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MERGELYAN TYPE THEOREMS FOR SOME FUNCTION SPACES

Arne Stray

Abstract

Let F be a relatively closed subset of the unit disc D. If A is any of the Hardy spaces $H^p(D)$, $0 < p < \infty$, $\overline{A|_F}$ denotes the functions on F being uniform limits of elements from $H^p(D)$. Let $\tilde{F}$ consist of all $z \in D$ such that $|f(z)| \le \sup\{|f(z)| z \in F\}$ for any bounded analytic function in D. It is proved that $\overline{A|_F}$ consist of all functions f that can be decomposed as $f = u + v$, where u belongs to $H^p(D)$ and v is a uniformly continuous function on the set $\tilde{F}$, analytic at interior points of $\tilde{F}$.

Let A be a linear space of analytic functions and F a subset of the complex plane $\mathcal{C}$ such that each $f \in A$ is defined on F. We denote by $\overline{A|_F}$ the functions being uniformly approximable on F by sequences from A. The aim with this paper is to give a partial solution to problem 8.5 no. 2 in [**7**]. If A is any of the classical Hardy spaces $H^p(D)$, $0 < p < \infty$, our main result is that $\overline{A|_F}$ coincides (modulo the approximating space) with a well defined algebra of uniformly continuous analytic functions on F.

Before giving a precise formulation of the main result, we need some definitions.

Let $C_{ua}(F)$ denote the functions on F being analytic in its interior F^0 and admitting continuous extension to the extended complex plane $\mathcal{C} \cup \{\infty\}$. If F is a compact subset of $\mathcal{C}$ and P consists of the polynomials, a famous theorem of S. N. Mergelyan [**7**] can be formulated as

$$\overline{P|_F} = C_{ua}(\tilde{F})$$

where $\tilde{F}$ is the union of F and the bounded components of C/F.

Suppose now we replace P by the set $H(\mathcal{C})$ consisting of all entire functions. Also allow F to be a closed but possibly unbounded subset of $\mathcal{C}$. Then it can be proved that

$$\text{(I):} \qquad \overline{H(\mathcal{C})|_F} = H(\mathcal{C}) + C_{ua}(\tilde{F})$$

where $\tilde{F}$ is the union of F and certain components of C/F. A component V is to be included in $\tilde{F}$ if and only if $V \cup \{\infty\}$ is not arcwise connected in $C \cup \{\infty\}$. For details see [**8**], [**9**] and [**10**].

In general A may contain unbounded functions. For this reason it is natural to look for an identity like (I) if we seek to describe $\overline{A|F}$ in terms of uniformly continuous analytic functions. Let us use the notation $\|g\|_B = \sup\{|g(x)| : z \in B\}$ if g is a function defined on the set B. We also define the hull of F with respect to A:

$$\hat{F}^A = \{z : |f(z)| \le \|f\|_F, \quad f \in A\}.$$

We look for spaces A satisfying the following:

$$(*): \qquad \overline{A_{|F}} = A + C_{ua}(\hat{F}^A).$$

Our main result is the $(*)$ is valid for the classical Hardy sapces $H^P(D)$ in the unit disc D, $0 < p < \infty$, when F is any relatively closed subset of D. Also note that the two introductory examples are special cases of $(*)$.

We refer to [**3**] or [**5**] for the basic theory of $H^p(D)$, $0 < p \le \infty$. In particular $H^\infty(D)$ denotes the bounded analytic functions in D. If $F \subset D$ is relatively closed, let

$$\hat{F} = \{z \in D : |f(z)| \le \|f\|_F, \quad f \in H^\infty(D)\}.$$

Our main result is

Theorem 1. *If* $0 < p < \infty$, *then* $\overline{H^p(D)|_F} = H^p(D) + C_{ua}(\hat{F})$.

Proof of Theorem 1: If $f \in H^p(D)$ it is easy to find $\{f_n\} \subset H^\infty(D)$ such that $|f_n(z)| \le |f(z)|$ and $f_n(z) \to f(z)$ for $z \in D$. This shows that $\hat{F}^A = \hat{F}$ if $A = H^p(D)$, $0 < p < \infty$.

