NORMAL BASES FOR THE SPACE
OF CONTINUOUS FUNCTIONS
DEFINED ON A SUBSET OF $\mathbb{Z}_p$

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Abstract

Let $K$ be a non-archimedean valued field which contains $\mathbb{Q}_p$ and suppose that $K$ is complete for the valuation $|\cdot|$, which extends the $p$-adic valuation. $V_q$ is the closure of the set \{aq^n | n = 0, 1, 2, \ldots \} where $a$ and $q$ are two units of $\mathbb{Z}_p$, $q$ not a root of unity. $C(V_q \to K)$ is the Banach space of continuous functions from $V_q$ to $K$, equipped with the supremum norm. Our aim is to find normal bases $(r_n(x))$ for $C(V_q \to K)$, where $r_n(x)$ does not have to be a polynomial.

1. Introduction

The main aim of this paper is to find normal bases $(r_n(x))$ for the space of continuous functions on $V_q$, where $r_n(x)$ does not have to be a polynomial.

Therefore we start by recalling some definitions and some previous results.

Let $E$ be a non-archimedean Banach space over a non-archimedean valued field $L$.

Let $f_1, f_2, \ldots$ be a finite or infinite sequence of elements of $E$. We say that this sequence is orthogonal if $\|\alpha_1 f_1 + \cdots + \alpha_k f_k\| = \max\{\|\alpha_i f_i\| : i = 1, \ldots, k\}$ for all $k$ in $\mathbb{N}$ (or for all $k$ that do not exceed the length of the sequence) and for all $\alpha_1, \ldots, \alpha_k$ in $L$. If the sequence is infinite, it follows that $\left\|\sum_{i=1}^{\infty} \alpha_i f_i\right\| = \max\{\|\alpha_i f_i\| : i = 1, 2, \ldots\}$ for all $\alpha_1, \alpha_2, \ldots$ in $L$ for which $\lim_{i \to \infty} \alpha_i f_i = 0$. An orthogonal sequence $f_1, f_2, \ldots$ is called orthonormal if $\|f_i\| = 1$ for all $i$.

This leads us to the following definition:
If $E$ is a non-archimedean Banach space over a non-archimedean valued field $L$, then a family $(f_i)$ of elements of $E$ is a (ortho)normal basis of $E$ if the family $(f_i)$ is orthonormal and also a basis.

An equivalent formulation is (see [1, Propositions 50.4 and 50.6])

If $E$ is a non-archimedean Banach space over a non-archimedean valued field $L$, then a family $(f_i)$ of elements of $E$ is a (ortho)normal basis of $E$ if each element $x$ of $E$ has a unique representation $x = \sum_i x_i f_i$ where $x_i \in L$ and $x_i \to 0$ if $i \to \infty$, and if the norm of $x$ is the supremum of the norms of the $x_i$.

Let $\mathbb{Z}_p$ be the ring of $p$-adic integers, $\mathbb{Q}_p$ the field of $p$-adic numbers, and $K$ is a non-archimedean valued field, $K$ containing $\mathbb{Q}_p$, and we suppose that $K$ is complete for the valuation $| \cdot |$, which extends the $p$-adic valuation. Let $a$ and $q$ be two units of $\mathbb{Z}_p$, $q$ not a root of unity. We define $V_q$ to be the closure of the set $\{aq^n | n = 0, 1, 2, \ldots \}$. The set $V_q$ has been described in [3]. Let $C(V_q \to K)$ (resp. $C(\mathbb{Z}_p \to K)$) be the Banach space of continuous functions from $V_q$ to $K$ (resp. $\mathbb{Z}_p$ to $K$) equipped with the supremum norm. $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{N}_0$ is the set of natural numbers without zero.

