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IMPROVED MUCKENHOUPT-WHEEDEN INEQUALITY AND WEIGHTED INEQUALITIES FOR POTENTIAL OPERATORS

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Abstract

By a variant of the standard good λ inequality, we prove the Muckenhoupt-Wheeden inequality for measures which are not necessarily in the Muckenhoupt class. Moreover we can deal with a general potential operator, and consequently we obtain a suitable approach to the two weight inequality for such an operator when one of the weight functions satisfies a reverse doubling condition.

0. Introduction

In this paper $d\mu$, $d\omega$ are locally finite positive Borel measures of R^n, $n \geq 1$. For a nonnegative locally-$d\mu$ integrable function $K(x,y)$ (a.e. continuous in the first variable) we define the potential operator

$$(Tf\mu)(x) = \int_{y\in\mathbb{R}^n} K(x,y)f(y)d\mu(y).$$

For each $C_1 > 0$, we assume the existence of a $C_2 > 0$ so that

$$(\mathcal{H})\ \ K(x,y) \leq C_2 K(z,y), \text{ for each } x,y,z \text{ with } 0 < |z-y| < C_1|x-y|.$$

The dual operator T^* is the operator defined by the kernel $K^*(x,y) = K(y,z)$. The usual fractional integral operator I_s, with $0 < s < n$, is given by $K(x,y) = |x-y|^{s-n}$. Other examples of operators T are those introduced by Chanillo-Stromberg-Wheeden [**Ch-St-Wh**] and given by kernels $K(x,y) = \frac{a(y,|x-y|)}{|x-y|^n}$. Here a is considered as a function defined on balls of $\mathbb{R}^n$ and which satisfies some growth conditions we precise below.

We are interested in finding a constant $C > 0$ for which

(P_T) $\qquad \|Tf\mu\|_{L^q_\omega} \leq C\|f\|_{L^p_\mu}$ for all nonnegative functions f

with $1 < p, q < \infty$. The constant C depends only on n, p, q, ω, μ and K; and when it is necessary we denote this dependance by writing $C = C(n, p, q, K, \omega, \mu)$. Here $\|g\|_{L^r_\nu} = \Big(\int_{\mathbb{R}^n} |g|^r d\nu\Big)^{\frac{1}{r}}$. The inequality (P_T) includes the usual two weight norm inequality

$$\|Tf\|_{L^q_u} \leq C\|f\|_{L^p_v}$$

since it is sufficient to replace f by $fv^{\frac{1}{p-1}}$, and to take $d\omega = udx$, $d\mu = v^{-\frac{1}{p-1}}dx$, where dx is the usual Lebesgue measure on $\mathbb{R}^n$. Inequality (P_T) with $T = I_s$ has been studied extensively by many authors (see for instance [**Ke-Sa**], [**Sa-Wh**] and [**Pe**] and the reference given by them). Kerman and Sawyer [**Ke-Sa**] solved the problem (P_{I_s}) with $d\omega = dx$. This particular case is first interesting since it is the usual form which appears in many mathematic and physic areas. It also appears that the case $d\omega = dx$ is naturally suitable to be treated. In fact using a good λ-inequality, they proved that the left member of (P_T) is majorized by the L^q norm of the fractional maximal function. So (P_T) is reduced to a weighted inequality for maximal operator, whose study was done by the first author [**Sa**]. Problem (P_T) with general measures $d\mu$ and $d\omega$ was solved by Sawyer and Wheeden [**Sa-Wh**].

Let us consider the operator $T = I_s$ with $0 < s < n$. We have the pointwise inequality

$$M_s g \leq c(s, n)\, I_s g$$

where M_s is the fractional maximal operator defined by

$$(M_s f)(x) = \sup\{|Q|^{\frac{s}{n}-1}\|f1\!\!1_Q\|_{L^1(dy)};\, Q \text{ a cube with } Q \ni x\}.$$

We generally use the letter Q to denote a cube of $\mathbb{R}^n$, and by which we mean a product of n intervals $[a_i, a_i+t]$ $(0 < t < \infty)$. The Muckenhoupt-Wheeden inequality [**Mu-Wh**] yields a sort of converse (in norm) of the above inequality, and asserts that for $0 < q < \infty$:

$$\|I_s f\|_{L^q_\omega} \leq c(s, n, q, \omega)\|M_s f\|_{L^q_\omega} \text{ for all functions } f$$

whenever the measure $d\omega$ satisfies the Muckenhoupt condition A_∞, i.e. there are $c = c(\omega), \delta > 0$ such as

$$\frac{|E|_\omega}{|Q|_\omega} \leq c\Big(\frac{|E|}{|Q|}\Big)^\delta \text{ for all cubes } Q \text{ and all measurables sets } E \subset Q.$$

In his thesis Perez [**Pe**] gave a weaker condition than the A_∞'s. He proved the above Muckenhoupt-Wheeden inequality for measures $d\omega$ satisfying D_∞ and B_ρ conditions with $\rho > 1 - \frac{s}{n}$, and which can be noted as $d\omega \in D_\infty \cap B_\rho$. These conditions respectively mean:

$$|2Q|_\omega = \int_{2Q} \omega \leq C(\omega)\,|Q|_\omega \text{ for all cubes } Q,$$

(here $2Q$ is the cube having the same center as Q and the length expanded twice)

$$\frac{|Q'|\omega}{|Q|_\omega} \leq C'(\omega)\Big(\frac{|Q'|}{|Q|}\Big)^\rho \text{ for all cubes } Q, Q' \text{ with } Q' \subset Q.$$

Contrary to the Muckenhoupt-Wheeden technique [**Mu-Wh**], the Perez's analysis [**Pe**] is not based on the standard good-λ inequalities. This last author used some estimates obtained by Frazier and Jawerth [**Fr-Ja**] for local maximal operator, and moreover he was able to treat the problem with a general convolution operator.

In this paper we also prove the Muckenhoupt-Wheeden inequality for measures which are not necessarily in the Muckenhoupt class (see Corollary 4), and with the general potential operator T described above. We do this, with a sort of a variant of the standard good λ inequality and by introducing a suitable maximal operator $M_{T,\omega}$ (see Theorem 1). The additional conditions on the measure $d\omega$ arise only in order to relate this "exotic" maximal operator to a more standard one like M_s (see Theorem 3). Consequently we obtain a suitable approach to the two weight inequality for such an operator when one of the weight functions satisfies a reverse doubling condition (see Theorem 5).