Let us first prove that $\overline{H^p(D)|_F} \subset H^p(D) + C_{ua}(\hat{F})$. If $g \in \overline{H^p(D)|_F}$ is bounded, we may assume

$$g = \sum_n f_n, \quad f_n \in H^p(D)$$

in the sense that $\sum_n \|f_n\|_F < \infty$.

There are two special classes of sets F where a short proof of the decomposition of g can be found. It may be instructive to consider these cases prior to the general proof.

Let us firs assume that F is a Farrell set for $H^p(D)$. (See [**8**] for definition and various properties of these sets). Then we can find polynomials p_n, $n = 1, 2$ such that

$$\|p_n\|_F \leq \|f_n\|_F + 2^{-n}$$

and

$$\|p_n - f_n\|_{H^p} \leq 2^{-n}$$

for $n = 1, 2, \ldots$. This gives a decomposition

$$g = \sum_n (f_n - p_n) + \sum_n p_n|_{\hat{F}}$$

as claimed.

In our second example, we assume that F can be written as a Blaschke sequence $S = \{\zeta_\nu\}$, meaning that

$$\sum_\nu 1 - |\zeta_\nu| < \infty. \tag{1}$$

Then it is well known that the Blashcke product

$$B(z) = \Pi \frac{|\zeta_\nu|}{\zeta_\nu} \frac{\zeta_\nu - z}{1 - \bar{\zeta}_\nu z}$$

converge in D. Using cofinite subproducts B_n of B, we can obtain

$$\|(1 - B_n) f_n\|_{H^p(D)} < 2^{-n}, \quad n = 1, 2, \ldots$$

and again we have a decomposition

$$g = \sum_n (1 - B_n) f_n + \sum_n B_n f_n|_F.$$

The geometric properties of the set F are quite different in the two cases just discussed. To clarify this, let $\overline{F}_{nt}$ denote the non tangential closure of F on the unit circle T. So $z \in \overline{F}_{nt}$ if $z \in T$ and z is a limit of a sequence $\{z_n\} \subset F$ satisfying $|z - z_n| \leq C(1 - |z_n|)$, $n = 1, 2, \ldots$, where C may depend only on z. We also define $\overline{F}_t = \overline{F} \cap \overline{T} \backslash \overline{F_{nt}}$.

It is well known that F is a Farrell set for $H^p(D)$ if and only if the linear measure $|\overline{F}_t|$ of $\overline{F}_t$ is zero ([**8**]). On the other hand, the condition (1) is easily seen to imply that $|\overline{F}_{nt}| = 0$.

We have thus obtained the decomposition of $\overline{H^p(D)|_F}$ in two rather different situations. The general proof will be divided into parts reflecting the "geometry" of the cases considered above. We shall argue as in the proof where F was a Farrell set, but the polynomials p_n will be replaced by functions from $H^p(D)$ having a uniformly continuous restriction to F. ■

The key part of the proof is an approximation argument related to the set $\overline{F}_{nt}$:

Lemma 1. *Given $f \in H^p(D)$, $0 < p < \infty$, and $\epsilon > 0$, there is an open set V and $f_1 \in H^p(D)$ with the following properties:*

(i) $\|f - f_1\|_{H^p(D)} < \epsilon$
(ii) $\|f_1\|_F < \|f\|_F + \epsilon$
(iii) f_1 *extends continuously to* $\overline{F} \cap V$
(iv) $|\overline{F}_{nt} \backslash V| = 0$.

For the moment we take Lemma 1 for granted. Consider the "tangential" part $\overline{F}_t$ of $\overline{F} \cap T$. Let K be a compact subset of $\overline{F}_t$. We assume there is a number $\delta = \delta(K)$ such that $I_z \cap K = \phi$ if $z \in F$, $|z| > 1 - \delta$, and I_z denotes the arc $I_z = \{e^{i\theta} : |z - e^{i\theta}| \leq 2(| - |z|)\}$.

By a construction due to J. Detraz ([**2**, Prop. 3.1]), we can find an outer function $G_K \in H^\infty(D)$ such that

$|G_K| \leq 1$ and $G_K(z) \to 0$ if $z \to K$ and $z \in F$. Moreover, G_K extends to be continuous and non zero at any $e^{i\theta} \in T \backslash K$.

