Publicacions Matemàtiques, Vol **39** (1995), 5–21.

POLAR DECOMPOSITION IN RICKART C^*-ALGEBRAS

Dmitry Goldstein

Abstract

A new proof is obtained to the following fact: a Rickart C^*-algebra satisfies polar decomposition. Equivalently, matrix algebras over a Rickart C^*-algebra are also Rickart C^*-algebras.

Introduction.

In this paper we give new proof of the following result: all Rickart C^*-algebras satisfy polar decomposition. This fact was established in [**2**] by P. Ara and author by using a suitable factorization of the elements in the regular overring of a finite Rickart C^*-algebra.

New proof also uses the construction of the regular overring, but in a different way. In particular, we don't need the result of Goodearl, Lawrence and Handelman about algebras without one-dimensional representations.

The regular ring of measurable operators of a finite AW^*-algebra was constructed by S. K. Berberian in [**3**]. Later Saito modified Berberian's approach for general AW^*-algebras [**11**]. E. Christensen constructed and investigated a $*$-algebra of measurable operators, associated to MSC C^*-algebras [**5**].

Handelman found a regular extension $Q(T)$ for a finite Rickart C^*-algebra T, using a technique of the module-homomorphisms on the essential countably generated ideals (instead of Berberian's coordinated sequences) [**9**]. It was established ([**1**], [**10**]) that a finite Rickart C^*-algebra T satisfies polar decomposition iff the bounded elements of $Q(t)$ belong to T. Goodearl, Handelman and Lawrence have proved that T satisfies polar decomosition in the case where T has no one-dimensional representations (see [**8**]).

P. Ara in [**1**], using his special construction, proved that left and right projections of element in a Rickart C^*-algebra are equivalent and the polar decomposition problem in general Rickart C^*-algebras can be reduced to the finite case. In this work Ara also proved an equivalence of the following conditions for a Rickart C^*-algebra T:

(i) T satisfies polar decomposition;
(ii) The matrix algebras $M_n(T)$ over T are the Rickart C^*-algebras for all n;
(iii) The partial isometries of T are $\aleph_0$-addable.

Finally, by development of the methods of [**1**], [**8**], it was proved in [**2**] that conditions (i)-(iii) are always fulfiled in Rickart C^*-algebras.

In [**6**], [**7**] was constructed a $*$-algebra of measurable operators for a finite Rickart C^*-algebra and were proved some algebraic properties of this $*$-algebra. We continue to develope this approach in order to solve the polar decomposition problem.

1. Preliminaries.

A $*$-algebra A is *Rickart*, if for all $x \in A$ there exists a projection $e \in A$ such that $R(x) = \{a \in A | xa = 0\}$ is eA. Because of the involution, $L(x) = \{a \in A | ax = 0\} = Tf$ for some projection f. We shall write $e = RA(x)$, $f = LA(x)$, $1 - e = RP(x)$, $1 - f = LP(x)$ and $P(A)$ for the set of all projections of A.

The projections e and f are equivalent ($e \sim f$) in a $*$-algebra A if $e = uu^*$, $f = u^*u$ for some partial isometry $u \in A$. A is finite if $p \sim 1$ implies $p = 1$. A Rickart C^*-algebra is a C^*-algebra that is also a Rickart $*$-algebra. We recall some properties of the Rickart C^*-algebras.

Theorem 2.1. *A Rickart C^*-algebra satisfies the following properties:*

(i) *$P(T)$ is $\aleph_0$-complete lattice partially ordered by $p \geq q$ iff $pq = q$ (see* [**4**]*).*

If in addition T is finite then the lattice $P(T)$ is $\aleph_0$-continuous [**9**, Cor. 1.1].

(ii) *$LP(x) \sim RP(x)$ for all $x \in T$* [**1**, Th. 2.5].
(iii) *For given sequences (e_n) and (f_n) of ortogonal projections such that $e_n \sim f_n$ for all $n \in \mathbf{N}$, we have $\bigvee_n e_n \sim \bigvee_n f_n$* [**1**].

The partial isometries are $\aleph_0$-addable in a Rickart C^*-algebra T if for every sequence of partial isometries $\{w_n\}$ such that $\{w_n w_n^*\}$ and $\{w_n^* w_n\}$ are the sequences of ortogonal projections there exists a partial isometry w such that $ww_n^*w_n = w_n w_n^* w = w_n$.

2. Strongly dense domains.

Through this paper T denotes (if the opposite is not specified) a finite Rickart C^*-algebra.

A sequence of projections $(e_n) \subset P(T)$ is a strongly dense domain (SDD) in case $e_n \uparrow 1$. Let $e \in P(T)$, $x \in T$. We define $x^{-1}(e) = RA[(1-e)x]$.

Proposition 2.1. *Let (e_n) and (f_n) are SDD, $x_n \in T$ such that $m \leq n$ implies $x_n e_m = x_m e_m$. Then a sequence $(t_n = x_n^{-1}(f_n) \bigwedge e_n)$ is a SDD.*

Proof: Let $d_n = x_n^{-1}(f_n)$. If $m \leq n$ then

$$(1-e_n)x_n t_m = (1-e_n)(1-e_m)x_n e_m t_m = (1-e_n)(1-e_m)x_m t_m = 0,$$

so that $t_m \leq t_n$. Let $p \in P(T)$. We show that there exists a number k such that $t_k \bigwedge p \neq 0$. For that choose a number i such that $q = p \bigwedge e_i \neq 0$. If $x_i q = 0$, then $q \leq t_i$. Now let $x_i q \neq 0$. There exists $a \in T$ such that $h = x_i q a$ is non-zero projection [**4**, par. 8]. Observe that $x_i q = x_n q$ for all $n \geq i$ and $h' = f_k \bigwedge h \neq 0$ for sufficiently large k. For $k \geq i$ we have

$$(1-f_k)x_k q a h' = (1-f_k)x_i q a h' = (1-f_k)hh' = 0.$$

Therefore $g = LP(qah') \leq d_k$. In addition, $g \leq q \leq e_i \leq e_k$, hence $g \leq t_k$. Thus $p \bigwedge t_k \geq g \neq 0$. ■

Corollary 2.2. *If (e_n) and (f_n) are SDD, then $(e_n \bigwedge f_n)$ is also SDD.*

Proof: Put in Proposition 2.1 $x_n = 1$ for all n. ■

3. A ring of measurable operators.

An essentially measurable operator (EMO) is a pair of sequences (x_n, e_n) with $x_n \in T$, (e_n) an SDD, and such that $m \leq n$ implies $x_n e_m = x_m e_m$ and $x_n^* e_m = x_m^* e_m$. Two (EMO) (x_n, e_n) and (y_n, f_n) are equivalent, if there exists an SDD (g_n) such that $x_n g_n = y_n g_n$, $g_n x_n = g_n y_n$ for all $n \in \mathbf{N}$. Clearly that this relation is indeed equivalence relation (By Corollary 2.2). If (x_n, e_n) is (EMO), $[x_n, e_n]$ denotes its equivalence class. We call $[x_n, e_n]$ a measurable operator (MO) and denote by $S(T)$ the set of all (MO), and use the letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ for the elements of $S(T)$. Now we define the algebraic operations on $S(T)$. We put

$$[x_n, e_n] + [y_n, f_n] = \left[x_n + y_n, e_n \bigwedge f_n\right]$$

$$\lambda[x_n, e_n] = [\lambda x_n, e_n]$$

$$[x_n, e_n][y_n, f_n] = [x_n y_n, k_n],$$

$$[x_n, e_n] = [x_n^*, e_n],$$

where $k_n = f_n \bigwedge y_n^{-1}(e_n) \bigwedge e_n \bigwedge (x_n^*)^{-1}(f_n)$.