We introduce the following:

If $x$, is an element of $\mathbb{Q}_p$, $x$ can be written in the following way:

$$x = \sum_{j=-\infty}^{+\infty} a_j p^j$$

where $a_{-i}$ is zero for $i$ sufficiently large ($i \in \mathbb{N}$) (see [1, section 3 and section 4]). This is called the Hensel development of the $p$-adic integer $x$. We then define the $p$-adic entire part $[x]_p$ of $x$ by

$$[x]_p = \sum_{j=-\infty}^{-1} a_j p^j$$

and we put $x_n = p^n [p^{-n} x]_p = \sum_{j=-\infty}^{-1} a_j p^j$ ($n \in \mathbb{N}$).

We write $m \prec x$, if $m$ is one of the numbers $x_0, x_1, \ldots \ldots$. We then say that "$m$ is an initial part of $x$" or "$x$ starts with $m$" (see [1, section 62]).

If $n$ belongs to $\mathbb{N}_0$, $n = \sum_{j=0}^{s-1} a_j p^j$ where $a_s \neq 0$, then we put $n_- = \sum_{j=0}^{s-1} a_j p^j$. We remark that $n_- \prec n$.

In [1, Theorem 62.2], we find the following result which is due to van der Put:

**Theorem.**

The functions $g_0, g_1, \ldots$ defined by

$$g_n(x) = 1 \quad \text{if } n \prec x,$$

$$= 0 \quad \text{otherwise},$$
form a normal basis for $C(\mathbb{Z}_p \to K)$. If $f$ is an element of $C(\mathbb{Z}_p \to K)$, then $f$ can be written as a uniformly convergent series $f(x) = \sum_{k=0}^{\infty} \gamma_k g_k(x)$ where $\gamma_0 = f(0)$ and $\gamma_n = f(n) - f(n_-)$ if $n \in \mathbb{N}_0$.

We now survey the content of this paper:

In Theorem 1 of section 2, our aim is to find a basis $(e_n(x))$ analogous to van der Put's basis, but with the space $C(\mathbb{Z}_p \to K)$ replaced by $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$, then there exist elements $a_k$ of $K$ such that $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ where the series on the right-hand-side is uniformly convergent. We are able to give an expression for the coefficients $a_k$.

In Theorem 2 of section 3, we prove the following result:

Define $r_n(x) = \sum_{j=0}^{n} c_{n;j} e_j(x)$, $c_{n;j} \in K$, $c_{n;n} \neq 0$ ($(e_n(x))$ as in Theorem 1 below).

Then $(r_n(x))$ forms a normal basis for $C(V_q \to K)$ if and only if for all $n \|r_n\| = 1$ and $|c_{n;n}| = 1$.

In Theorem 3 of section 3, we give an extension of Theorem 2:

Let $(r_n(x))$ be such a sequence which forms a normal basis for $C(V_q \to K)$, and let $(s_n(x))$ be a sequence such that $s_n(x) = \sum_{j=0}^{n} d_{n;j} r_j(x)$, $d_{n;j} \in K$, $d_{n;n} \neq 0$. Then $(s_n(x))$ forms a normal basis for $C(V_q \to K) \iff \|s_n\| = 1$, $|d_{n,n}| = 1 \iff |d_{n;j}| \leq 1$, $|d_{n;n}| = 1$.

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2. Proof of the first theorem

We start with some lemmas and some definitions.

Definition.

If $b$ and $x$ are elements of $\mathbb{Z}_p$, $b \equiv 1 \pmod{p}$, then we put $b^x = \lim_{n \to x} b^n$.

The mapping: $\mathbb{Z}_p \to \mathbb{Z}_p : x \to b^x$ is continuous.

For more details, we refer the reader to [1, section 32].

Notation.

Take $m \geq 1$, $m$ the smallest integer such that $q^m \equiv 1 \pmod{p}$.

We have $1 \leq m \leq p - 1$ and $(q^m)^x$ is defined for all $x$ in $\mathbb{Z}_p$. 
Definition.

Let \( k \) be a natural number prime to \( p \). We denote by \( \mathbb{Z}_p(k) \) the projective limit \( \mathbb{Z}_p(k) = \lim_{j} (\mathbb{Z}/kp^j\mathbb{Z}) \cong (\mathbb{Z}/k\mathbb{Z}) \times \mathbb{Z}_p \).