1. Statements of results

Let us define the dyadic maximal operator

$$(M^d_{T,\omega,\mu}f)(x) = \sup\{|Q|_\omega{}^{-1}\Big\|f(T^*1\!\!1_Q\omega)1\!\!1_{3Q}\Big\|_{L^1(d\mu)};\quad Q \text{ a dyadic cube with } Q \ni x\}.$$

A dyadic cube Q is a product of n intervals of the form $[2^k a_i, 2^k(a_i+1)]$, where k and a_i are integers. Fix $q \geq 1$. By using the Holder inequality we can observe that $(M^d_{T,\omega,\mu}f)(x)$ is a.e. finite for all bounded functions with compact supports whenever measures $d\omega$ and $d\mu$ satisfy the condition

$$(S_T) \qquad \|T1\!\!1_{|x|<R}\mu\|_{L^q_\omega} \leq c(R) < \infty \text{ for all } R > 0.$$

Our first result is as follow:

Theorem 1.
Let $0 < q < \infty$ and K be a nonnegative kernel satisfying the hypothesis $\mathcal{H}$. Assume the measures $d\omega$ and $d\mu$ satisfy the condition (S_T). Then there is $C = C(n, q, K) > 0$ so that

$$\|Tf\mu\|_{L^q_\omega} \leq C\|M^d_{T,\omega,\mu}f\|_{L^q_\omega}.$$

If M^d_ω is the dyadic maximal operator defined by

$$(M^d_\omega f)(x) = \sup\{|Q|_\omega{}^{-1}\left\|f\,\mathbb{1}_Q\right\|_{L^1(d\omega)};\, Q \text{ a dyadic cube with } Q \ni x\},$$

then (see Lemma 1 below)

$$(M^d_{T,\omega,\mu}g)(.) \leq (M^d_\omega(Tg\mu))(.)$$

and consequently we get

Proposition 2.
Let K, $d\omega$, $d\mu$ be as above Then for $q > 1$ we have

$$\|Tf\mu\|_{L^q_\omega} \approx \|M^d_{T,\omega,\mu}f\mu\|_{L^q_\omega}.$$

Moreover this equivalence also holds for the range of $q \in]0, 1]$ whenever

$$M^d_\omega(Tg)(.) \leq c(n, K, \omega)\,(Tg)(.)$$

for all f nonnegative functions g.

The above equivalence means

$$C_1(n, q, K)\|Tf\mu\|_{L^q_\omega} \leq \|M^d_{T,\omega,\mu}f\mu\|_{L^q_\omega} \leq C_2(n, q, \omega)\|Tf\mu\|_{L^q_\omega}.$$

The extra assumption in this result is satisfied for instance for the kernel $K(x, y) = |x - y|^{s-n}$, with $d\omega = dx$ the Lebesgue measure, and more generally for measures $d\omega \in D_\infty \cap B_\rho$ with $1 - \frac{s}{n} < \rho$.

Thus in view of Theorem 1, the inequality (P_T) is reduced to the following one, related for $M^d_{T,\omega,\mu}$

$$(\tilde{P}_T) \qquad \|M^d_{T,\omega,\mu}f\mu\|_{L^q_\omega} \leq C\|f\|_{L^p_\mu}.$$

In order to get this last one, we impose more hypothesis on the kernel K. So as to simplify, we only deal with kernels

$$K(x, y) = K_a(x, y) = \frac{a(y, |x - y|)}{|x - y|^n} = \frac{a(B(y, |x - y|))}{|x - y|^n}$$

where a is a function defined on balls satisfying the following hypotheses H:

(i) $a(B_1) \le c_1(n,a)\, a(B_2)$ for all balls B_1, B_2 with $B_1 \subset B_2$;

there are $\lambda, \sigma > 0$ so that
(ii)
$c_1'(n,a)\, t^{n\lambda}\, a(B) \le\ a(tB) \le c_2'(n,a)\, t^{n\sigma}\, a(B)$ for all balls B and $t \ge 1$.

We also define the function a on cubes by $a(Q) = a(B)$, where B is the smallest ball which contains the cube Q. A suitable dyadic maximal operator related to the potential operator $T = T_a$ (with kernel $K = K_a$) is

$$(M^d_\Phi f)(x) = \sup\{a(Q)|Q|^{-1}\|f\mathbb{1}_Q\|_{L^1(dy)};\, Q \text{ a dyadic cube with } Q \ni x\}.$$

The nondyadic version of M^d_Φ is merely denoted by M_Φ. The measure $d\omega$ satisfies the condition RD_ρ with $\rho > 0$ (and we write as $d\omega \in RD_\omega$) when there $c = c(\omega, n) > 0$ for which

$$t^{n\rho}\,|B|_\omega \le c\,|tB|_\omega \text{ for all balls } B \text{ and } t > 1.$$

Our second result ensures the link between the two maximal operators we have defined above.

Theorem 3.
Let $K = K_a$ be a kernel satisfying $\mathcal{H}$i)-ii) with $0 < \lambda, \sigma \le 1$. Suppose $d\omega \in RD_\rho$ with $1 - \lambda < \rho$. Then

$$C_1(M^d_\Phi f\mu)(.) \le (M^d_{T_a,\omega,\mu} f)(.) \le C_2(M_\Phi f\mu)(.) \text{ for all functions } f$$

here $C_1 = C_1(n,a) > 0$ and $C_2 = C_2(n,a,\omega)$.

In fact C_2 does not depend on the individual measure $d\omega$ but only on the RD_ρ constant of $d\omega$. The claim we announced in the introduction can be stated as

Corollary 4.
Let $0 < s < n$ and $0 < q < \infty$. Suppose $d\omega \in RD_\omega$ with $(1 - \frac{s}{n}) < \rho$. Then

$$\|I_s g\|_{L^q_\omega} \approx \|M_s g\|_{L^q_\omega} \text{ for all nonnegative functions } g$$

whenever

$$\int_{R<|x|} |x|^{(s-n)q} d\omega(x) < c(R) < \infty \text{ for all } R > 0.$$

This is an immediate consequence of Theorems 1 and 2. Indeed since for all $R > 0$:
$(I_s 1\!\!\mathrm{I}_{B(0,R)})(.) 1\!\!\mathrm{I}_{B(0,2R)}(.) \approx R^s 1\!\!\mathrm{I}_{B(0,2R)}(.)$ and $(I_s 1\!\!\mathrm{I}_{B(0,R)})(x) 1\!\!\mathrm{I}_{|x|>2R}(x) \approx |x|^{s-n} 1\!\!\mathrm{I}_{|x|>2R}(x)$ so the condition $\|I_s 1\!\!\mathrm{I}_{B(0,R)}\|_{L^q_\omega} < \infty$ is reduced to the one written in this corollary. Note also in studying the two weight inequality $\|I_s f\|_{L^q_\omega} \leq c\|f\|_{L^q_\nu}$ it is necessary that $\|I_s 1\!\!\mathrm{I}_{B(0,R)}\|_{L^q_\omega} \leq c\|1\!\!\mathrm{I}_{B(0,R)}\|_{L^p_\nu} < \infty$.