Summarizing,

Theorem 3.1. *The set $S(T)$ of all MO is a $*$-algebra. The mapping $x \mapsto [x, 1] (x \in T)$ is a $*$-isomorphism of T into Q, and $[1, 1]$ is a unity element for $S(T)$.*

We write $\overline{x} = [x, 1]$, for $x \in T$. The image of T in $S(T)$ is $\overline{T}$.

We recall the construction by Handelman of the $*$-regular ring associated to a finite Rickart C^*-algebra. Let A be a unital ring. A right (left) ideal $E \subseteq A$ is *essential* if E has nontrivial intersection with any nonzero right (left) ideal of A. We say that E is *essential countably generated* (ecg) right ideal if there exist a sequence $\{t_n\}_{n \in \mathbf{N}} \subseteq A$ such that $\sum t_i A$ is essential in A. Similarly, we define left ecg ideal. It was proved in [**9**] that every ecg ideal of a finite Rickart C^*-algebra is generated by SDD.

Let T be a finite Rickart C^*-algebra. Consider the following pairs of mappings $[f, E; f_1, E_1]$, where f is right T-module homomorphisms from essential countably generated right ideal E, f_1 is left T-module homomorphism from essential countably generated left ideal E_1, and they are balanced by the following condition: $e_1 f(e) = f_1(e_1) e$ for all $e \in E$ and all $e_1 \in E_1$. Two pairs $[f, E; f_1, E_1]$ and $[g, J; g_1, G_1]$ are equivalent if $f(x) = g(x)$ and $f_1(y) = g_1(y)$ for all $x \in E \bigcap J$ and all $y \in E_1 \bigcap J_1$. Let Q be the set of equivalence classes of just defined pairs. It was shown in [**9**] that Q is endowed with algebraic operations, and with respect to these operation Q becomes a $*$-regular algebra.

Define mapping from $S(T)$ to Q. If $[x_n, e_n]$ is MO then $E = \bigcup_{n=1}^{\infty} e_n T$ ($E_1 = \bigcup_{n=1}^{\infty} T e_n$) is an essential countably generated right(left) ideal in T correspondently. Define a right T-module homomorphism $f : f(e_n t) = x_n e_n t$, where $e_n t \in E$. Obviously, $f(e_n t x) = f(e_n t) x$ for all $x \in T$. Let $e_n t = e_m s$ $(m \leq n)$. Then

$$f(e_m s) = f(e_m) s = x_m e_m s = x_n e_m s = x_n e_n t = f(e_n t).$$

Thus this definition is correct. Similarly, we define a left T-module homomorphism $f_1 : E_1 \to T$, $f(t e_n) = t e_n x_n$. Now let $e \in E$, $e_1 \in E_1$, $e = e_m t$, $e_1 = t_1 e_n$. If $m \leq n$ then

$$\begin{aligned} e_1 f(e) &= t_1 e_n f(e_m t) = t_1 e_n x_m e_m t = t_1 e_n x_n e_m t \\ &= f_1(t_1 e_n) e_m t = f_1(e_1) e. \end{aligned}$$

By a similar argument $e_1 f(e) = f_1(e_1) e$. Therefore $[f, E, f_1, E_1] \in Q$. We shall denote just defined mapping by π. Then $\pi([x_n, e_n]) =$

$[f, E, f_1, E_1]$. Let $[x_n, e_n] = [x'_n, e'_n]$, $\pi([x'_n, e'_n]) = [f', E', f'_1, E'_1]$. Choose an SDD (p_n) such that $x_n p_n = x'_n p_n$, $p_n x_n = p_n x'_n$ for all $n \in \mathbf{N}$. Put $q_n = p_n \bigwedge e_n \bigwedge e'_n$. Note that $q_n \in E \bigcap E_1 \bigcap E' \bigcap E'_1$. We have $f(q_n) = x_n q_n = x_n p_n q_n = x'_n q_n = f(q_n)$. Thus $f = f'$ on $\bigcup\limits_{n=1}^{\infty} q_n T$. In the same way we obtain $f_1 = f'_1$ on $\bigcup\limits_{n=1}^{\infty} T q_n$.

Theorem 3.2. *The mapping π is a $*$-isomorphism from $S(T)$ onto Q.*

Proof: Let $[f, E, f_1, E_1] \in Q$, $E = \bigcup_{n=1}^{\infty} e_n T$, $E_1 = \bigcup_{n=1}^{\infty} T e_n$, (e_n) an SDD,

$$f(ex) = f(e)x,\ f_1(xe_1) = x f_1(e_1)$$

for all $e \in E$, $e_1 \in E_1$, $x \in T$. Put $f(e_n) = y_n$, $f_1(e_n) = z_n$. Obviously, $y_n e_n = y_n$, $e_n z_n = z_n$. Set

$$x_n = y_n + z_n - z_n e_n = y_n + z_n - e_n y_n$$

so that $x_n e_n = y_n$, $e_n x_n = z_n$ for all $n \in \mathbf{N}$. It is easy to see that $[x_n, e_n]$ is MO. Set $\pi(x_n, e_n]) = [g, E, g_1, E_1]$, where $g(e_n) = x_n e_n$, $g_1(e_n) = e_n x_n$. Then $g(e_n) = y_n = f(e_n)$, $g_1(e_n) = z_n = f_1(e_n)$, hence $[f, E, f_1, E_1] = [g, E, g_1, E_1]$. Thus π is surjective. Now we show that the mapping π preserves the algebraic operations. Let $[x_n, e_n]$, $[y_n, k_n] \in S(T)$. Put

$$\pi([x_n, e_n]) = [f, E, f_1, E_1],\ \pi([y_n, k_n]) = [g, J, g_1, J_1],$$

where

$$E = \bigcup_{n=1}^{\infty} e_n T,\ E_1 = \bigcup_{n=1}^{\infty} T e_n,$$
$$J = \bigcup_{n=1}^{\infty} k_n T,\ J_1 = \bigcup_{n=1}^{\infty} T k_n.$$

We have (see [**9**, Section 2])

$$[f, E, f_1, E_1] + [g, J, g_1, J_1] = \left[f + g, E \bigcap J, f_1 + g_1, E_1 \bigcap J_1\right],$$
$$[x_n, e_n] + [y_n, k_n] = \left[x_n + y_n, e_n \bigwedge k_n\right].$$

Let $p_n = e_n \bigwedge k_n$ and $\pi([x_n+y_n, e_n \bigwedge k_n]) = [r, L, r_1, L_1]$. We can regard that

$$L = \bigcup_{n=1}^{\infty} p_n T,\ L_1 = \bigcup_{n=1}^{\infty} T p_n,$$
$$r(p_n) = (x_n + y_n)p_n, r_1(p_n) = p_n(x_n + y_n).$$

Since $(e_n \bigwedge k_n)T = (e_n T) \bigcap (k_n T)$ (see [**9**]) it follows $L = J \bigcap E$, $L_1 = J_1 \bigcap E_1$. In addition

$$r(p_n) = (x_n+y_n)p_n = (f+g)(p_n),\ r_1(p_n) = p_n(x_n+y_n) = (f_1+g_1)(p_n).$$