In the following lemma we use the fact that \( \mathbb{Z}_p(m) = (\mathbb{Z}/m\mathbb{Z}) \times \mathbb{Z}_p \) to denote an element of \( \mathbb{Z}_p(m) \) as \( x = (r, y) \). Also, if \( n \in \mathbb{N} \), \( n = r + mk \) \((0 \leq r < m)\) then the map \( n \to (r, k) \) imbeds \( \mathbb{N} \) in \( \mathbb{Z}_p(m) \).

Lemma 1.

The mapping \( \varphi : \mathbb{Z}_p(m) \to V_q : (r, y) \to aq^r(q^m)^y \) is a homeomorphism.

The proof of this lemma can be found in [2, p. 377].

Corollary.

If \( q \equiv 1 \pmod{p} \), i.e. \( m = 1 \), then the mapping: \( \mathbb{Z}_p \to V_q : x \to aq^x \) is a homeomorphism.

Let \( \beta \) be an element of \( \mathbb{Z}_p\setminus\{0\} \). We want to know the valuation of the \( p \)-adic integer \((q^m)^\beta - 1\). Therefore we need two lemmas:

The following lemmas (2 and 3) are proved in [3]:

Lemma 2.

Let \( \alpha \) be an element of \( \mathbb{Z}_p \), \( \alpha \equiv 1 \pmod{p^r} \), \( \alpha \not\equiv 1 \pmod{p^{r+1}} \), \( r \geq 1 \).

If \( (p, r) \neq (2, 1), \beta \in \mathbb{Z}_p\setminus\{0\} \) then \( \alpha^\beta \equiv 1 \pmod{p^{r+\text{ord}_p\beta}}, \alpha^\beta \not\equiv 1 \pmod{p^{r+1+\text{ord}_p\beta}} \).

Corollary.

Let \( q^m \equiv 1 \pmod{p^{k_0}}, q^m \not\equiv 1 \pmod{p^{k_0+1}} \). If \( (p, k_0) \neq (2, 1), \beta \in \mathbb{Z}_p\setminus\{0\} \) then \( (q^m)^\beta \equiv 1 \pmod{p^{k_0+\text{ord}_p\beta}}, (q^m)^\beta \not\equiv 1 \pmod{p^{k_0+1+\text{ord}_p\beta}} \).

In Lemma 2 we excluded the case where \( (p, r) = (2, 1) \). This case will be handled in the following lemma:

Lemma 3.

Let \( \alpha \) be an element of \( \mathbb{Z}_2 \), \( \alpha \equiv 3 \pmod{4} \). Define a natural number \( n \) by \( \alpha = 1 + 2 + 2^2\varepsilon, \varepsilon = \varepsilon_0 + \varepsilon_1 + \varepsilon_2 2^2 + \ldots, \varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{n-1} = 1, \varepsilon_n = 0 \).

If \( \beta \in \mathbb{Z}_2\setminus\{0\}, \text{ord}_2 \beta = 0 \) then \( \alpha^\beta \equiv 1 \pmod{2}, \alpha^\beta \not\equiv 1 \pmod{4} \).
If $\beta \in \mathbb{Z}_2 \setminus \{0\}$, $\text{ord}_2 \beta = k \geq 1$ then $\alpha^\beta \equiv 1 \pmod{2^{n+2+\text{ord}_2 \beta}}$, $\alpha^\beta \not\equiv 1 \pmod{2^{n+3+\text{ord}_2 \beta}}$.

**Corollary.**

If $q \equiv 3 \pmod{4}$, we define a natural number $N$ by $q = 1 + 2 + 2^2 \varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$.

If $\beta \in \mathbb{Z}_2 \setminus \{0\}$, $\text{ord}_2 \beta = 0$ then $q^\beta \equiv 1 \pmod{2}$, $q^\beta \not\equiv 1 \pmod{4}$.