By Theorems 1 and 3, the problem (P_T) is then reduced to the following maximal inequality

$$\|M_\Phi f\mu\|_{L^q_\omega} \leq c\|f\|_{L^p_\mu}.$$

By the study of this last case (see [**Ra1**], or adapt the proof given in [**Sa**]) then we get

Theorem 5.

Let $1 < p, q < \infty$, and $K = K_a$ be a kernel satisfying $\mathcal{H}i)$-$ii)$ with $0 < \lambda, \sigma < 1$. Suppose $d\omega \in RD_\omega$ with $(1 - \lambda) < \rho$. Then the inequality

$$(P_T) \qquad \|Tf\mu\|_{L^q_\omega} \leq c\|f\|_{L^p_\mu}$$

holds if and only if

$$(1) \qquad \int_{|x|>R} \Big[\frac{a(x, |x|)}{|x|^n}\Big]^q d\omega(x) < c(R) < \infty \text{ for all } R > 0,$$

and

$$(2) \qquad \Big\|(T\sum_k \varepsilon_k 1\!\!\mathrm{I}_{Q_k}\mu) 1\!\!\mathrm{I}_{\bigcup Q_k}\Big\|_{L^q_\omega} \leq C\Big\|\sum_k \varepsilon_k 1\!\!\mathrm{I}_{Q_k}\Big\|_{L^p_\mu},$$

where $C > 0$ is a constant which does not depend of each sequence $(Q_k)_k$ of cubes and $(\varepsilon_k)_k$ of nonnegative reals ε_k.
Moreover in the case $1 < p \leq q$ the condition (2) can be replaced by

$$(2') \qquad \|(T 1\!\!\mathrm{I}_Q \mu) 1\!\!\mathrm{I}_Q\|_{L^q_\omega} \leq C\|1\!\!\mathrm{I}_Q\|_{L^p_\mu} \text{ for all cubes } Q$$

Sawyer and Wheeden [**Sa-Wh**] proved that for $1 < p \leq q$ and for all general measures $d\omega$ and $d\mu$, then (P_T) is equivalent to $(2')$ and

$$(2'') \qquad \|(T^* 1\!\!\mathrm{I}_Q \omega) 1\!\!\mathrm{I}_Q\|_{L^{p'}_\mu} \leq c\|1\!\!\mathrm{I}_Q\|_{L^{q'}_\omega} \quad p' = \frac{p}{p-1}, q' = \frac{q}{q-1}.$$

With an additional hypothesis on the measure $d\mu$ we can simplify the conditions in Theorem 5. We first consider the case $p \leq q$.

Proposition 6.

Let $1 < p \leq q < \infty$, and $K = K_a$ be a kernel satisfying $\mathcal{H}$i)-ii) with $0 < \lambda, \sigma < 1$. Suppose $d\omega \in RD_\rho$, $d\mu \in RD_{\rho'}$ with $(1-\lambda) < \rho$ and $\rho' > 0$. Then the inequality (P_T) holds if and only if

$$\|(T\mathbb{1}_Q\mu)\mathbb{1}_Q\|_{L^q_\omega} \leq C\|\mathbb{1}_Q\|_{L^p_\mu}.$$

If moreover $d\mu \in RD_{\rho'}$ with $(1-\lambda) < \rho'$ or $d\mu \in A_\infty$ then, an easy necessary and sufficient condition for (P_T) is

$$\frac{a(Q)}{|Q|}|Q|_\mu^{1-\frac{1}{p}}|Q|_\omega^{\frac{1}{q}} \leq C \text{ for all cubes } Q.$$

This second part is already known [**Sa-Wh**], and here we deduce it by using results on maximal functions (see [**Ra2**] and [**Pe**]). To deal with the range of $q < p$, we introduce the two conditions $d\mu \in \widetilde{RD}(p)$, $d\omega \in D_{\varepsilon,q}$ with $\varepsilon \in [1,\infty[$ (see [**Ch-St-Wh**]) and which mean respectively

$$\Big\|\sum_{j\geq 0}\sum_k \varepsilon_k\Big(\frac{|Q_k|_\mu}{|2^jQ_k|_\mu}\Big)\mathbb{1}_{2^jQ_k}\Big\|_{L^p_\mu} \leq c(\mu)\Big\|\sum_k \varepsilon_k\mathbb{1}_{Q_k}\Big\|_{L^p_\mu}$$

$$\Big\|\sum_k \varepsilon_k\mathbb{1}_{tQ_k}\Big\|_{L^q_\omega} \leq c(\omega)\, t^{n\varepsilon\frac{1}{q}}\Big\|\sum_k \varepsilon_k\mathbb{1}_{Q_k}\Big\|_{L^q_\omega}$$

for all $t \geq 1$, $\varepsilon_k > 0$ and all cubes Q and Q_k. Thus we can state

Proposition 7.

Let $1 < q < p < \infty$, and $K = K_a$ be a kernel satisfying $\mathcal{H}$i)-ii) with $0 < \lambda, \sigma < 1$, and $d\omega \in RD_\rho$, with $(1-\lambda) < \rho$.

Suppose $d\mu \in \widetilde{RD}(p)$. Then the inequality (P_T) holds if and only if for some $m \geq 4$ and $C > 0$

$$\Big\|\sum_k \varepsilon_k(T\mathbb{1}_{Q_k})\mathbb{1}_{(mQ_k)}\Big\|_{L^q_\omega} \leq C\Big\|\sum_k \varepsilon_k\mathbb{1}_{Q_k}\Big\|_{L^p_\mu}$$

for all cubes Q_k and all $\varepsilon_k > 0$.

For $d\mu \in RD_{\rho'} \cap D_{\varepsilon',p}$ with $\max(1-\lambda, \frac{1}{p}\varepsilon') < \rho'$, a necessary and sufficient condition for (P_T) is

$$\Big\|\sum_k \varepsilon_k\Big(\frac{a(Q_k)}{|Q_k|}|Q_k|_\mu\Big)\mathbb{1}_{Q_k}\Big\|_{L^q_\omega} \leq C\Big\|\sum_k \varepsilon_k\mathbb{1}_{Q_k}\Big\|_{L^p_\mu}.$$

This equivalence is also true when $d\mu \in RD_\rho \cap D_\infty$, $d\omega \in D_{\varepsilon,q}$ *with* $1-\lambda < \rho'$ *and* $\varepsilon < q(1-\sigma)$.