Consequently

$$[r, L, r_1, L_1] = \left[f + g, E \bigcap J, f_1 + g_1, E_1 \bigcap J_1\right].$$

Further, $[x_n, e_n][y_n, k_n] = [x_n y_n, t_n]$, where t_n is a suitable SDD. On the other hand,

$$[f, E, f_1, E_1][g, J, g_1, J_1] = [fg, g^{-1}E, g_1 f_1, f_1^{-1} J_1].$$

We shall establish that $\bigcup_{n=1}^{\infty} t_n T$ is an essential subideal in $g^{-1}E$. By the definition, $t_n = h_n \bigwedge g_n$, where

$$h_n = e_n \bigwedge (x_n^*)^{-1}(k_n),\ g_n = k_n \bigwedge y_n^{-1}(e_n),\ g^{-1}E = \{x \in J : g(x) \in E\}.$$

We have $t_n \le g_n \le k_n$, therefore $t_n \in J$ for all $n \in \mathbf{N}$. It remains to prove that $g(t_n) \in E$ for all n. Really,

$$\begin{aligned} g(t_n) &= g\,(g_n)t_n = g(g_n k_n y_n^{-1}(e_n))t_n = g(k_n)y_n^{-1}(e_n)g_n t_n \\ &= y_n k_n y_n^{-1}(e_n) g_n t_n = y_n y_n^{-1}(e_n) k_n g_n t_n \\ &= e_n y_n y_n^{-1}(e_n) k_n g_n t_n = e_n y_n y_n^{-1}(e_n) t_n, \end{aligned}$$

therefore $g(t_n) \in E$. So $t_n \in g^{-1}E$ for all n.

Similarly, we obtain $\bigcup_{n=1}^{\infty} T t_n \subset f^{-1} J_1$. Thus

$$[fg, g^{-1}E, f_1 g_1, f_1^{-1} J_1] = \left[fg, \bigcup_{n=1}^{\infty} t_n T, g_1 f_1, \bigcup_{n=1}^{\infty} T t_n\right].$$

It implies

$$\pi([x_n, e_n][y_n, k_n]) = \pi([x_n, e_n])\pi([y_n, k_n]).$$

Obviously,

$$\pi(\lambda[x_n, e_n]) = \lambda\pi([x_n, e_n]).$$

It is sthrightforward to check that $\pi([x_n, e_n]^*) = \pi([x_n, e_n])^*$. ■

Corollary 3.3. *$S(T)$ is a Rickart $*$-algebra and $\aleph_0$-continuous ring.*

Proof: It follows from Theorem 3.2 and [**9**, Th. 2.1]. ■

4. Some algebraic properties of $S(T)$. Cayley transform.

Lemma 4.1. *If $\mathbf{x} = [x_n, e_n] \in Q$ and the x_n are all invertible then $\mathbf{x}$ is invertible and $\mathbf{x}^{-1} = [x_n^{-1}, h_n]$ for a suitable SDD (h_n).*

Proof: Set $f_n = LP(x_n e_n)$. We show that (f_n) is a SDD. If $m \le n$ then $f_n(x_m e_m) = f_n x_n e_m = f_n x_n e_n e_m = x_n e_n e_m = x_m e_m$, $f_m \le f_n$. Since x_n is invertible then $RP(x_n e_n) = e_n$. We have $f_n \sim e_n$ [**1**, Th. 2.5]. As $p \sim q$ implies $1 - p \sim 1 - q$ for the projections p and q in a finite Rickart C^*-algebra, so $e_{n+1} - e_n \sim f_{n+1} - f_n$. Then by $\aleph_0$-additivity,

$$1 = [\sup_n (e_{n+1} - e_n)] \bigvee e_1 \sim \left[\sup_n (f_{n+1} - f_n) \bigvee f_1\right] = f.$$

Set $y_n = x_n^{-1}$. If $m \le n$, then $y_n f_m = y_m f_m$. Really,

$$y_n x_m e_m = y_n x_n e_m = e_m = y_m x_m e_m,$$

hence $(y_n - y_m) x_m e_m = 0$ and $(y_n - y_m) f_m = 0$. Similary on setting $g_n = LP(x_n^* e_n)$, we have that (y_n) is a SDD and $y_n^* g_m = y_m^* g_m$ when $m \le n$. Put $h_n = f_n \bigwedge g_n$, then $\mathbf{y} = [y_n, h_n]$ is MO, and $\mathbf{xy} = \mathbf{yx} = 1$. ■

Corollary 4.2. *For any $\mathbf{x} \in S(T)$ an element $1 + \mathbf{x}^*\mathbf{x}$ is invertible.*

Proof: It follows immediately from Lemma 4.1. ■

Lemma 4.3. *If $\mathbf{x} = \mathbf{x}^*$, one can write $\mathbf{x} = [x_n, e_n]$ with $x_n^* = x_n$.*

Proof: If $\mathbf{x} = [y_n, f_n]$, then $\mathbf{x} = 1/2(\mathbf{x}^* + \mathbf{x}) = [1/2(y_n^* + y_n), f_n]$. ■

Corollary 4.4. *If $\mathbf{x} = \mathbf{x}^*$, then $\mathbf{x} + i$ is invertible.*

Proof: Let $\mathbf{x} = [x_n, e_n]$, $x_n^* = x_n$; then $x + i = [x_n + i, e_n]$ and each $x_n + i$ is invertible. ■

Theorem 4.5. *The formulas*

$$\mathbf{u} = (\mathbf{x} - i)(\mathbf{x} + i)^{-1}, \ \mathbf{x} = i(1 + \mathbf{u})(1 - \mathbf{u})^{-1}$$

define mutually inverse one-one correspondences between the self-adjoint elements $\mathbf{x} \in Q$, and the unitary elements $\mathbf{u}$ for which $1 - \mathbf{u}$ is invertible.

Proof: It follows from Corollary 4.4. ■

We call this $\mathbf{u}$ the Cayley transform of $\mathbf{x}$.

Lemma 4.6. *Let* $\mathbf{x} = [x_n, e_n] \in S(T)$ *and* $x_n \longrightarrow x$ *in norm, then* $\mathbf{x} = \overline{x}$.

Proof: Evidently, $\|xe_n - x_n e_n\| = \|xe_n - x_k e_n\|$ for all $k \geq n$. Then $\|xe_n - x_n e_n\| \leq \|x - x_k\|$ for all $k \geq n$, $\|xe_n - x_n e_n\| = 0$, $xe_n = x_n e_n$. In just the same way, $e_n x = e_n x_n$. ■

Lemma 4.7. *Let* $\mathbf{x} = [x_n, e_n] \in S(T)$. *Then* $\mathbf{x}e_n = \overline{x}_n e_n$.

Proof: Obvious. ■

5. The bounded measurable operators.

Let T be a finite Rickart C^*-algebra, $Q = S(T)$ denotes a $*$-algebra of measurable operators of T.

An element $\mathbf{x} = [x_n, e_n] \in Q$ is bounded, if $\sup_n \|x_n\| \leq \infty$. Let B be a set of all bounded MO. It is clear that B is $*$-algebra. Since $P(Q) \subset B$ hence B is Rickart $*$-algebra. We define the mapping $\|\cdot\|_1 : B \ni \mathbf{x} \mapsto \|\mathbf{x}\|_1 = \inf \sup_n \{\|x_n\| | (x_n, e_n) \in \mathbf{x}\} \in \mathbf{R}$.