We can find an increasing sequence of such sets $K_n \subset \overline{F}_t \backslash V$ with corresponding outer functions G_n, such that $|\overline{F}_t \backslash V \backslash K_n| \to 0$ and such that $G = \prod_n G_n$ has the following properties

(i) $0 < |G(z)| \leq 1$, $z \in D$
(ii) G extends to be continuous at any $e^{i\theta} \in V \cap T$.

It also follows from the construction of $\{G_n\}$ that

$$G(z) \to 0 \text{ if } z \in F \text{ and } z \to z_0 \in \cup_n K_n.$$

Consider finally the set

$$L = (\overline{F} \bigcap T) \backslash V \backslash \bigcup_n K_n.$$

It is evident that the linear meausre $|L|$ of L is zero. By a general version of the Rudin-Carleson theorem ([**2**]) there is $H \in H^\infty(D)$ with continuous extension to $L \cup (T \setminus \overline{L})$ such that $H \neq 0$ in D and $H = 0$ on L.

For $n = 1, 2, \ldots$ we consider the functions U_n in $H^p(D)$ given by

$$U_n = G^{\frac{1}{n}} H^{\frac{1}{n}} f_1$$

where f_1 satisfies the conclusions of Lemma 1.

It follows from the construction of f_1, G and H, that $U_n|_F$ is uniformly continuous. This implies that $U_n|_{\hat{F}} \in C_{ua}(\hat{F})$, by the maximum principle. To be a little bit more specific, suppose $z_0 \in \overline{F} \cap T$ and that $U_n(z) \to 0$ as $z \to z_0$ and $z \in F$. Then $|U_n| < \epsilon$ in $F \cap \Delta(z_0)$, for some disc centered at z_0. Choose a polynomial p peaking at z_0 such that $|U_n p| < \epsilon$ on F. Then if $p(z_0) = 1$, we have

$$\limsup_{\substack{z \to z_0 \\ z \in \hat{F}}} |U_n(z)| = \limsup_{\substack{z \to z_0 \\ z \in \hat{F}}} |U_n(z) p(z)| \leq \epsilon$$

since $\|U_n p\|_{\hat{F}} \leq \|U_n p\|_{\overline{F}} \leq \epsilon$.

We turn to the proof of Theorem 1. If $f \in H^p(D)$ and $\epsilon > 0$ is given, we have shown (modulo proving Lemma 1) that there is a function $U = U_n \in H^p(D) \cap C_{ua}(\hat{F})$ with n so large that

$$\|f - U\|_{H^p(D)} < \epsilon$$
$$\|U\| \leq \|f\|_F + \epsilon.$$

The proof of Theorem 1 now follows the introductory argument we gave in the special case where F is a Farrell set.

Let us finally prove Lemma 1. We may assume that f is bounded in D.

So given $f \in H^\infty(D)$, and $\epsilon > 0$, we consider a compact set $K \subset \overline{F}_{nt}$. We shall require several properties of K related to f. If $0 < \alpha < \pi$, $T(\theta, \alpha)$ denotes the cone in D with opening angle α, terminating at $e^{i\theta}$, and being symmetric with respect to the radius $\{re^{i\theta}, 0 \leq r < 1\}$. We assume that

$$f(e^{i\theta}) = \lim f(z)$$

holds uniformly in $e^{i\theta} \in K$ as $z \to e^{i\theta}$ inside $T(\theta, \alpha)$. Now fix $p \in (0, \infty)$. Since $f \in H^p(D)$, the radial limits $f(e^{i\theta})$, $0 \leq \theta < 2\pi$, belong to $L^p(d\theta)$. We assume that K is included in the Lebesgue set for f and that

$$\frac{1}{2r} \int_{\theta - r}^{\theta + r} |f(e^{i\varphi}) - f(e^{i\theta})|^p \, d\varphi \to 0 \tag{3}$$

uniformly in $e^{i\theta} \in K$ as $r \to 0$. Such a set K can be found with $|\overline{F}_{nt} \backslash K|$ as small we please.

