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ON LOCALLY PSEUDOCONVEX SQUARE ALGEBRAS

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Abstract

Let A be an algebra over the field of complex numbers with a (Hausdorff) topology given by a family $\mathcal{Q} = \{q_\lambda | \lambda \in \Lambda\}$ of square preserving r_λ-homogeneous seminorms ($r_\lambda \in (0,1]$). We shall show that $(A, T(\mathcal{Q}))$ is a locally m-convex algebra. Furthermore we shall show that A is commutative.

Introduction. Let A be a locally pseudoconvex algebra over the field of complex numbers. Let $\mathcal{Q} = \{q_\lambda | \lambda \in \Lambda\}$ be a family of r_λ-homogeneous seminorms defining a Hausdorff topology on A. For each $\lambda \in \Lambda$ the number $r_\lambda \in (0,1]$ is fixed. By r_λ-homogeniousity we mean that $q_\lambda(\alpha x) = |\alpha|^{r_\lambda} q_\lambda(x)$ for all $x \in A$ and $\alpha \in \mathbb{C}$. We shall say that the seminorm q_λ is submultiplicative if $q_\lambda(xy) \leq q_\lambda(x)q_\lambda(y)$ for all x and y in A. If every $q_\lambda \in \mathcal{Q}$ is submultiplicative, then $(A, T(\mathcal{Q}))$ is called a locally m-pseudoconvex algebra. If each $q_\lambda \in \mathcal{Q}$ is square preserving in other words if $q_\lambda(x^2) = q_\lambda(x)^2$ for all $x \in A$ and $\lambda \in \Lambda$ we shall say that $(A, T(\mathcal{Q}))$ is a square algebra. Note that locally pseudoconvex algebras include as a special case better known locally convex algebras. Namely for locally convex algebras we have $r_\lambda = 1$ for every $\lambda \in \Lambda$. For properties of commutative locally convex square algebras see [**1**], [**4**] or [**14**]. Commutative locally pseudoconvex square and star algebras have been studied in [**2**]. It is known that a commutative locally convex square algebra is automatically locally m-convex. See [**1**] and [**5**] and [**16**]. It was claimed in [**2**] that the corresponding result is valid also for locally pseudoconvex algebras. In this paper we shall show that indeed this claim is true and even the assumption of commutativity is superfluous.

Main results. If $\| \ \|$ is a r-homogeneous submultiplicative norm on a complex associatve algebra A, then $(A, \| \ \|)$ is called a locally bounded algebra (more precisely a r-normed algebra). See [**18**]. First we shall prove a locally bounded version of Theorem of [**8**] and Corollary 16.8 of [**7**]. See also [**3**], [**12**], [**13**] and [**16**].

Lemma 1. *Suppose that $(A, \|\|)$ is a r-Banach algebra for which there is a constant $K > 0$ such that*

$$\|x\|^2 \leq K\|x^2\| \text{ for all } x \in A. \tag{1}$$

Then A is commutative.

Proof: Let x be a given element of A. Furthermore, let B be a maximal commutative subalgebra of A including the element x. Then also $(B, \| \|)$ is a r-Banach algebra. By Theorems 3.3, 4.4 and 4.8 of [**18**] we have $s_B(y)^r = \lim_{n\to\infty} \|y^n\|^{\frac{1}{n}}$ for all $y \in B$. (Here s_B stands for the spectral radius of y in B.) It follows from (1) that there is some constant $M := M(K) > 0$ such that $\|y\| \leq M s_B(y)^r$ for all $y \in B$. Since B is a maximal commutative subalgebra of A we have $s_B(y) = s_A(y)$ for all $y \in B$. (See for ex. [**17**, p. 46].) Since the above mentioned x is in B we can see that we have $\|x\|^{\frac{1}{r}} \leq M^{\frac{1}{r}} s_A(x)$ and since x was chosen arbitrarily we can see that this same holds for all $x \in A$. But now we have

$$s_A(xy) \leq \|xy\|^{\frac{1}{r}} \leq \|x\|^{\frac{1}{r}}\|y\|^{\frac{1}{r}} \leq M^{\frac{2}{r}} s_A(x) s_A(y) \text{ for all } x \text{ and } y \in A.$$

By Theorem 1 of [**11**] it follows that $A/\operatorname{Rad} A$ is commutative. ($\operatorname{Rad} A$ stands for the Jacobson radical of A). But it follows from (1) that $\operatorname{Rad} A = \{0\}$ and thus we can see that A is commutative. ■

Note that the topological dual of A was not used in [**11**] in proving the commutativity of $A/\operatorname{Rad} A$.

We shall now prove the generalization of the results of [**5**] and [**6**]. For a seminorm q on an algebra A denote by $N_q = \ker q = \{x \in A | q(x) = 0\}$.