The bounded elements of $S(T)$ play a crucial role in the following discussion of the polar decomposition problem (or $\aleph_0$-addability of the partial isometries, see Introduction) in a finite Rickart C^*-algebra. It is easy to see that the partial isometries of B are $\aleph_0$-addable (Corollary 7.3). On the other hand, it is well known that the algebras B and $\overline{T}$ coincide if T is AW^*-algebra [**3**]. We shall prove a similar result for a general Rickart C^*-algebra.

Theorem 5.1. *The mapping* $\|\cdot\|_1$ *is a* C^**-norm.*

Proof: Let $\mathbf{x} = [x_n, e_n] \in B$. Clearly, $\|\mathbf{x}\|_1 \geq 0$. If $\|\mathbf{x}\|_1 = 0$ then for any $\varepsilon \geq 0$ there exists EMO $(x_n, e_n) \in \mathbf{x}$ such that $\sup_n \|x_n\| \leq \varepsilon$. Let $\mathbf{y} = [y_n, e_n] \in B$, $\sup_n \|y_n\| = \alpha$. We can choose $(x'_n, e'_n) \in \mathbf{x}$ with $\|x'_n\| \leq \varepsilon/\alpha$ for all $n \in \mathbf{N}$. Therefore $\|x_n y_n\| \leq \varepsilon$, $\|\mathbf{xy}\|_1 = 0$.

Now assume that there exists $\mathbf{x} \in B$ such that $\mathbf{x} \neq 0$ and $\|\mathbf{x}\|_1 = 0$. Choose number n such that $\mathbf{x}e_n \neq 0$. By Lemma 4.7 $\mathbf{x}e_n = \overline{x}_n e_n$. As it was shown above $\|\mathbf{x}e_n\|_1 = \|\overline{x}_n e_n\|_1 = 0$. Let $a = x_n e_n$. By the definition of the norm $\|\cdot\|_1$ we have $\|a\|_1 = \inf \sup_n \{\|a_n\| | (a_n, k_n) \in \overline{a}\}$. For any $(a_n, k_n) \in \overline{a}$ there exists an SDD (p_n) such that $ap_n = a_n p_n$. Note $\|ap_n\| = \|a_n p_n\| \leq \|a_n\|$. Choose b such that $ba = e$ is a non-zero projection [**4**, par. 8]. Since $P(T)$ is $\aleph_0$-continuous, there exists $k \in \mathbf{N}$ such that $q = e \bigwedge p_k \neq 0$.

Consequently

$$1 = \|q\| \leq \|ep_k\| = \|bap_k\| \leq \|ap_k\| \|b\|,$$

hence $\|ap_k\| \geq 1/\|b\|$. It follows $\|a_k\| \geq 1/\|b\|$, hence $\|\overline{a}\|_1 \neq 0$, a contradiction. Thus $\|\mathbf{x}\|_1 = 0$ implies $\mathbf{x} = 0$.

Obviously $\|\lambda\mathbf{x}\|_1 = \lambda\|\mathbf{x}\|_1$ for each $\mathbf{x} \in B$.

Further, let $\mathbf{x}$, $\mathbf{y} \in B$, $\mathbf{x} = [x_n, e_n]$, $\mathbf{y} = [y_n, f_n]$. Then

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_1 &= \inf\sup_n\{\|c_n\|\,|(c_n, g_n) \in \mathbf{x}+\mathbf{y}\}\\ &\leq \inf\sup_n\{\|x'_n + y'_n\|\,|(x'_n, e'_n) \in \mathbf{x}\,(y'_n, f'_n) \in \mathbf{y}\}\\ &\leq \inf\sup_n\{\|x'_n\| + \|y'\|\,|(x'_n, e'_n) \in \mathbf{x},\ (y'_n, f'_n) \in \mathbf{y}\}\\ &= \|\mathbf{x}\|_1 + \|y\|_1.\end{aligned}$$

In just the same way, we get

$$\|\mathbf{x}\mathbf{y}\|_1 \leq \|\mathbf{x}\|_1\|\mathbf{y}\|_1.$$

From previous property we have $\|\mathbf{x}^*\mathbf{x}\|_1 \leq \|\mathbf{x}\|_1^2$. On the other hand, let $(b_n, q_n) \in \mathbf{x}^*\mathbf{x}$, $(s_n, k_n) \in \mathbf{x}$ for a suitable SDD k_n.

Hence there exists SDD (p_n) such that

$$_np_n = s_n^* s_n p_n,\ p_n b_n = p_n s_n^* s_n.$$

Let $t_n = p_n \bigwedge k_n \bigwedge q_n$. Then $(t_n s_n^*)(s_n t_n) = t_n b_n t_n$ and so $\|t_n s_n^* s_n t_n\| \leq \|b_n\|$. In addition, $[t_n s_n^*, f_n] = [s_n^*, k_n]$ for suitable SDD (f_n), therefore $(t_n s_n^*, f_n) \in \mathbf{x}^*$. Hence for any EMO $(b_n, q_n) \in \mathbf{x}^*\mathbf{x}$ there exists EMO $(z_n, f_n) \in \mathbf{x}\,(z_n = s_n t_n)$ such that $\|z_n\|^2 \leq \|b_n\|$. Thus $\|\mathbf{x}\|_1^2 \leq \|\mathbf{x}^*\mathbf{x}\|_1$. ■

Corollary 5.2. *The norms $\|\cdot\|$ and $\|\cdot\|_1$ coincide on T.*

Proof: Let x be a positive element of T. By the definition, we have

$$\|\overline{x}\|_1 = \inf\sup_n\{\|x_n\|\,|(x_n, e_n) \in \overline{x}\}.$$

Obviously $\|\overline{x}\|_1 \leq \|x\|$. Set $(x_n, e_n) \in \overline{x}$. Then there exists SDD p_n such that $x_n p_n = x p_n$ for all n. Therefore $\|xp_n\| = \|x_n p_n\| \leq \|x_n\|$. Choose a sequence of the positive numbers ε_n with $\varepsilon_n \uparrow \|x\|$. Set $\{x\}'' = C(K)$, for some Hausdorff space K. Put $U_n = \{a \in K : x(a) > \varepsilon_n\}$,

$$b_n(a) = \begin{cases} \dfrac{1}{x(a)}, & a \in \overline{U} \\ 0, & \text{otherwise.}\end{cases}$$

Since $\overline{U}_n$ is clopen [**4**, par. 8] so $b_n(a) \in C(K)$ and $\|b_n(a)\| \leq \frac{1}{\varepsilon_n}$. As it was shown in Theorem 5.1, we can obtain that for each $n \in \mathbf{N}$ there exists a number $m(n)$ such that $\|x_m\| \geq 1/\|b_n\| \geq \varepsilon_n$ if $m \geq m(n)$. Therefore $\sup_m \|x_m\| \geq \varepsilon_n$ for all n. It follows that $\|\overline{x}\|_1 \geq \varepsilon_n$ for all $n \in \mathbf{N}$. Therefore $\|\overline{x}\|_1 \geq \|x\|$. Thus $\|\overline{x}\|_1 = \|x\|$ for all positive $x \in T$. For arbitrary $x \in T$ we have $\|\overline{x}\|_1^2 = \|\overline{x}^*\overline{x}\|_1 = \|x^*x\| = \|x\|^2$.