If $\beta \in \mathbb{Z}_2 \setminus \{0\}$, $\text{ord}_2 \beta = k \geq 1$ then $q^\beta \equiv 1 \pmod{2^{N+2+\text{ord}_2 \beta}}$, $q^\beta \not\equiv 1 \pmod{2^{N+3+\text{ord}_2 \beta}}$.

We remark that is possible to write each $x$ and element of $V_q$ in the following way: $x = aq^{i_x}(q^m)^{\alpha_x}$ where $i_x$ is a natural number, $0 \leq i_x < m$, and where $\alpha_x$ is an element of $\mathbb{Z}_p$. This immediately follows from Lemma 1. This leads us to the following definition:

**Definition.**

We now define a sequence of functions $e_k$ in the following way. Write $k(\in \mathbb{N})$ in the form $k = i + mj$, $0 \leq i < m$, $(i, j \in \mathbb{N})$. The functions $e_k$ are defined by

$$e_k(x) = e_{i+mj}(x) = 1 \text{ if } x = aq^{i_x}(q^m)^{\alpha_x} \text{ where } i_x = i, j < \alpha_x.$$  

$$= 0 \text{ otherwise.}$$

Let us use the notation $B(b, r^-)$ for the 'open' disc with radius $r$ and with center $b$, i.e. $B(b, r^-) = \{x \in V_q | |x-b| < r\}$, and $B(b, r)$ for the 'closed' disc with radius $r$ and with center $b$, i.e. $B(b, r) = \{x \in V_q | |x-b| \leq r\}$.

In the following lemmas we will show that the functions $e_k(x)$ are characteristic functions of discs. There exists a $k_0$ such that $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$. We distinguish two cases: $(p, k_0) \neq (2, 1)$ (Lemma 4), and $(p, k_0) = (2, 1)$ i.e. $q \equiv 3 \pmod{4}$ (Lemma 5). If we use the same notation in Lemmas 4 and 5 as in the definition, we have

**Lemma 4.**

Let $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$ and suppose $(p, k_0) \neq (2, 1)$.

If $0 \leq i < m$ then $e_i(x)$ is the characteristic function of the closed disc $B(aq^i, p^{-k_0})$, and if $0 \leq i < m$, $j \geq 1$ then $e_k(x) = e_{i+jm}(x)$ is the characteristic function of the open disc $B \left(aq^i(q^m)^j, \left(\frac{p^{-k_0}}{j}\right)^-\right)$. 
Proof:

Let \( j = \sum_{i=0}^{s} a_i p^i \) be the Hensel development of \( j \in \mathbb{N}_0 \), with \( a_s \) different from zero.

If we use the notation \( x = aq^i (q^m)^{a_x} (0 \leq i_x < m) \) for an element \( x \) of \( V_q \), we will show the following:

a) if \( 0 \leq i < m : |x - aq^i| \leq p^{-k_0} \) if and only if \( i_x = i \).

b) if \( 0 \leq i < m, j \geq 1 : |x - aq^i (q^m)^j| < \frac{p^{-k_0}}{j} \) if and only if \( i_x = i \).

We first prove a). If \( i_x = i \), then \( |x - aq^i| = |aq^i (q^m)^{a_x} - aq^i| = |(q^m)^{a_x} - 1| \leq p^{-k_0} \) by the corollary to Lemma 2.

If \( i_x \neq i \), then

\[
|x - aq^i| = |aq^i (q^m)^{a_x} - aq^i| = \max \{ |aq^i (q^m)^{a_x} - aq^i|, |aq^i - aq^i| \} = 1,
\]

since \( |aq^i (q^m)^{a_x} - aq^i| \leq p^{-k_0}, |aq^i - aq^i| = 1 \). This proves a).

Now we prove b).

Suppose \( i_x = i, j < \alpha_x \). Then \( |x - aq^i (q^m)^j| = |(q^m)^{a_x-j} - 1| \leq p^{-k_0-(s+1)} \) by the corollary following Lemma 2, since \( j \) is an initial part of \( \alpha_x \). Since \( j \) is strictly smaller than \( p^{(s+1)} \), we conclude that \( |x - aq^i (q^m)^j| < \frac{p^{-k_0}}{j} \).