Remark. Now we show that the use of the sharp maximal $M^\#$ (see [**Ya**] for a definition) is not well adapted to weaken the weight condition in the Muckenhoupt-Wheeden inequality

$$\|I_s f\|_{L^q_\omega} \le c\|M_s f\|_{L^q_\omega}. \tag{1}$$

Indeed such a purpose is based on the two inequalities:

$$(I_s f)^\# \le c(M_s f); \tag{2}$$

$$\|g\|_{L^q_\omega} \le c\|g^\#\|_{L^q_\omega}. \tag{3}$$

Inequality (2) is valid for all functions f with $(I_s f) \in L^1_{\text{loc}}$ and was proved in [**Ad**]. Although (3) is well known to be true for $w \in A_\infty$, Yabuta [**Ya**] had obtained such an inequality with a weak condition he denoted as $w \in C_r$ (with $r > q$). Thus we think get (1) with this last condition. But since $M_s f \le c I_s f = h$ then

$$\|h^\#\|_{L^q_\omega} \le c\|h\|_{L^q_\omega} \tag{4}$$

It was proved in [**Ya**] that condition like (4) implies necessarily $w \in A_\infty$.

2. Some Lemmas

We first state two Lemmas we need and then we give their proofs.

Lemma 1. *Let* f *be a nonnegative (*$d\mu$*-locally integrable) function. Then*

$$(M^d_{T,\omega,\mu} f)(.) \le (M^d_\omega(Tf\mu))(.).$$

Lemma 2. *Let* $T = T_a$ *be an operator with the kernel* $K = K_a$ *satisfying* $\mathcal{H}i)$*-ii), and let* $d\nu$ *be a positive Borel measure.*

A) If $0 < \sigma \le 1$ *then there is* $C = C(a,n) > 0$ *so that for all cubes* Q

$$\Big(\frac{a(Q)}{|Q|}|Q|_\nu\Big)\mathbb{1}_Q(.) \le C\,(T\mathbb{1}_Q\nu)(.)\mathbb{1}_Q(.).$$

B) Let $m \ge 1$. *There is* $C = C(a,n,m) > 0$ *so that*

$$(T\mathbb{1}_Q\nu)(.)\mathbb{1}_{mQ}(.) \le C\,[S_1(.) + S_2(.)]$$

where

$$S_1(.) = \Big(\frac{a(Q)}{|Q|}|Q|_\nu\Big)\,1\!\!1_{mQ}(.)$$

and

$$S_2(.) = \frac{a(Q)}{|Q|}\Big(\sum_{j\geq 0} 2^{-jn[\lambda-1]}\int_{Q\cap\{|y-.|\sim 2^{-j}|Q|^{\frac{1}{n}}\}}\nu\Big)\,1\!\!1_{mQ}(.)$$

C) Let $m \geq 4$. There is $C = C(a,n,m) > 0$ so that

$$(T1\!\!1_Q\nu)(.)1\!\!1_{(mQ)^c}(.) \leq C\,|Q|_\nu\Big(\sum_{j\geq 0}\frac{a(2^jQ)}{|2^jQ|}\,1\!\!1_{2^jQ}\Big).$$

Proof of Lemma 1:

Let Q be a dyadic cube. Then we have

$$\int_Q (Tf\mu)d\omega = \int_{\mathbb{R}^n}[T^*1\!\!1_Q\omega]fd\mu \geq \int_{3Q}[T^*1\!\!1_Q\omega]fd\mu.$$

Diving by $|Q|_\omega$ this inequality and taking the supremum, we obtain the conclusion. ■

Proof of Lemma 2:

A) Let Q be a cube with center x_0 and length $2R > 0$. Then $|x-y| \leq c2R$ for all $x,y \in Q$ with $c = c(n)$. We obtain

$$\begin{aligned}\Big(\frac{a(Q)}{|Q|}|Q|_\nu\Big)1\!\!1_Q(y) &\leq c_1(a,n)\Big(\int_{x\in Q}\frac{a(y,c2R)}{R^n}d\nu\Big)1\!\!1_Q(y)\\ &\leq c_2(a,n)\Big[\int_{x\in Q}\Big(\frac{|x-y|}{c2R}\Big)^n\frac{1}{|x-y|^n}a\\ &\qquad\Big(y,\frac{c2R}{|x-y|}|x-y|\Big)d\nu(x)\Big]1\!\!1_Q(y)\\ &\leq c_3(a,n)\Big[\int_{x\in Q}\Big(\frac{|x-y|}{c2R}\Big)^{n[1-\sigma]}\frac{a(y,|x-y|)}{|x-y|^n}d\nu(x)\Big]\\ &\qquad 1\!\!1_Q(y).\end{aligned}$$

Since $0 < \sigma \leq 1$ and $|x-y| < 2cR$ we get

$$\Big(\frac{a(Q)}{|Q|}|Q|_\nu\Big)1\!\!1_Q(y) \leq c(a,n)\,(T_a1\!\!1_Q\nu)(y)1\!\!1_Q(y).$$

B) Let Q be a cube as in part A, and let $m \geq 1$. We can write

$$(T_a 1\!\!1_Q \nu)(y) 1\!\!1_{mQ}(y) = S_1(y) + S_2(y)$$

where

$$S_1(y) = \left(\int_{Q \cap \{R \leq |y-x|\}} \frac{a(y, |x-y|)}{|x-y|^n} d\nu(x) \right) 1\!\!1_{mQ}(y)$$

and

$$S_2(y) = \left(\int_{Q \cap \{|y-x| < R\}} \frac{a(y, |x-y|)}{|x-y|^n} d\nu(x) \right) 1\!\!1_{mQ}(y).$$

For $S_1(y)$ we can observe that for $y \in (mQ)$ and $x \in Q$ then $B(y, |x-y|) \subset c(n)Q$ and consequently we get

$$\begin{aligned} S_1(y) &= \left(\int_{Q \cap \{R \leq |y-x|\}} \frac{a(y, |x-y|)}{|x-y|^n} d\nu(x) \right) 1\!\!1_{mQ}(y) \\ &\leq c_1(a,n,m) \frac{a(Q)}{|Q|} \left(\int_{Q \cap \{R \leq |y-x|\}} d\nu(x) \right) 1\!\!1_{mQ}(y) \\ &\leq c_1(a,n,m) \left(\frac{a(Q)}{|Q|} |Q|_\nu \right) 1\!\!1_{mQ}(y). \end{aligned}$$