We shall use a notation $\tilde{B}$ for a completion of B in the norm $\|\cdot\|_1$. In this connection $\tilde{\mathbf{x}}$ is an image of $\mathbf{x} \in B$ in $\tilde{B}$. ■

Lemma 5.3. *If* $\mathbf{x} \in B$ *and* $\|\mathbf{x}\|_1 < 1$ *then the series* $\sum_{n\geq 0} \mathbf{x}^n$ *converges to* $(1-\mathbf{x})^{-1} \in B$ *in the norm* $\|\cdot\|_1$.

Proof: We can choose $(x_n, e_n) \in \mathbf{x}$ such that $\sup_n \|x_n\| < 1$. Then all the $1-x_n$ are invertible. By Lemma 5.2 it follows that $1-\mathbf{x}$ is invertible in Q and $(1-\mathbf{x})^{-1} = [(1-x_n)^{-1}, k_n]$ for suitable SDD (k_n). Observe

$$\|(1-x_n)^{-1}\| \leq \sum_{k\geq 0} \|x_n^k\| \leq \sum_{k\geq 0} \mu^k < \infty$$

for all n, where $\mu = \sup_n \|x_n\| < 1$. Thus $(1-\mathbf{x})^{-1} \in B$. Identifying $\mathbf{x}$ and $(1-\mathbf{x})$ with their images $\tilde{\mathbf{x}}$ and $\widetilde{(1-\mathbf{x})}^{-1}$ in C^*-algebra $\tilde{B}$, we get the statement of Lemma. ■

Lemma 5.4. *If* $\mathbf{x} \in B$ *then* $\rho(\mathbf{x}) = \sup\{|\lambda| | \lambda \in \sigma(\mathbf{x})\} \leq \|\mathbf{x}\|_1$, *where* $\sigma(\mathbf{x})$ *is a spectrum of* $\mathbf{x}$.

Proof: Let $|\lambda| > \|\mathbf{x}\|_1$, then applying Lemma 5.3 we obtain that the series $\lambda^{-1}\sum_{m\geq 0}(\mathbf{x}\lambda^{-1})^m$ converges to $(\lambda 1-\mathbf{x})^{-1}$ in the norm $\|\cdot\|_1$ and lemma follows. ■

Lemma 5.5. *Let* $\mathbf{u}$ *be a unitary element in* B. *Then*

$$\sigma(\mathbf{u}) \subset \{\lambda \in \mathbf{C} | \lambda| = 1\}.$$

Proof: By Lemma 5.4 $\sigma(\mathbf{u}) \subseteq \{\lambda \in \mathbf{C} | |\lambda| \leq 1$. Since $\mathbf{u}$ is invertible we have $\sigma(\mathbf{u}) = \overline{\sigma(\mathbf{u}^*)} = \overline{\sigma(\mathbf{u})^{-1}}$. It follows that $\sigma(\mathbf{u}) \subseteq \{\lambda \in \mathbf{C} | |\lambda| = 1\}$. ■

Lemma 5.6. *If* $\mathbf{x} \in B$, $\mathbf{x} = \mathbf{x}^*$ *then* $\sigma(\mathbf{x}) \subseteq [-\|\mathbf{x}\|_1, \|\mathbf{x}\|_1]$.

Proof: The proof is similar to the case of C^*-algebras. ■

6. Module of a self-adjoint element from B.

We call an element $x \in B$ positive, $\mathbf{x} \geq 0$, if $\tilde{\mathbf{x}} \geq 0$.

The goal of this section is to prove that for any self-adjoint element $\mathbf{x} \in B$ there exists an unique positive $\mathbf{y} \in B$ such that $\mathbf{y}^2 = \mathbf{x}^2$.

Theorem 6.1. *Let $\mathbf{u}$ be a unitary element of B. Then the mapping $\overline{T} \ni \overline{x} \mapsto \mathbf{u}\overline{x}\mathbf{u}^* \in B$ is a $*$-automorphism of a finite Rickart C^*-algebra $\overline{T}$.*

Proof: Set $A = \mathbf{u}\overline{T}\mathbf{u}^*$. Obviously A is a $*$-algebra with a C^*-norm $\|\cdot\|_1$. Let $\{\mathbf{x}_n\}$ be a $\|\cdot\|_1$-fundamental sequence in A. Then there exists a sequence $\{t_n\}$ such that $x_n = \mathbf{u}\overline{t}_n\mathbf{u}^*$. Since

$$\|t_n - t_m\| = \|\overline{t}_n - \overline{t}_m\|_1 = \|\mathbf{u}^*(\mathbf{u}\overline{t}_n\mathbf{u}^* - \mathbf{u}\overline{t}_m\mathbf{u}^*)\mathbf{u}\|_1 \leq \|\mathbf{x}_n - \mathbf{x}_m\|_1,$$

hence the sequence $\{\overline{t}_n\}$ is fundamental in T. Let $t = \lim_{n\to\infty} t_n$. Then clearly that the sequence $\{\mathbf{u}\overline{t}_n\mathbf{u}^*\}$ converges to $\mathbf{u}\overline{t}\mathbf{u}^*$ in the norm $\|\cdot\|_1$. Thus A is a C^*-algebra. Clearly, that $P(A) \subset P(Q)$. On the other hand, any projection $e \in P(T)$ can be written as $\mathbf{u}(\mathbf{u}^*e\mathbf{u})\mathbf{u}^*$. Since $\mathbf{u}^*e\mathbf{u} \in P(Q) = P(T)$ ([**9**]) we conclude that $P(A) = P(T)$. By spectral theory, it follows that $A = \overline{T}$. ∎

The next Corollary is a key result in proving an existence of a module of self-adjoint element of B.

Corollary 6.2. *Let $\mathbf{u}$ be a unitary element of B, $t \in T$. Then $\overline{t}\mathbf{u} \in \overline{T}$ implies $\mathbf{u}\overline{t} \in \overline{T}$.*

Proof: Since $\mathbf{u}\overline{t} = \mathbf{u}(\overline{t}\mathbf{u})\mathbf{u}^*$, by using Theorem 6.1 we have $\mathbf{u}\overline{t} \in \overline{T}$. ∎

Proposition 6.3. *Let $\mathbf{x} \in B$ and SDD (e_n) such that $\mathbf{x}e_n$, $e_n\mathbf{x} \in \overline{T}$ for all $n \in \mathbf{N}$. Then $\mathbf{x} = [y_n, e_n]$, where $\overline{y}_n = \mathbf{x}e_n + e_n\mathbf{x} - e_n\mathbf{x}e_n$.*

Proof: Let $\mathbf{x} = [x_n, p_n]$, $q_n = p_n \bigwedge e_n$. By using Lemma 4.7,

$$\overline{x}_n q_n = \mathbf{x}q_n = (\mathbf{x}e_n + e_n\mathbf{x} - e_n\mathbf{x}e_n)q_n = \overline{y}_n q_n,\ x_n q_n = y_n q_n.$$

In analogy, $q_n x_n = q_n y_n$. ∎

Lemma 6.4. *Let $\mathbf{u} = [u_n, e_n]$ be a unitary element of B. Then for any $k \in \mathbf{N}$ one can write $\mathbf{u}^k = [x_n, e_n]$ for a suitable sequence $\{x_n\}$.*