For the converse, suppose \( |x - aq^i (q^m)^j| < \frac{p^{-k_0}}{j} \). Then we must have that \( i_x \) equals \( i \), since otherwise \( |x - aq^i (q^m)^j| = 1 \):

\[
|x - aq^i (q^m)^j| = |aq^i (q^m)^{a_x} - aq^i (q^m)^j| = \max \{ |aq^i (q^m)^{a_x} - aq^i|, |aq^i - aq^i|, |aq^i - aq^i (q^m)^j| \} = 1
\]

since \( |aq^i (q^m)^{a_x} - aq^i| \leq p^{-k_0}, |aq^i - aq^i (q^m)^j| \leq p^{-k_0} \) (corollary to Lemma 2) and \( |aq^i - aq^i| = 1 \) if \( i_x \) is different from \( i \).

So we have \( |(q^m)^{a_x-j} - 1| < \frac{p^{-k_0}}{j} \) and from this it follows that \( |(q^m)^{a_x-j} - 1| \leq p^{-k_0-(s+1)} \) since \( j \) is at least \( p^s \). This means that \( \text{ord}_p (\alpha_x - j) \) is at least \( s+1 \) (again by the corollary to Lemma 2) and so we conclude that \( j \) is an initial part of \( \alpha_x \). \( \blacksquare \)

Lemma 5.

If \( q \equiv 3 \pmod{4} \), with \( q = 1 + 2 + 2^2 \varepsilon \), where \( \varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots \), \( \varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{N-1} = 1, \varepsilon_N = 0 \), then \( e_0(x) \) is the characteristic
function of $V_q$, and $e_j(x)$ is the characteristic function of the open disc $B\left(aq^j, \left(2^{-(N+2)}\right)^{-j}\right)$ if $j \geq 1$.

Proof:
In this case $m$ equals one and we use the notation $x = aq^{\alpha_x}$ for an element $x$ of $V_q$.

It is clear that $e_0(x)$ is the characteristic function of $V_q$.

If $j$ is at least one, we prove: $|x - aq^j| < \frac{2^{-(N+2)}}{j}$ if and only if $j < \alpha_x$.

Suppose $j < \alpha_x$. Then $|x - aq^j| = |q^{\alpha_x-j} - 1| < 2^{-(N+2)-(s+1)}$ (corollary following Lemma 3), and since $j$ is strictly smaller than $2^{s+1}$, we conclude $|x - aq^j| < \frac{2^{-(N+2)}}{j}$.

For the converse, suppose $|x - aq^j| < \frac{2^{-(N+2)}}{j}$. Then $|q^{\alpha_x-j} - 1| < \frac{2^{-(N+2)}}{j}$ and so $|q^{\alpha_x-j} - 1| < 2^{-(N+2)-(s+1)}$ since $j$ is at least $2^s$. By the corollary to Lemma 3, we have that $\text{ord}_2(\alpha_x - j)$ is at least $s+1$ and so $j$ is an initial part of $\alpha_x$.

Corollary.
The functions $(e_k(x))$ are continuous functions on $V_q$.

In the following theorem we prove that the sequence $(e_k(x))$ forms a normal basis for $C(V_q \to K)$. This implies that if $f$ is an element of $C(V_q \to K)$, there exists elements $a_k$ of $K$ such that $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ where the right-hand-side is uniformly convergent. We are able to give an expression for the coefficients $a_k$. The proof of this theorem is analogous to the proof of Theorem 62.2 in [1].