Theorem 1. *Let q be a r-homogeneous square preserving seminorm on a (complex associative) algebra A. Then q is submultiplicative, N_q is an ideal of A, $q^{\frac{1}{r}}$ is a seminorm on A, and the quotient algebra A/N_q is commutative.*

Proof: Define the "Jordan product" $\circ$ of A by $x \circ y = \frac{1}{2}(xy + yx)$ x, $y \in A$. Now $4(x \circ y) = (x + y)^2 - (x - y)^2$ for all x and y in A. Thus, if x and $y \in A$ then

$$\begin{aligned} 4^r q(x \circ y) &= q(4(x \circ y)) \\ &= q((x + y)^2 - (x - y)^2) \leq q((x + y)^2) + q((x - y)^2) \\ &= (q(x + y))^2 + (q(x - y))^2 \leq (q(x) + q(y))^2 + (q(x) + q(y))^2 \\ &= 2(q(x) + q(y))^2. \end{aligned}$$

So we have $q(x \circ y) \le \frac{2}{4^r}(q(x)+q(y))^2$ for all x and y in A. Let x, $y \in A$ and $\epsilon > 0$ be arbitrary. Denote by $\alpha = q(x)+\epsilon$ and $\beta = q(y)+\epsilon$. Then

$$\alpha^{-1}\beta^{-1}q(x\circ y) = q((\alpha^{-\frac{1}{r}}x)\circ(\beta^{-\frac{1}{r}}y)) \le \frac{2}{4^r}(q(\alpha^{-\frac{1}{r}}x)+q(\beta^{-\frac{1}{r}}y))^2$$
$$= \frac{2}{4^r}(\alpha^{-1}q(x)+\beta^{-1}q(y))^2 \le \frac{2}{4^r}(1+1)^2 = \frac{8}{4^r}.$$

This shows that $q(x\circ y) \le \frac{8}{4^r}q(x)q(y)$ for all x and y in A. If we now define $p = \frac{8}{4^r}q$, then p is a r-homogeneous seminorm on A satisfying $p(x\circ y) \le p(x)p(y)$ for all x and y in A and $p(x)^2 \le \frac{8}{4^r}p(x^2)$ for all x in A. For x, $y \in A$ denote $[x,y] = xy - yx$. As in the proof of Proposition 1 of [**9**] it can be shown that there is some constant K for which $p([x,y])^2 \le Kp(x)^2p(y)^2$ for all x and y in A. Since $xy = x\circ y + \frac{1}{2}[x,y]$ x, $y \in A$, we can see that there is some constant R such that $p(xy) \le Rp(x)p(y)$ for all x and y in A. Thus there is some constant M for which $q(xy) \le Mq(x)q(y)$ for all x and y in A. It follows from this inequality that N_q is an ideal of A. Write $B := A/N_q$ and let $\dot{q}$ denote the induced r-homogeneous norm on B. Then $\|.\| := M\dot{q}$ is a submultiplicative r-homogeneous norm on B satisfying $\|x\|^2 \le M\|x^2\|$ for all $x \in A$. Applying Lemma 1 to the completion of $(B, \|\ \|)$, we can see that B is commutative. To prove that $\dot{q}^{\frac{1}{r}}$ is a norm on B let $\dot{s}_q$ be the spectral norm of $(B,\dot{q})$ i.e. $\dot{s}_q(x) = \lim_{n\to\infty}\sqrt[n]{\dot{q}(x^n)^{\frac{1}{n}}}$. By Theorem 3.3 of [**18**] $\dot{s}_q$ satisfies the triangle inequality and on the otherhand we have $\dot{q}(x) = \dot{s}_q(x)$ for all $x \in B$ (since $\dot{q}$ is square preserving). By Corollary 3.5 of [**18**] $\dot{q}^{\frac{1}{r}}$ is a usual (1-homogeneous) norm on B. Thus $q^{\frac{1}{r}}$ is a seminorm on A. See also [**2**, Lemma 9]. Note also that $q^{\frac{1}{r}}$ is submultiplicative. See [**1**] or [**5**]. ∎

Corollary 1. *Suppose that $(A, T(\mathcal{Q}))$ is a square algebra. Then $(A, T(\mathcal{Q}))$ is a locally m-convex commutative algebra.*

Proof: It follows from Theorem 1 that each quotient algebra A/N_λ is commutative ($N_\lambda = \ker q_\lambda$). This implies that $q_\lambda(xy - yx) = 0$ for all $\lambda \in \Lambda$ and since we assumed that $T(\mathcal{Q})$ is a Hausdorff topology this implies that A is commutative. By Theorem 1 for each $\lambda \in \Lambda$, $q^{\frac{1}{r_\lambda}}$ is a usual 1-homogeneous submultiplicative seminorm on A. Thus $T(\mathcal{Q})$ is equivalent with a locally m-convex topology $T(\mathcal{P})$ where $\mathcal{P} = \{q^{\frac{1}{r_\lambda}} | \lambda \in \Lambda\}$. ∎

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