Proof: By Lemma 4.7, $\mathbf{u}e_n = \overline{u}_n e_n$ for all n. Let $f = \mathbf{u}e_n\mathbf{u}^*$, then $f\mathbf{u} \in \overline{T}$. By using Corollary 6.2 we obtain that $\mathbf{u}f \in \overline{T}$. Therefore

$\mathbf{u}^2 e_n = \mathbf{u}f\mathbf{u} = \mathbf{u}ff\mathbf{u} \in \overline{T}$. Now let $g = \mathbf{u}f\mathbf{u}^*$. Obviously $g\mathbf{u} \in \overline{T}$. By Corollary 6.2 it follows $\mathbf{u}g \in \overline{T}$. Hence

$$\mathbf{u}^3 e_n = \mathbf{u}\mathbf{u}^2 e_n = \mathbf{u}\mathbf{u}f\mathbf{u} = \mathbf{u}g\mathbf{u}f\mathbf{u} \in \overline{T}.$$

Inductively, applying the same k times, we obtain that $\mathbf{u}^k e_n \in \overline{T}$ and so (Corollary 6.2) $e_n \mathbf{u}^k \in \overline{T}$ for all n.

Now we can get the sequence $\{x_n\}$. As it was shown above,

$$\mathbf{u}^k e_n + e_n \mathbf{u}^k - e_n \mathbf{u}^k e_n \in \overline{T}.$$

Put $\overline{x}_n = \mathbf{u}^k e_n + e_n \mathbf{u}^k - e_n \mathbf{u}^k e_n$, where $x_n \in T$. By Proposition 6.3, $[x_n, e_n] = \mathbf{u}^k$. ■

Lemma 6.5. *Let $\{\mathbf{x}^{(k)}\}$ be a $\|\cdot\|$-fundamental sequence in B. And let a SDD (e_n) such that $\mathbf{x}^{(k)} e_n$, $e_n \mathbf{x}^{(k)} \in \overline{T}$ for all n and k. Then the sequence $\{\mathbf{x}^{(k)}\}$ converges to some element $\mathbf{x} \in B$ in the norm $\|\cdot\|_1$.*

Proof: Let $\|\mathbf{x}^{(k)} - \mathbf{x}^{(l)}\|_1 \leq \varepsilon/3$. By Proposition 6.3, $\mathbf{x}^{(k)} = [y_n^{(k)}, e_n]$, where $\overline{y}_n^{(k)} = \mathbf{x}^{(k)} e_n + e_n \mathbf{x}^{(k)} - e_n \mathbf{x}^{(k)} e_n$. For fixed n, we have

$$\begin{aligned} \|y_n^{(k)} - y_n^{(l)}\| &= \|\overline{y}_n^{(k)} - \overline{y}_n^{(l)}\|_1 \\ &= \|(\mathbf{x}^{(k)} - \mathbf{x}^{(l)}) e_n + e_n(\mathbf{x}^{(k)} - \mathbf{x}^{(l)}) - e_n(\mathbf{x}^{(l)} - \mathbf{x}^{(k)}) e_n\|_1 \leq \varepsilon. \end{aligned}$$

Thus, $\{y_n^{(k)}\}_k$ is fundamental in T. Set $y_n = \lim_{k\to\infty} y_n^{(k)}$. Now we show that $[y_n, e_n]$ is a MO. Let $m \leq n$, then

$$\|y_n e_m - y_m e_m\| = \|(y_n - y_n^{(k)}) e_m + (y_n^{(k)} - y_m^{(k)}) e_m + (y_m^{(k)} - y_m) e_m\| \leq \delta(k).$$

Since $\|(y_n e_m - y_m e_m\|$ das not depend on k we can conclude $\|y_n e_m - y_m e_m\| = 0$, $y_n e_m = y_m e_m$. In just the same way, $e_m y_n = e_m y_m$. Put $\mathbf{x} = [y_n, e_n]$. It remains to prove that the sequence $\mathbf{x}^{(k)}$ converges to $\mathbf{x}$ in the norm $\|\cdot\|_1$. We have

$$\|\mathbf{x} - \mathbf{x}^{(k)}\|_1 = \|[y_n - y_n^{(k)}, e_n]\|_1 = \|[y_n e_n - y_n^{(k)} e_n, p_n]\|_1$$

for a suitable SDD (p_n). Note that $\overline{y}_n^{(k)} e_n = \mathbf{x}^{(k)} e_n$. Identifying $\overline{y}_n^{(k)}$, e_n, $\overline{y}_n$ and $\mathbf{x}^{(k)}$ with their images $\tilde{\overline{y}}_n^{(k)}$, $\tilde{e_n}$, $\tilde{\overline{y}}_n$ and $\tilde{\mathbf{x}}^{(k)}$ in $\tilde{B}$, we obtain the following relations:

$$\tilde{\overline{y}}_n^{(k)} \tilde{e_n} = \tilde{\mathbf{x}}^{(k)} \tilde{e_n},\ \tilde{y_n} \tilde{e_n} = \|\cdot\|_1 - \lim_{k\to\infty} \tilde{\overline{y}}_n^{(k)} \tilde{e_n}.$$

Since $\{\tilde{\mathbf{x}}^{(k)}\}$ is a $\|\cdot\|_1$-fundamental, there exists $\tilde{y}\in\tilde{B}$ such that $\tilde{y} = \|\cdot\|_1 - \lim\limits_{k\to\infty}\tilde{y}^{(k)}$. Hence,

$$\widetilde{y_n}\widetilde{e_n} = \|\cdot\|_1 - \lim_{k\to\infty}\widetilde{y_n}^{(k)}\widetilde{e_n} = \|\cdot\|_1 - \lim_{k\to\infty}\tilde{\mathbf{x}}^{(k)}\tilde{e}_n = \tilde{y}\tilde{e}_n.$$

It yields

$$\begin{aligned}\|\mathbf{x}-\mathbf{x}^{(k)}\|_1 &\leq \sup_n\|y_ne_n - y_n^{(k)}e_n\|\\ &= \sup_n\|\widetilde{y_n}\widetilde{e_n} - \widetilde{y_n}^{(k)}\widetilde{e_n}\|_1 = \sup_n\|\tilde{y}\widetilde{e_n} - \tilde{\mathbf{x}}^{(k)}\widetilde{e_n}\|\\ &\leq \sup_n\{\|\tilde{y}-\tilde{\mathbf{x}}^{(k)}\|_1\|e_n\|\} = \|\tilde{y}-\tilde{\mathbf{x}}^{(k)}\|_1 \to 0\end{aligned}$$

for $k\to\infty$. ■

Theorem 6.6. *If $\mathbf{x} = \mathbf{x}^*\in B$, then there exists a positive element $\mathbf{a}\in B$ such that $\mathbf{a}^2 = \mathbf{x}^2$.*