For $S_2(y)$ with $R = |Q|^{\frac{1}{n}}$, we have

$$\begin{aligned} S_2(y) &= \left[\sum_{j \geq 0} \int_{Q \cap \{|y-x| \sim 2^{-j} R\}} \frac{a(y, |x-y|)}{|x-y|^n} d\nu(x) \right] 1\!\!1_{mQ}(y) \\ &\leq c_1(a,n) \sum_{j \geq 0} (2^{-j}R)^{-n} a(y, 2^{-j}R) \left(\int_{Q \cap \{|y-x| \sim 2^{-j} R\}} d\nu(x) \right) 1\!\!1_{mQ}(y) \\ &\leq c_2(a,n,m) \frac{a(Q)}{|Q|} \left[\sum_{j \geq 0} 2^{-jn[\lambda - 1]} \left(\int_{Q \cap \{|y-x| \sim 2^{-j} R\}} d\nu(x) \right) \right] 1\!\!1_{mQ}(y). \end{aligned}$$

The proof of the part C) lies on the same ideas. We leave the detail for the reader. ■

3. Proofs of main results

Preliminaries for the proof of Theorem 1.

Let f be a nonnegative function bounded and with support compact. By the hypothesis (S_T) then we can observe that

$$\|Tf\mu\|_{L^q_\omega} < \|f\|_{L^\infty} \|T 1\!\!1_{|x|<R} \mu\|_{L^q_\omega} < \infty.$$

Since $Tf\mu$ is semicontinuous, so for each $k \in \mathbb{Z}$ the open set $\Omega_k = \{Tf\mu > 2^k\}$ can be written as $\bigcup_j Q_{jk}$ where the Q_{jk} are the dyadic cubes maximal among those dyadic cubes Q satisfying $(RQ) \subset \Omega_k$. Choosing $R \geq 3$ sufficiently large (depending only on the dimension n), we obtain

$$\Omega_k = \bigcup_j Q_{jk} \text{ where } \operatorname{int}(Q_{jk}) \cap \operatorname{int}(Q^k_{j'}) = \emptyset \text{ if } j \neq j'$$

and

$$(RQ_{jk}) \subset \Omega_k \text{ and } (3RQ_{jk})\Omega_k^c \neq \emptyset \text{ for all } k, j \text{ (Whitney condition).}$$

Let fix $m \geq 2$ which we will choose later, and let define $E_{jk} = Q_{jk} \cap (\Omega_{k+m-1} \backslash \Omega_{k+m})$. Using the hypothesis $\mathcal{H}$ on the kernel K, and the Whitney condition we get

Lemma.
1) There is $C = C(K, R) > 0$ *so that for all* k, j

$$(Tf\mu)(.)\mathbb{1}_{(3Q_{jk})^c}(.) \leq C2^k \mathbb{1}_{Q_{jk}}(.).$$

2) For a suitable choice of the integer m

$$2^k \mathbb{1}_{E_{jk}}(.) \leq (Tf\mu\mathbb{1}_{Q_{jk}})(.)\mathbb{1}_{E_{jk}}(.)$$

therefore

$$|E_{jk}|_\omega \leq 2^{-k} \int_{3Q_{jk}} [T^*\omega\mathbb{1}_{Q_{jk}}] f d\mu.$$

By the Whitney condition one can find at least one z which belongs to $(3RQ_{jk}) \cap \Omega_k^c$. It first implies: $(Tf\mu)(z) \leq 2^k$.
Also for $x \in Q_{jk}$ and $y \in (3Q_{jk})^c$ $|z - y| \leq C_1|x - y|$ for some $C_1 = C_1(n, R) > 0$. So for another constant $C_2 = C_2(C_1) > 0$: $K(x, y) \leq C_2\, K(z, y)$.

The conclusion 1) appears from these two inequalities, indeed we have

$$\begin{aligned}(Tf\mu\mathbb{1}_{(3Q_{jk})^c})(x)\mathbb{1}_{Q_{jk}}(x) &= \Big(\int_{(3Q_{jk})^c} K(x, y)f(y)d\mu(y)\Big)\mathbb{1}_{Q_{jk}}(x) \\ &\leq C_2 \Big(\int_{\mathbb{R}^n} K(z, y)f(y)d\mu(y)\Big)\mathbb{1}_{Q_{jk}}(x) \\ &\leq C_2(Tf\mu)(z)\mathbb{1}_{Q_{jk}}(x) \leq C_2 2^k \mathbb{1}_{Q_{jk}}(x).\end{aligned}$$

The part 2) will be a direct consequence of 1). Since $E_{jk} = Q_{jk} \cap (\Omega_{k+m-1} \backslash \Omega_{k+m})$ and $\Omega_k = \{Tf\mu > 2^k\}$ then choosing $m \geq 2$ and $2^{m-2} \geq C_2$ we get

$$\begin{aligned}(Tf\mu 1\!\!1_{(3Q_{jk})})(x)1\!\!1_{E_{jk}}(x) &= (Tf\mu)(x)1\!\!1_{E_{jk}}(x) - (Tf\mu 1\!\!1_{(3Q_{jk})^c})(x)1\!\!1_{E_{jk}}(x)\\ &> \Big(2^{k+m-1} - 2^k C_2\Big)(x)1\!\!1_{E_{jk}}(x)\\ &> \Big(2^{k+m-1} - 2^{k+m-2}\Big)(x)1\!\!1_{E_{jk}}(x) > 2^k 1\!\!1_{E_{jk}}(x).\end{aligned}$$

So, by integration with respect to the measure $d\omega$, this involves

$$\begin{aligned}2^k|E_{jk}|_\omega &\leq \int_{E_{jk}} [Tf\mu 1\!\!1_{3Q_{jk}}]d\omega\\ &\leq \int_{3Q_{jk}} [T^*\omega 1\!\!1_{E_{jk}}]fd\mu \leq \int_{3Q_{jk}} [T^*\omega 1\!\!1_{Q_{jk}}]fd\mu\end{aligned}$$

Proof of the Theorem 1:

Using this Lemma, now we prove the inequality

$$\|Tf\mu\|_{L^q_\omega} \leq c\,\|M^d_{T,\omega,\mu}\mu\|_{L^q_\omega}.$$

Let $\beta \in]0,1[$ whose value is to be specified later in the course of the proof. Then we get

$$\begin{aligned}\|Tf\mu\|^q_{L^q_\omega} &\leq c\sum_{k,j} 2^{kq}|E_{jk}|_\omega \quad c = c(q,m)\\ &\leq c\Bigg[\sum_{k,j;\,|E_{jk}|_\omega \leq \beta|Q_{jk}|_\omega} + \sum_{k,j;\,\beta|Q_{jk}|_\omega < |E_{jk}|_\omega}\Bigg]2^{kq}|E_{jk}|_\omega\\ &\leq c\beta\sum_{k,j} 2^{kq}|Q_{jk}|_\omega + c\sum_{k,j;\,\beta|Q_{jk}|_\omega < |E_{jk}|_\omega}\\ &\quad |E_{jk}|_\omega\left(\frac{1}{|E_{jk}|_\omega}\int_{3Q_{jk}}[T^*\omega 1\!\!1_{Q_{jk}}]fd\mu\right)^q\\ &\leq c'\beta\|Tf\mu\|^q_{L^q_\omega} + c\beta^{-q}\sum_{k,j}\\ &\quad |E_{jk}|_\omega\left(\frac{1}{|Q_{jk}|_\omega}\int_{3Q_{jk}}[T^*\omega 1\!\!1_{Q_{jk}}]fd\mu\right)^q\\ &\leq c'\beta\|Tf\mu\|^q_{L^q_\omega} + c\beta^{-q}\sum_{k,j}\int_{E_{jk}}(M^d_{T,\omega,\mu}f\mu)^q\end{aligned}$$

$$\leq c'\beta\|Tf\mu\|^q_{L^q_\omega} + c\beta^{-q}\|M^d_{T,\omega,\mu}f\mu\|^q_{L^q_\omega}.$$

Since $\|Tf\mu\|^q_{L^q_\omega} < \infty$, then choosing $\beta \in]0,1[$ and $c'(q,m)\beta < \frac{1}{2}$, we have

$$\|Tf\mu\|^q_{L^q_\omega} \leq c(q,m)\|M^d_{T,\omega,\mu}f\mu\|^q_{L^q_\omega}.$$

Therefore the Theorem is proved for each bounded function f with support compact. For a general nonnegative function f, we can also obtain the same conclusion by using the monotone convergence theorem. ■

Proof of proposition 2:

Since the first inequality is proved in Theorem 1, then we are reduced to get the converse inequality

$$(*) \qquad \|M^d_{T,\omega,\mu}f\mu\|_{L^q_\omega} \leq C\|Tf\mu\|_{L^q_\omega}.$$

By Lemma 1: $(M^d_{T,\omega,\mu}f)(.) \leq (M^d_\omega Tf\mu)(.)$, then the conclusion appears if we have

$$\|M^d_\omega g\|_{L^q_\omega} \leq C\|g\|_{L^q_\omega}.$$

By the well known arguments (covering lemma using dyadic cubes and interpolation) then this last maximal inequality is valid for all $q > 1$. The same inequality $(*)$ is also true for all q with $0 < q \leq 1$ by the means of the extra-hypothesis $M^d_\omega(Tg)(.) \leq c(n,K,\omega)\,(Tg)(.)$. ■

Proof of Theorem 3:

Let f be a nonnegative $d\mu$-locally integrable function. Since $(T^*\nu) \approx (T\nu)$ then taking $d\nu = d\omega$ in part A) of Lemma 2, then it appears that for each dyadic cube Q

$$\frac{a(Q)}{|Q|}\int_Q f d\mu = |Q|_\omega{}^{-1}\int_Q\Big(\frac{a(Q)}{|Q|}|Q|_\omega\Big)f d\mu$$
$$\leq c(a,n)|Q|_\omega{}^{-1}\int_{3Q}\Big(T^*_a\omega 1\!\!1_Q\Big)f d\mu.$$

Hence, we have $(M^d_\Phi f\mu) \leq c(a,n)\,(M^d_{T_a,\omega,\mu}f)$.

Conversely in order to get $(M^d_{T_a,\omega,\mu}f\mu) \leq C(a,n)\,(M_\Phi f\mu)$, it is sufficient to get

$$(\$) \qquad (T^*_a 1\!\!1_Q\omega)(.)1\!\!1_{3Q}(.) \leq C(a,n)\Big(\frac{a(cQ)}{|cQ|}|cQ|_\omega\Big)1\!\!1_{cQ}(.)$$

where $c = c(n) \geq 3$. By part B) of Lemma 3, the first member of ($) is essentially dominated by the sum of

$$S_1(.) = \Big(\frac{a(Q)}{|Q|}|Q|_\omega\Big)1\!\!1_{3Q}(.)$$

and

$$S_2(.) = \frac{a(Q)}{|Q|}\Big(\sum_{j\geq 0} 2^{-jn[\lambda-1]}\int_{Q\cap\{|y-.|\sim 2^{-j}R\}} d\omega(y)\Big)\mathbb{1}_{3Q}(.).$$

So it is clear, that it remains to estimate $S_2(.)$.

If $\lambda = 1$, then we immediately get

$$\begin{aligned} S_2(x) &= \frac{a(Q)}{|Q|}\Big(\sum_{j\geq 0}\int_{Q\cap\{|y-x|\sim 2^{-j}R\}} d\omega(y)\Big)\mathbb{1}_{3Q}(x) \\ &\leq c(a,n)\Big(\frac{a(Q)}{|Q|}|Q|_\omega\Big)\mathbb{1}_{3Q}(x). \end{aligned}$$

Now for $\lambda \in]0,1[$ we use the hypothesis $d\omega RD_\rho$ with $1-\lambda < \rho$. We also note that for $x \in (3Q)$ then $B(x,R) \subset (c_1Q)$ for a constant $c_1 = c_1(n) \geq 3$. Therefore we obtain

$$\begin{aligned} S_2(x) &\leq \frac{a(Q)}{|Q|}\Big(\sum_{j\geq 0} 2^{-jn[\lambda-1]}|B(x,2^{-j}R)|_\omega\Big)\mathbb{1}_{3Q}(x) \\ &\leq c(\omega)\frac{a(Q)}{|Q|}\Big(\sum_{j\geq 0} 2^{-jn[\lambda-1+\rho]}\Big)|B(x,R)|_\omega\mathbb{1}_{3Q}(x) \\ &\leq c'(a,\omega)\frac{a(Q)}{|Q|}|c_1Q|_\omega\mathbb{1}_{3Q}(x) \\ &\leq c''(a,\omega)\frac{a(c_1Q)}{|c_1Q|}|c_1Q|_\omega\mathbb{1}_{c_1Q}(x). \ \blacksquare \end{aligned}$$