Theorem 1.
The functions $(e_k(x))$ form a normal basis for $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$ then $f$ can be written as a uniformly convergent series $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ where

\[ a_k = f(aq^k) \quad \text{if } 0 \leq k < m \]
\[ a_k = a_{i+jm} = f(aq^i(q^m)^j) - f(aq^i(q^m)^{j-}) \quad \text{if } 0 \leq i < m, j > 0. \]

Proof:
Let $f$ be an element of $C(V_q \to K)$, and let $a_k$ be defined as $a_k = f(aq^k)$ if $0 \leq k < m$, $a_k = a_{i+jm} = f(aq^i(q^m)^j) - f(aq^i(q^m)^{j-})$ if $0 \leq i < m, j > 0$. 


We first prove that \(a_k\) tends to zero if \(k\) tends to infinity: for all \(\varepsilon > 0\), there exists a \(J\) such that \(k > J\) implies \(|a_k| \leq \varepsilon\). To prove this, we distinguish two cases:

i) Let \(q^m \equiv 1 \pmod{p^{k_0}}, q^m \not\equiv 1 \pmod{p^{k_0+1}}\), with \((p, k_0) \neq (2, 1)\).

Since the function \(f\) is continuous on \(V_q\), it is uniformly continuous on \(V_q\), and so there exist an \(S\), such that \(|x - y| \leq p^{-(k_0+S)}\) implies \(|f(x) - f(y)| < \varepsilon\). We then put \(J = p^S m\).

If \(k > J\), and \(k\) equals \(i + jm\) with \(0 \leq i < m\), then we have that \(j \geq p^S\) and so (corollary to Lemma 2) \(|aq^i(q^m)^j - aq^i(q^m)^j - 1| = |(q^m)^j - 1| \leq p^{-(k_0+S)}\) and this implies that \(|a_k| = |f(aq^i(q^m)^j) - f(aq^i(q^m)^j - 1)| < \varepsilon\).

ii) Let \(q \equiv 3 \pmod{4}, q = 1 + 2 + 2^2 \varepsilon, \varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots, \varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1, \varepsilon_N = 0\). We remark that \(m\) equals one in this case.

Since the function \(f\) is continuous on \(V_q\), it is uniformly continuous on \(V_q\), and so there exist an \(S\), such that \(|x - y| \leq 2^{-(N+2+S)}\) implies \(|f(x) - f(y)| < \varepsilon\). We then put \(J = 2^S\).

If \(k > J\), then (corollary to Lemma 3) \(|q^k - q^k - 1| = |q^{k-k-} - 1| \leq 2^{-(N+2+S)}\) and this implies that \(|a_k| = |f(q^k) - f(q^k - 1)| < \varepsilon\).

We conclude that \(a_k\) tends to zero if \(k\) tends to infinity.

If \(f\) is an element of \(C(V_q \rightarrow K)\), we introduce a function \(g(x)\) defined by \(g(x) = \sum_{k=0}^{\infty} a_k e_k(x)\) with \(a_k\) as in \((*)\). Since \(||a_k e_k|| \leq |a_k| \to 0\), the series on the right-hand-side converges uniformly, so the function \(g\) is continuous as a uniformly limit of continuous functions. We can prove that \(g(aq^k) = f(aq^k)\) if \(0 \leq k < m\) and that \(g(aq^i(q^m)^j) - g(aq^i(q^m)^j - 1) = f(aq^i(q^m)^j) - f(aq^i(q^m)^j - 1)\) for \(0 \leq i < m, j > 0\). Then we have \(g(aq^k) = f(aq^k)\) for all natural numbers \(k\) and by continuity, we conclude that \(f(x) = g(x)\).

So we have \(f(x) = \sum_{k=0}^{\infty} a_k e_k(x)\), with \(a_k\) as in \((*)\).

It is clear that \(||f|| \leq \max_{0 \leq k} \{|a_k|\}\), but we also have \(||f(aq^k)|| \leq ||f||\) and \(||(aq^i(q^m)^j) - f(aq^i(q^m)^j - 1)|| \leq ||f||\), so we conclude \(||f|| = \max_{0 \leq k} \{|a_k|\}\).

Finally we prove the uniqueness of the coefficients.