Proof: We have $\mathbf{x} = i(1+\mathbf{u})(1-\mathbf{u})^{-1}$, where $\mathbf{u} = [u_n, e_n]$ is the Cayley transform of $\mathbf{x}$. Then $\mathbf{x}^2 = \mathbf{x}\mathbf{x}^* = (2+\mathbf{u}+\mathbf{u}^*)(2-\mathbf{u}-\mathbf{u}^*)^{-1}$. Observe the sequence

$$\mathbf{y}^{(l)} = \|2+\mathbf{u}+\mathbf{u}^*\|_1^{\frac{1}{2}}\left(1+\sum_{k=1}^{l}c_k(1-(2+\mathbf{u}+\mathbf{u}^*)/\|2+\mathbf{u}+\mathbf{u}^*\|_1)^k\right),$$

where c_k are coefficients of Taylor series for a function $f(a)=\sqrt{1-a}$ on $[0,1]$. Since $\widetilde{(2+\mathbf{u}+\mathbf{u}^*)}\geq 0$ the sequence $\{\tilde{\mathbf{y}}^{(l)}\}$ is $\|\cdot\|_1$-fundamental in $\tilde{B}$ and therefore so is $\{\mathbf{y}^{(l)}\}$ in B. But all the members of the sum

$$\sum_{k=1}^{l}c_k(1-(2+\mathbf{u}+\mathbf{u}^*)/\|2+\mathbf{u}+\mathbf{u}^*\|_1)^k$$

are linear combinations of the degrees of $\mathbf{u}$, $\mathbf{u}^*$ and 1. By combining Lemma 6.4 and Lemma 6.5 the sequence $\{\mathbf{y}^{(l)}\}$ $\|\cdot\|_1$-converges to some element $\mathbf{y}\in B$. Clearly that $\tilde{\mathbf{y}}^2 = \widetilde{(2+\mathbf{u}+\mathbf{u}^*)}$, hence $\mathbf{y}^2 = 2+\mathbf{u}+\mathbf{u}^*$.

Similarly, we can find an element $\mathbf{z}\in B$ such that $\mathbf{z}^2 = 2-\mathbf{u}-\mathbf{u}^*$. Note that all elements $\mathbf{y}$, $(2-\mathbf{u}-\mathbf{u}^*)$, $\mathbf{z}$, $(2+\mathbf{u}+\mathbf{u}^*)$, $(2-\mathbf{u}-\mathbf{u}^*)^{-1}$ mutually commute. Consequently, $\mathbf{z}\mathbf{z}(2-\mathbf{u}-\mathbf{u}^*)^{-1} = \mathbf{z}(2-\mathbf{u}-\mathbf{u}^*)^{-1}\mathbf{z} = 1$, i.e. $\mathbf{z}$ is invertible. Finally, puting $\mathbf{a} = \mathbf{y}\mathbf{z}^{-1}$, we obtain $\mathbf{a}^2 = \mathbf{x}^2$. Evidently, that $\mathbf{a}\in B$ and such $\mathbf{a}$ is positive and unique. ■

7. Polar decomposition.

In this section we prove the main result of the paper: all Rickart C^*-algebras satisfy polar decomposition.

Theorem 7.1. *Let T be a finite Rickart C^*-algebra. Then the algebras B and $\overline{T}$ coincide.*

Proof: We shall prove this statement as a spectral theorem for self-adjoint element of B. Each operator $\mathbf{x} = \mathbf{x}^* \in B$ will be approximated (in norm $\|\cdot\|_1$) by means of simple operators of $\overline{T}$.

For self-adjoint $\mathbf{x} \in B$ write $|\mathbf{x}| = (\mathbf{x}^2)^{\frac{1}{2}}$, $\mathbf{x}_+ = (|\mathbf{x}| + \mathbf{x})/2$, $\mathbf{x}_- = (|\mathbf{x}| - \mathbf{x})/2$. Note that $\mathbf{x}_+ - \mathbf{x}_- = \mathbf{x}$, $\mathbf{x}_+ + \mathbf{x}_- = |\mathbf{x}|$, $\mathbf{x}_+\mathbf{x}_- = 0$. If $\mathbf{x} = \mathbf{x}^* \in B$ then $\{\mathbf{x}\}''_B = A$ is a commutative Rickart $*$-algebra (see [**4**, p. 17] with C^*-norm $\|\cdot\|_1$. It is easy to see that $|\mathbf{x}|, \mathbf{x}_+, \mathbf{x}_- \in A$. ■

Lemma 7.2. *Let $\mathbf{x} \in B$, $\mathbf{x} = \mathbf{x}^*$. The family of the projections $e_\lambda = s[(\lambda 1 - \mathbf{x})_+]$ holds the following properties:*

(a) *$e_\mu \geq e_\lambda$ for $\mu \geq \lambda$;*
(b) *$\sup_\lambda e_\lambda = 1$;*
(c) *$\inf_\lambda e_\lambda = 0$;*
(d) *If $\mu_1 \geq \mu_2 \geq \lambda_1 \geq \lambda_2$ then $(e_{\mu 1} - e_{\mu 2})(e_{\lambda 1} - e_{\lambda 2}) = 0$.*

Proof: (a) Let $\lambda \leq \mu$, then $\lambda 1 - \mathbf{x} \leq \mu 1 - \mathbf{x}$. Set $\{\mathbf{x}\}''_B = A$, $\mathbf{a} = (\lambda 1 - \mathbf{x})_+$, $\mathbf{b} = (\mu 1 - \mathbf{x})_+$. Then $\mathbf{a}, \mathbf{b} \in A$. Put $s(\mathbf{a}) = e$, $s(\mathbf{b}) = f$. By [**4**, p. 17], e, $f \in A$. Since $\tilde{A}$ is commutative C^*-algebra we have $\mathbf{a} \leq \mathbf{b}$. Observe,

$$\mathbf{a}(1-f) = (1-f)\mathbf{a}(1-f) \leq (1-f)\mathbf{b}(1-f) = 0,$$

hence $f \geq e$.

(b) Let $e = \sup_\lambda e_\lambda$. Then $\lambda(1-e) \leq \lambda(1-e_\lambda)$ for all $\lambda \in \mathbf{R}_+$. Further,

$$\lambda 1 - \mathbf{x} = (\lambda 1 - \mathbf{x})_+ - (\lambda 1 - \mathbf{x})_- \leq (\lambda 1 - \mathbf{x})_+.$$

In addition,

$$e_\lambda(\lambda 1 - \mathbf{x}) = e_\lambda[(\lambda 1 - \mathbf{x})_+ - (\lambda 1 - \mathbf{x})_-] = (\lambda 1 - \mathbf{x})_+.$$

Hence $\lambda(1-e_\lambda) \leq (1-e_\lambda)\mathbf{x}$ and thus $\lambda(1-e) \leq (1-e_\lambda)\mathbf{x}$.

Note since

$$(1-e_\lambda)\mathbf{x} \leq \|x\| 1,$$

consequently, $1-e \leq \frac{\|x\|}{\lambda}$. If $e \neq 1$ then

$$1 = \|1-e\| \leq \||\mathbf{x}\||_1/\lambda$$

for all $\lambda > 0$, a contradiction.

(c) Using the inequality $|\lambda| e_\lambda \leq e_\lambda \mathbf{x}$, repeat the proof of (b).

(d) It follows immediatly from (a). ■

Now one can begin to approximate an operator $\mathbf{x}$.

