( \beta_n \in K_{8.4^n-1}(A; \mathbb{Z}/4^n) \) of (2.4).

We have a commutative diagram of spectra:

\[
\begin{array}{ccccccc}
P(4^n) & \xrightarrow{\chi} & P(4^n) \wedge P(4^n) & \xrightarrow{\beta_n^i g} & KA \wedge KX & \xrightarrow{\mu} & KX \\
\downarrow & & \downarrow \beta_n^i g & & \downarrow \beta_n^i g & & \downarrow \\
A' & \xrightarrow{A_n^i} & P(4^n) \wedge P(4^n) & \xrightarrow{a_n^i g} & \mathcal{S}^0 \wedge KX & \xrightarrow{1 \wedge g} & \mathcal{S}^0 \wedge P(4^n) \\
\downarrow & & \downarrow j \wedge g & & \downarrow j \wedge g & & \Downarrow \\
P(4^n) & \xrightarrow{\sim} & \mathcal{S}^0 \wedge P(4^n)
\end{array}
\]

where the composite of the top row represents the product

\[ \beta_n \cdot [g] \in K_{i+d}(X; \mathbb{Z}/4^n), \]
$S^0$ is the sphere spectrum, $\chi$ is the copairing of Moore spectra of $[4]$, $\mu$ is the multiplication induced by the action of $A$ on $X$, $A_n$ and $j$ are the maps of spectra of (2.15) and $a_n \simeq j \cdot A_n$ in (2.15), and $D$ is the unit of $KA$.

It follows that $A'_n$ is also an Adams map between Moore spectra.

From the commutativity of this diagram it follows that:

$$\beta_n \cdot [g] = [g \cdot A'_n] = A''_n[g] \in K_{i+d}(X; \mathbb{Z}/4^n)$$

i.e., multiplication by $\beta_n$ is precomposition with an Adams map $A'_n$.

From this remark, we obtain the analogue of Snaith's theorem [16, 3.22]:

**2.17. Theorem.** Let $X$ be a commutative $\mathbb{Z}[1/2, \zeta]$-algebra. Suppose that there exists a map of Moore spaces $A_n : P_{q+d}(4^n) \to P^q(4^n)$ with $d = 8 \cdot 4^{n-1}$, such that its stable homotopy class is $A'_n : P(4^n) \to P(4^n)$ an Adams map of Moore spectra as in (2.11). Suppose $i \geq q$. Then:

$$K_i(\mathbb{Z}/4^n)[1/\beta_n] \approx \lim_{k} (K_{i+kd}(X; \mathbb{Z}/4^n) \xrightarrow{(\Sigma^{i+kd-q}A_n)^*} K_{i+(k+1)d}(X; \mathbb{Z}/4^n))$$

**Proof:** First, recall that there exist Adams maps

$$A_n : P_{q+d}(4^n) \to P^q(4^n)$$

for $d = \max(8, 2^{2n-1})$ and $q$ large enough.

Now, by choosing appropriate compositions of suspensions of these Adams maps we get maps

$$A'_n : P_{q+8 \cdot 4^{n-1}}(4^n) \to P^q(4^n)$$

that still induce isomorphisms in $K$-theory, i.e. they are Adams maps.

Now, by the remark (2.16)

$$\beta_n \cdot [g] = A''_n[g] = [g \cdot A'_n]$$

and since the isomorphisms

$$K_i(X; \mathbb{Z}/4^n) = [P^i(4^n), BGLX^+] \approx [\Sigma^i P(4^n), KX]$$

are such that the following diagram commutes
\[ [P_i(4^n), \text{BGL}X^+] \xrightarrow{\approx} [\Sigma^i P(4^n), KX] \]

\[ (\Sigma^i A_n)^* \xrightarrow{\approx} (A'_n)^* \]

\[ [P_i(4^n), \text{BGL}X^+] \xrightarrow{\approx} [\Sigma^{i+d} P(4^n), KX] \]

provided \( i \geq q \), since we are assuming that the stable homotopy class of the map \( A_n \) is \( A'_n \). Therefore the result follows.

References

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