Fix $\delta > 0$ so small that $|f(e^{i\theta}) - f(z)| < \epsilon$ if $e^{i\theta} \in K$, $|z| > 1 - \delta$ and $z \in T(\theta, \alpha)$. We are now in a convenient position for applying Vitushkin's scheme for approximation (see [**13**] or [**4**]). Let $\{\Delta_j\}_{j=1}^N$ be a finite collection of open discs with centers $z_j \in K$ and a common radius $r < \delta$. Following Vitushkin's scheme, let $\varphi_j \in C_0^1(\Delta_j)$ be chosen such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in $\Delta_j^1 = \left\{z : |z - z_j| < \frac{r}{2}\right\}$. As a preliminary approximation to f we define

$$f_K = f - G_K$$

where $G_K = \sum_j T_{\varphi j}(f - f(z_j)) - r_j$ is a finite sum which we shall explain in some detail.

We assume f is defined outside of D by $f(\overline{z}^{-1}) = f(z)$. For general properties of the T_φ-operator we refer to [**11**] or [**4**, page 30]. Here we only note that

$$\begin{aligned} T_{\varphi j}(f - f(z_j))(\varsigma) &= \varphi_j(\varsigma)(f(\varsigma) - f(z_j)) \\ &\quad - \frac{1}{\pi} \int_{\Delta_j}\!\!\int \frac{f(z) - f(z_j)}{z - \varsigma} \frac{\partial \varphi_j}{\partial \overline{z}} \, dx \, dy(z) \\ &= U_j + V_j \text{ say.} \end{aligned}$$

We assume $\left|\frac{\partial \varphi_j}{\partial \overline{z}}\right| \le \frac{A}{r}$, where A is a numerical constant. Since $f \in H^p(D)$, we have in particular that $f \in L^p(dx\,dy)$ locally. Therefore the convolution term V_j is continuous as a function of ς. If α is close to π, Hölders inequality gives that

$$|V_j(\varsigma)| \le \epsilon, \quad \varsigma \in \mathcal{C} \quad j = 1, 2 \dots N.$$

Note also that V_j is analytic outside Δ_j. According to Vitushkin's scheme, the functions r_j should be analytic outside a compact subset of $\Delta_j \backslash \overline{D}$ and with the property that $(V_j - r_j)(\varsigma)$ has a zero of order 3 at ∞. In addition we should require

(4): $$\|r_j\|_\infty \le A_1 \|V_j\|_\infty \le A_1 \epsilon, \quad j = 1, 2, \dots, N$$

where A_1 is a numerical constant. In our simple situation, the existence of $\{r_j\}$ is rather evident ([**4**, page 210–214]). From the individual bounds (3), it is part of Vitushkin's scheme that

(5): $$\left\| \sum_{j=1}^N (V_j - r_j) \right\|_\infty \le A_2 \epsilon$$

for some numerical constant A_2. We have not claimed that $\{\Delta_j^1\}_{j=1}^N$ cover all of K. In fact we shall assume that $\Delta_j \cap \Delta_k = \phi$ if $j \neq k$. In addition we assume that

$$\left| K \cap \bigcup_1^N \Delta_j^1 \right| \geq A_3 |K|$$

for some numerical constant A_3, where $\Delta_j^1 = \left\{ z : |z - z_j| \leq \frac{r}{2} \right\}$. We remark that $f_K = f - G_K$ is analytic near $\Delta_j^1 \cap T$ for $1 \leq j \leq N$. This is seen by writing

$$f_k = (f - U_j) - V_j - \sum_{i \neq j} (U_i + V_i) + \sum_{i=1}^n r_i$$

and inspecting these four terms separately.

Note that the (Fatou) boundary values $f(z_j)$ satisfy $|f(z_j)| \leq \|f\|_F$. Since

$$f_K = f\left(1 - \sum \varphi_j\right) + \sum_j \varphi_j f(z_j) + \sum_j (V_j - r_j)$$

we have

$$\|f_K\|_F \leq \|f\|_F + A_2 \epsilon. \tag{6}$$

From (3) and (5) we also get

$$\|f - f_K\|_{H^p(D)} = \|G_K\|_{H^p(D)} \leq (1 + A_2)\epsilon$$

if r is sufficiently small.