By Lemma $\sigma(\mathbf{x}) \subseteq [-\|\mathbf{x}\|_1,\ \|\mathbf{x}\|_1]$. Let $\alpha \in \mathbf{R}$, $\|\mathbf{x}\|_1 \leq \alpha$. Take an arbitrary partition of the segment $[-\alpha, \alpha]$:

$$-\alpha = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k-1} \leq \lambda_k = \alpha.$$

Consider the elements $u_n = \lambda_n(e_{\lambda n} - e_{\lambda n-1})$. Observe that

$$\lambda(e_\mu - e_\lambda) \leq (e_\mu - e_\lambda)\mathbf{x} \leq \mu(e_\mu - e_\lambda)$$

for $\mu \geq \lambda$. It follows that

$$\overline{u}_n - \mathbf{x}(e_{\lambda n} - e_{\lambda n-1}) \leq \delta(e_{\lambda n} - e_{\lambda n-1}),$$

where $\delta = \max_k\{\lambda_i - \lambda_{i-1}\}$. Note $\overline{u}_n - \mathbf{x}(e_{\lambda n} - e_{\lambda n-1}) \geq 0$. Construct an integral sum

$$\sigma = \sum_{n=1}^{k} \lambda_n(e_{\lambda n} - e_{\lambda n-1}).$$

Set $\lambda \geq \alpha$, then $\lambda 1 - \mathbf{x} \geq \varepsilon 1$ for some $\varepsilon \geq 0$. Consequently, $(\lambda 1 - \mathbf{x})_+ = \lambda - \mathbf{x}$. Since $\lambda \notin \sigma(\mathbf{x})$, we obtain $s((\lambda - \mathbf{x})_+) = 1$. So $e_\lambda = 1$ for $\lambda \geq \alpha$. In analogy, $e_\lambda = 0$ for $\lambda \leq -\alpha$. We have,

$$\overline{\sigma} - \mathbf{x} = \sum_{n=1}^{k}(u_n - \mathbf{x}(e_{\lambda n} - e_{\lambda n-1})) \leq \sum_{n=1}^{k} \delta(e_{\lambda n - e_\lambda n-1}) = \delta 1.$$

Therefore, $0 \leq \overline{\sigma} - \mathbf{x} \leq \delta 1$, so $\|\overline{\sigma} - \mathbf{x}\|_1 \leq \delta$. Thus, each self-adjoint operator $\mathbf{x} \in B$ can be approximated by the simple elements from $\overline{T}$ in the norm $\|\cdot\|_1$. It follows that $\overline{T}$ is dense in B and therefore these C^*-algebras coincide.

Corollary 7.3. *The partial isometries are $\aleph_0$-addable.*

Proof: Let (e_i) and (f_i) are sequences of ortogonal projections such that $e_i = u_i u_i^*$ and $f_i = u_i^* u_i$. Put

$$v_n = \sum_{i=1}^{n} u_i,\ k_n = \sum_{i=1}^{n} e_i,\ t_n = \sum_{i=1}^{n} f_i,\ e = \bigvee_i e_i,\ f = \bigvee_i f_i.$$

Then the sequences $(p_n = e^\perp + k_n)$ and $(q_n = f^\perp + t_n)$ are SDD. Set $d_n = p_n \bigwedge q_n$. Clearly that $\mathbf{v} = [v_n, d_n]$ is MO from B. By previous theorem, there exists $v \in T$ such that $\overline{v} = \mathbf{v}$. It is easy to see that $v u_i^* u_i = u_i$, $u_i u_i^* v = u_i$ for all $i \in \mathbf{N}$. ■

Corollary 7.4. *All Rickart C^*-algebras satisfy polar decomposition.*

Proof: By [**1**, Th. 3.4], this assertion is reduced to a finite case. Now combine Corollary 7.3 and [**1**, Prop. 2.1] and the Corollary follows. ∎

Corollary 7.5. *Let T be a Rickart C^*-algebra, then the matrix algebras $M_n(T)$ over T are also Rickart C^*-algebras for all $n \in \mathbf{N}$.*

Proof: See [**1**, Th. 3.5]. ∎

8. Axiom (PSR) in Q.

Using Theorem 7.1 and the methods of [**3**], [**11**] or [**6**], we can describe the self-adjoint elements in Q.

Theorem 8.1. *Let $\mathbf{x} = \mathbf{x}^* \in Q$, $\mathbf{u} = \overline{u}$ $(u \in T)$ its Cayley transform. One can write $\mathbf{x} = [x_n, e_n]$ with x_n, $e_n \in \{u\}''$, $x_n^* = x_n$, $x_n e_n = x_n$, $x_n^2 \uparrow$.*

Proof: See [**3**, Th. 4.2]. ∎

An element $\mathbf{x} \in Q$ is positive, written $\mathbf{x} \geq 0$, if $\mathbf{x} = \mathbf{y}^*\mathbf{y}$ for some $\mathbf{y} \in Q$.

Theorem 8.2. *Let $\mathbf{x} = \mathbf{x}^* \in B$, $\mathbf{u} = \overline{u}$ its Cayley transform. The following conditions are equivalent:*

a) $\mathbf{x} \geq 0$;
b) *one can write* $\mathbf{x} = [y_n, f_n]$ *with* $y_n \geq 0$;
c) *the spectrum of u contained in* $\{e^{i\Theta} : -\pi \leq \Theta \leq 0\}$;
d) *one can write* $\mathbf{x} = [x_n, e_n]$ *with* x_n, $e_n \in \{u\}''$, $x_n \geq 0$, $x_n e_n = x_n$.

Proof: See [**3**, Th. 6.1]. ∎

Corollary 8.2. *Q satisfies axiom (PSR).*

Proof: See [**3**, Cor. 6.2]. ∎

Acknowledgments. The author express his gratitude to V. I. Chilin for stating the problem and for his concern about the work, and also to P. Ara and D. Handelman for valuable discussions.

I am very grateful to Ben-Gurion University for the help in preparing this paper.

References

1. P. Ara, Left and right projections are equivalent in Rickart C^*-algebras, *J. Algebra* **120** (1989), 433–448.
2. P. Ara and D. Goldstein, A solution to the matrix problem for Rickart C^*-algebras, *Math. Nachr.* **164** (1993), 259–270.
3. S. K. Berberian, The regular ring of a finite AW^*-algebra, *Ann. Math.* **65** (1957), 224–240.
4. S. K. Berberian, "*Baer $*$ Rings*," Springer-Verlag, Berlin and New-York, 1972.
5. E. Christensen, Non commutative integration for monotone sequentially closed C^*-algebras, *Math. Scand.* **31** (1972), 171–190.
6. D. Goldstein, Rickart ordered $*$-algebras, *Dokl.Uzb. Acad. Nauk* **1** (1990) (in Russian).
7. D. Goldstein, $\aleph_0$-addable of partial isometries in the Rickart C^*-algebras, *Dokl. Uzb. Acad. Nauk* **10** (1990) (in Russian).
8. K. R. Goodearl, D. E. Handelman and J. W. Lawrence, Affine representations of Grotendieck groups and applications to Rickart C^*-algebras and $\aleph_0$-continuous regular rings, *Memoirs Amer. Math. Soc.* **234** (1980).
9. D. Handelman, "*Finite Rickart C^*-algebras and their properties*," Adv. in Math. Suppl. Studies **4**, Academic press, Orlando, FL, 1979, pp. 171–196.
10. D. Handelman, Rickart C^*-algebras, II, *Adv. in Math.* **48** (1983), 1–15.
11. K. Saito, On the algebra of measurable operators for a general AW^*-algebra, *Tohoku Math. Journ.* **21** (1969), 249–270.

Institute of Mathematics
Hebrew University of Jerusalem
ISRAEL

Primera versió rebuda el 4 de febrer de 1993,
darrera versió rebuda el 17 de febrer de 1995