If \(f(x) = \sum_{k=0}^{\infty} a_k e_k(x) = \sum_{k=0}^{\infty} b_k e_k(x)\), then \(\sum_{k=0}^{\infty} (a_k - b_k) e_k(x) = 0\). So \(\max_{0 \leq k} \{|a_k - b_k|\} = 0\), from which it follows that \(a_k = b_k\) for all \(k\). This proves the theorem. ■
3. More bases for \( C(V_q \rightarrow K) \)

We can make more normal bases, using the basis \((e_k(x))\) of Theorem 1:

**Theorem 2.**

Let \((e_n(x))\) be as above, and define \(r_n(x) = \sum_{j=0}^{n} c_{n,j} e_j(x), \ c_{n,j} \in K, \ c_{n;n} \neq 0\). Then \((r_n(x))\) forms a normal basis for \(C(V_q \rightarrow K)\) if and only if \(\|r_n\| = 1\) and \(|c_{n;n}| = 1\) for all \(n\).

The proof of this theorem will not be given here, since it is analogous to the proof of Theorem 2 in [3].

**Remark.**

An analogous result can be found on the space \(C(\mathbb{Z}_p \rightarrow K)\), if we replace the sequence \((e_n(x))\) by the van der Put basis \((g_n(x))\) from the introduction.

We can extend Theorem 2 to the following:

**Theorem 3.**

Let \((r_n(x))\) be a sequence as found in Theorem 2, which forms a normal basis for \(C(V_q \rightarrow K)\), and let \((s_n(x))\) be a sequence such that \(s_n(x) = \sum_{j=0}^{n} d_{n,j} r_j(x), \ d_{n,j} \in K, \ d_{n;n} \neq 0\).

Then the following are equivalent:

i) \((s_n(x))\) forms a normal basis for \(C(V_q \rightarrow K)\).

ii) \(\|s_n\| = 1\), \(|d_{n;n}| = 1\).

iii) \(|d_{n;j}| \leq 1\), \(|d_{n;n}| = 1\).

**Proof:**

i) \(\Leftrightarrow\) ii) follows from Theorem 2, using the expression \(r_n(x) = \sum_{j=0}^{n} c_{n,j} e_j(x)\), and ii) \(\Leftrightarrow\) iii) follows from the fact that \((r_n(x))\) forms a normal basis. \(\blacksquare\)

**Examples.**

1) If a sequence \((r_n(x))\), as found in Theorem 2, forms a normal basis of \(C(V_q \rightarrow K)\), then so does \((s_n(x))\), where \(s_n(x) = r_0(x) + r_1(x) + \cdots + r_n(x)\): apply iii).

2) If we put for \(0 \leq i < m\),

\[
   r_i(x) = 1 \quad \text{if } x = a q^{i_x}(q^m)^{\alpha_x} \text{ where } i_x = i \\
   = 0 \quad \text{otherwise},
\]
and for $k \geq m$ we put

$$r_k(x) = r_{i+mj}(x) \quad (0 \leq i < m) = 1 \text{ if } x = aq^{i\alpha}(q^m)^\alpha,$$

where $i_x = i$, $j \neq \alpha_x$.

$$= 0 \text{ otherwise.}$$

then $(r_n(x))$ forms a normal basis for $C(V_q \to K)$. We can apply iii) since $r_i(x) = e_i(x)$ for $0 \leq i < m$, $r_k(x) = e_i(x) - e_k(x)$ for $k = i + mj$, $0 \leq i < m$, $j > 0$. If $f \in C(V_q \to K)$, then there exists a uniformly convergent expansion of the form $f(x) = \sum_{k=0}^{\infty} c_k r_k(x)$, where

$$c_k = c_{i+jm} = f(aq^i(q^m)^j) - f(aq^i(q^m)^j) \text{ if } 0 \leq i < m, j > 0, \text{ and}$$

$$c_i = f(aq^i) - \sum_{j=1}^{\infty} c_{i+jm} \text{ if } 0 \leq i < m.$$ 

References


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