The function f_K satisfies the conditions for f_1 in Lemma 1 except that f_K is only analytic (and hence continuous) near a subset P_K of K. But since $|P_K| \geq A_3 |K|$, Lemma 1 follows by repeating our construction countably many times. The main reason why repetition works, is that the $T\varphi$-operator preserves continuity and analyticity ([**4**, page 30]).

It remains to show that $C_{ua}(\hat{F}) \subset \overline{H^p(D)|_{\hat{F}}}$. Let B denote the Banach algebra $\overline{H^\infty(D)|_{\hat{F}}}$. Also put $X = (\hat{F})$. If V is a component of $\mathcal{C} \backslash X$, there mus exist $h \in H^\infty(D)$ such that

$$1 = h(z_0) > \|h\|_F$$

for some $z_0 \in V$. But then $1 - h$ is invertible in B and since

$$1 - h = (z - z_0) g, \quad g \in H^\infty(D)$$

we conclude that $(z-z_0)^{-1}|_{\hat{F}} \in B$. This means that $R(X)|_{\hat{F}} \subset B$, where $R(X)$ is the uniform closure on X by the rational functions with poles off X.

But if $\{V_j\}$ are the components os $\mathcal{C}\backslash X$, the maximum principle gives $\partial V_j \cap T \neq \phi$, $j = 1, 2, \ldots$ and hence $\partial X = \cup_1^\infty \partial V_j$. For such sets X (with empty "inner boundary") Vitushkin has proved that $R(X) = C_{ua}(X)$ ([**4**, page 219]), and hence Theorem 1 is proved.

This solves completely problem 8.5 no. 2 in [**7**] for the space $H^p(D)$, $0 < p < \infty$. For $p = \infty$ the problem is still open.

For $p = \infty$, some information about $\overline{H^\infty|_F}$ can be obtained from the work by Carl Sundberg in [**12**]. If $f \in$ BMOA and $f|_F$ is bounded, Sundberg shows that $f \in \overline{H^\infty|_F}$. On the other hand, our proof above shows that any $f \in \overline{H^\infty|_F}$ can be written as $f = u + v$ with $u \in H^\infty$ and $v \in \cap_{p>0} H^p(D)$. Several questions arises from this. Here we only mention the following: Let $f \in$ BMOA be bounded on a relatively closed set $F \subset D$.

Is there $g \in H^\infty$ such that the restriction $(f-g)|_F$ is uniformly continuous on F?

References

1. A. M. DAVIE AND A. STRAY, Interpolation sets for analytic functions, *Pacific J. Math.* **42(1)** (1972).
2. J. DETRAZ, Algebres de fonctions analytiques dans le disque, *Ann. Sci. Ecole Norm. Sup. 4e serie* (1970), 313–352.
3. P. L. DUREN, "*Theory of H^p spaces*," Academic Press, 1970.
4. T. W. GAMELIN, "*Uniform Algebras*," Prentice Hall, Englewood Cliffs, N. J., 1969.
5. J. GARNETT, "*Bounded Analytic Functions*," Academic Press, 1980.
6. S. N. MERGELYAN, Uniform approximation to functions of a complex variable, *Urephi Mat. Nauk.* **7(2)** (1952), 31–122.
7. ?, "*Linear and Complex Analysis Problem Book*," Lecture Notes in Mathematics **1043**, Springer Verlag, 1984.
8. F. PEREZ-GONZALEZ AND A. STRAY, Farrell and Mergelyan sets for H^p-spaces, $0 < p < 1$, *Michigan Math. J.* **36** (1989), 379–386.
9. A. STRAY, Decomposition of approximable functions, *Ann. of Math.* **120** (1984), 225–235.
10. A. STRAY, Approximation by analytic functions which are uniformly continuous on a subset of their domain of definition, *American J. Math.* **99** (1977), 787–800.

11. A. Stray, Characterization of Mergelyan sets, *Proc. Amer. Math. Soc.* **44** (1974), 347–352.
12. C. Sundberg, Truncations of BMO functions, *Indiana University Math. I.* **33(5)** (1984), 749–779.
13. A. G. Vitushkin, The analytic capacity of sets and problems in approximation theory, *Russian Math. Surveys* **22** (1967), 139–200.

Department of Mathematics
University of Bergen
Allégt 55
5007 Bergen
NORWAY

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