Proof of Theorem 5:

It is clear that (2) is a necessary condition for (P_T). To get the condition (1) we first note that for $|x| > R$ $(R > 0)$ and $|y| < \frac{1}{2}R$ then $|x-y| \approx |x|$, and consequently taking $f = \mathbb{1}_{B(0,R)}$ in inequality (P_T) we have

$$\begin{aligned} \infty > |B(0,R)|_\mu^{\frac{1}{p}} &\geq C\Big\|\Big(\int_{|y|<\frac{1}{2}R}\frac{a(x,|x-y|)}{|x-y|^n}d\mu(y)\Big)\mathbb{1}_{|x|>R}\Big\|_{L^q_\omega} \\ &\geq C|B(0,\frac{1}{2}R)|_\mu\Big[\int_{|x|>R}\Big(\frac{a(x,|x|)}{|x|^n}\Big)^q d\omega(x)\Big]^{\frac{1}{q}}. \end{aligned}$$

Now we suppose the conditions (1) and (2) are satisfied. The keys for the converse are the following:

(i) $$\|T\mathbb{1}_{B(0,R)}\mu\|_{L^q_\omega} < c(R) < \infty \text{ for all } R > 0;$$

(ii) $$M_\Phi \mu : L^p_\mu \to L^q_\omega.$$

Indeed by (i) and (ii) we have

$$\begin{aligned}\|Tf\mu\|_{L^q_\omega} &\le c\|M_{T,\omega,\mu}f\|_{L^q_\omega} \text{ by Theorem 1 and by using (i)}\\ &\le c\|M_\Phi f\mu\|_{L^q_\omega} \text{ by Theorem 3 since } d\omega \in RD_\rho \text{ with } 1-\lambda<\rho\\ &\le c\|f\|_{L^p_\mu} \text{ by } (ii).\end{aligned}$$

To get the point (i), we note that $\|(T 1\!\!1_{B(0,R)}\mu)1\!\!1_{|x|<2R}\|_{L^q_\omega} < \infty$. On the otherhand, we have

$$\begin{aligned}\left\|(T 1\!\!1_{B(0,R)}\mu)1\!\!1_{|x|>2R}\right\|_{L^q_\omega} &\le c\left\|\left(\int_{|y|<R}\frac{a(x,|x-y|)}{|x-y|^n}d\mu(y)\right)1\!\!1_{|x|>2R}\right\|_{L^q_\omega}\\ &\le c'|B(0,R)|_\mu\left[\int_{|x|>2R}\Big(\frac{a(x,|x|)}{|x|^n}\Big)^q d\omega(x)\right]^{\frac{1}{q}}<\infty.\end{aligned}$$

By a result in [**Ra1**], a sufficient (and necessary) condition for the embedding (ii) is

$$\left\|(M_\Phi\sum_k \varepsilon_k 1\!\!1_{Q_k}\mu)1\!\!1_{\bigcup Q_k}\right\|_{L^q_\omega} \le C\left\|\sum_k \varepsilon_k 1\!\!1_{Q_k}\mu\right\|_{L^p_\mu}$$

and $\|(M_\Phi 1\!\!1_Q\mu)1\!\!1_Q\|_{L^q_\omega} \le C\|1\!\!1_Q\|_{L^p_\mu}$ if $p \le q$. By Lemma 2 A) then $(M_\Phi f\mu) \le c(Tf\mu)$ and consequently the condition (2) in Theorem 5 implies the above one. ■

Proof of Proposition 6:

To prove the first part of Proposition 6, we suppose

$$\|(T 1\!\!1_Q\mu)1\!\!1_Q\|_{L^q_\omega} \le A\|1\!\!1_Q\|_{L^p_\mu} \text{ for all cubes } Q.$$

Since $1<p\le q$ this condition implies $M_\Phi\mu : L^p_\mu \to L^q_\omega$. And as above to get (i) it is sufficient to prove

$$\|(T 1\!\!1_Q\mu)1\!\!1_{(mQ)^c}\|_{L^q_\omega} \le C\|1\!\!1_Q\|_{L^p_\mu}$$

with a constant $m \ge 4$. Using the fact that $d\mu \in RD_{\rho'}$ for some $\rho'>0$, then by Lemma 2 (part C) we get

$$\left\|(T 1\!\!1_Q\mu)1\!\!1_{(mQ)^c}\right\|_{L^q_\omega} \le c_1|Q|_\mu\sum_{j\ge 0}\frac{a(2^jQ)}{|2^jQ|}|2^jQ|_\omega^{\frac{1}{q}}$$

$$\begin{aligned}
&\leq c_2|Q|_\mu \sum_{j\geq 0} |2^jQ|_\mu^{-1} \left\| (T\mathbb{1}_{2^jQ}\mu)\mathbb{1}_{(2^jQ)} \right\|_{L^q_\omega} \\
&\leq c_2 A |Q|_\mu^{\frac{1}{p}} \sum_{j\geq 0} \left(\frac{|Q|_\mu}{|2^jQ|_\mu} \right)^{1-\frac{1}{p}} \\
&\leq c_3 A |Q|_\mu^{\frac{1}{p}}.
\end{aligned}$$

For the second part of this proposition, the point is to note that $M_\Phi\mu : L^p_\mu \to L^q_\omega$ is equivalent to

$$\frac{a(Q)}{|Q|} |Q|_\mu^{1-\frac{1}{p}} |Q|_\omega^{\frac{1}{q}} < A < \infty$$

whenever $d\mu \in A_\infty$ (see [**Pe**]) or $d\mu \in RD_{\infty'}$ with $1-\lambda < \rho'$ (see [**Ra2**]). ■

Proof of Proposition 7:

It is clear that a necessary condition for (P_T) is

$$(*) \qquad \left\| \sum_k \varepsilon_k (T\mathbb{1}_{Q_k}\mu)\mathbb{1}_{(mQ_k)} \right\|_{L^q_\omega} \leq A \left\| \sum_k \varepsilon_k \mathbb{1}_{Q_k} \right\|_{L^p_\mu}$$

with $m \geq 4$ and for all cubes Q, Q_k and all $\varepsilon_k > 0$. Conversely we suppose this condition be satisfied and $d\mu \in \widetilde{RD}(p)$. Once we have

$$(**) \qquad \left\| \sum_k \varepsilon_k (T\mathbb{1}_{Q_k}\mu)\mathbb{1}_{(mQ_k)^c} \right\|_{L^q_\omega} \leq cA \left\| \sum_k \varepsilon_k \mathbb{1}_{Q_k} \right\|_{L^p_\mu}$$

then (i) and (ii) hold as in proof of Theorem 5, and consequently the inequality (P_T) is satisfied. Now using Part C) of Lemma 2, the above condition $(*)$ and the hypothesis $d\omega \in \widetilde{RD}(p)$ we have

$$\begin{aligned}
\mathcal{S} &= \left\| \sum_k \varepsilon_k (T\mathbb{1}_{Q_k}\mu)\mathbb{1}_{(mQ_k)^c} \right\|_{L^q_\omega} \\
&\leq c_1 \left\| \sum_k \varepsilon_k \sum_{j\geq 0} \frac{a(2^jQ_k)}{|2^jQ_k|} |Q_k|_\mu \mathbb{1}_{(2^jQ_k)} \right\|_{L^q_\omega} \quad \text{by part C of Lemma 2} \\
&\leq c_2 \left\| \sum_k \varepsilon_k \sum_{j\geq 0} \left(\frac{|Q_k|_\mu}{|2^jQ_k|_\mu} \right) (T\mathbb{1}_{2^jQ_k}\mu)\mathbb{1}_{(2^jQ_k)} \right\|_{L^q_\omega}
\end{aligned}$$

$$\leq c_2 A \Big\| \sum_{j\geq 0} \sum_k \varepsilon_k \Big(\frac{|Q_k|_\mu}{|2^j Q_k|_\mu} \Big) 1\!\!1_{(2^j Q_k)} \Big\|_{L^p_\mu} \text{ by the condition } (*)$$

$$\leq c_3 A \Big\| \sum_k \varepsilon_k 1\!\!1_{Q_k} \Big\|_{L^p_\mu} \quad \text{since } d\mu \in \widetilde{RD}(p).$$

It is also clear that a necessary condition for (P_T) is

$$(**') \qquad \Big\| \sum_k \varepsilon_k \Big(\frac{a(Q_k)}{|Q_k|} |Q_k|_\mu \Big) 1\!\!1_{Q_k} \Big\|_{L^q_\omega} \leq A \Big\| \sum_k \varepsilon_k 1\!\!1_{Q_k} \Big\|_{L^p_\mu}.$$

Conversely we assume this condition be satified and $d\mu \in D_{\varepsilon',p} \cap RD_{\rho'}$ with $1-\lambda < \rho'$ and $\varepsilon' < p\rho'$. It is sufficient to get the conditions in the first part of the present Proposition. As in the proof of Theorem 3 by using part A) of Lemma 2) and since $d\mu \in D_\infty$ then

$$(T 1\!\!1_{Q_k} \mu) 1\!\!1_{(mQ_k)} \leq c \Big(\frac{a(Q_k)}{|Q_k|} |Q_k|_\mu \Big) 1\!\!1_{(mQ_k)}$$

and consequently

$$\Big\| \sum_k \varepsilon_k (T 1\!\!1_{Q_k} \mu) 1\!\!1_{(mQ_k)} \Big\|_{L^q_\omega} \leq c \Big\| \sum_k \varepsilon_k \Big(\frac{a(Q_k)}{|Q_k|} |Q_k|_\mu \Big) 1\!\!1_{(mQ_k)} \Big\|_{L^q_\omega}$$
$$\leq c \Big\| \sum_k \varepsilon_k 1\!\!1_{(mQ_k)} \Big\|_{L^p_\mu}.$$

Now using $d\mu \in D_{\varepsilon',p} \cap RD_{\rho'}$ with $\varepsilon' < p\rho'$ we can get the condition $d\mu \in \widetilde{RD}(p)$ as follow:

$$\mathcal{S} = \Big\| \sum_k \varepsilon_k \sum_{j\geq 0} \Big(\frac{|Q_k|_\mu}{|2^j Q_k|_\mu} |Q_k|_\mu \Big) 1\!\!1_{(2^j Q_k)} \Big\|_{L^p_\mu}$$
$$\leq c_1 \sum_{j\geq 0} 2^{-jn\rho'} \Big\| \sum_k \varepsilon_k 1\!\!1_{(2^j Q_k)} \Big\|_{L^p_\mu}$$
$$\leq c_2 \sum_{j\geq 0} 2^{-jn[\rho' - \frac{1}{p}\varepsilon']} \Big\| \sum_k \varepsilon_k 1\!\!1_{Q_k} \Big\|_{L^p_\mu} = c_3 \Big\| \sum_k \varepsilon_k 1\!\!1_{Q_k} \Big\|_{L^p_\mu}.$$

Finally we suppose $d\mu \in D_\infty \cap RD_{\rho'}$ and $d\omega \in D_{\varepsilon,q} \cap RD_\rho$ with $1-\lambda < \rho'$ and $\varepsilon < q(1-\sigma)$. It remains to get the above condition $(**)$. Thus we have

$$\mathcal{S} = \Big\| \sum_k \varepsilon_k (T 1\!\!1_{Q_k} \mu) 1\!\!1_{(mQ_k)^c} \Big\|_{L^q_\omega}$$

$$
\begin{aligned}
&\le c_1 \Big\| \sum_k \varepsilon_k \sum_{j\ge 0} \Big(\frac{a(2^jQ_k)}{|2^jQ_k|} |Q_k|_\mu \Big) \mathrm{1\!I}_{(2^jQ_k)} \Big\|_{L^q_\omega} \\
&\le c_2 \sum_{j\ge 0} 2^{-jn[1-\sigma]} \Big\| \sum_k \varepsilon_k \Big(\frac{a(Q_k)}{|Q_k|} |Q_k|_\mu \Big) \mathrm{1\!I}_{(2^jQ_k)} \Big\|_{L^q_\omega} \\
&\le c_3 \sum_{j\ge 0} 2^{-jn[1-\sigma-\frac{1}{q}\varepsilon]} \Big\| \sum_k \varepsilon_k \Big(\frac{a(Q_k)}{|Q_k|} |Q_k|_\mu \Big) \mathrm{1\!I}_{Q_k} \Big\|_{L^q_\omega} \\
&\le c_3 A \sum_{j\ge 0} 2^{-jn[1-\sigma-\frac{1}{q}\varepsilon]} \Big\| \sum_k \varepsilon_k \mathrm{1\!I}_{Q_k} \Big\|_{L^p_\mu} \\
&\le c_4 A \Big\| \sum_k \varepsilon_k \mathrm{1\!I}_{Q_k} \Big\|_{L^p_\mu}. \quad \blacksquare
\end{aligned}
$$

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