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ON THE ANALYTIC CAPACITY AND CURVATURE OF SOME CANTOR SETS WITH NON-σ-FINITE LENGTH

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Abstract

We show that if a Cantor set E as considered by Garnett in [**G2**] has positive Hausdorff h-measure for a non-decreasing function h satisfying $\int_0^1 r^{-3}\, h(r)^2\, dr < \infty$, then the analytic capacity of E is positive. Our tool will be the Menger three-point curvature and Melnikov's identity relating it to the Cauchy kernel. We shall also prove some related more general results.

1. Introduction. In this paper we shall study the analytic capacity of some Cantor sets with non-σ-finite length. The analytic capacity of a compact set E in the complex plane $\mathbb{C}$ is defined as

$$\gamma(E) = \sup_f \lim_{z\to\infty} |zf(z)|$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $|f| \le 1$ and $\lim_{z\to\infty} f(z) = 0$. Then $\gamma(E) = 0$ if and only if E is removable for bounded analytic functions. For this and other properties of the analytic capacity, see e.g. [**G2**] and [**M**].

Let Λ_h be the Hausdorff measure generated by a non-decreasing function $h : [0,\infty) \to [0,\infty)$, i.e.,

$$\Lambda_h(A) = \lim_{\delta\downarrow 0} \inf \left\{ \sum_{i=1}^{\infty} h\big(d(E_i)\big) : A \subset \bigcup_{i=1}^{\infty} E_i,\ d(E_i) \le \delta \right\}.$$

Here $d(E)$ denotes the diameter of E. For $h(r) = h_1(r) = r$, a classical result of Painlevé tells us that $\Lambda_{h_1}(E) = 0$ implies $\gamma(E) = 0$. Some of the sets E with $0 < \Lambda_{h_1}(E) < \infty$ have zero and some positive analytic capacity, and although a complete characterization within this class is lacking, pretty much is known, see [**G2**], [**M**] and [**MMV**]. It seems that some kind of rectifiability properties are required in order that a set of positive and finite length could have positive analytic capacity.

To the other direction we have an easy classical result saying that positive Newtonian capacity implies positive analytic capacity, cf. [**G2**] or [**M**]. The Newtonian capacity of E can be defined as

$$C_1(E) = \sup\left\{\mu(E) : \operatorname{spt}\mu \subset E,\ \int |x-y|^{-1}\,d\mu y \le 1 \text{ for } x \in \mathbb{C}\right\}$$

where the supremum is taken over all (non-negative) Borel measures μ, and $\operatorname{spt}\mu$ stands for the support of μ. It is fairly easy to see (cf. [**G2**, p. 73]) that for non-decreasing functions $h : [0,\infty) \to [0,\infty)$ and Borel sets $E \subset \mathbb{C}$,

$$\int_0^1 r^{-2}\,h(r)\,dr < \infty \text{ and } \Lambda_h(E) > 0 \text{ implies } C_1(E) > 0,$$

whence also $\gamma(E) > 0$. There are also functions h with $\int_0^1 r^{-2}\,h(r)\,dr = \infty$ for which $\Lambda_h(E) > 0$ implies $C_1(E) > 0$, see [**E**].

As in [**G2**] we shall consider Cantor sets $E(\lambda) \subset \mathbb{C} = \mathbb{R}^2$ for non-increasing sequences (λ_n) with $0 < \lambda_n < 1/2$. Each $E(\lambda)$ is a product set $E(\lambda) = K(\lambda) \times K(\lambda)$ where $K(\lambda) \subset \mathbb{R}$ is the linear Cantor set constructed as follows. Let $K_0 = [0,1]$, $K_1 = [0,\lambda_1] \cup [1-\lambda_1, 1]$, and at each stage n, K_n is obtained from K_{n-1} by replacing each component of K_{n-1} by its two endmost intervals of length λ_n times the length of the component. Then

$$K(\lambda) = \bigcap_{n=1}^{\infty} K_n$$

where K_n is a union of 2^n intervals of length

$$\sigma_n = \lambda_1 \cdot \ldots \cdot \lambda_n. \tag{1.1}$$

Thus

$$E(\lambda) = K(\lambda) \times K(\lambda) = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{4^n} E_{n,j}$$

where each $E_{n,j}$ is a square of side-length σ_n.

Let $h : [0,\infty) \to [0,\infty)$ be a non-decreasing function with

$$h(\sigma_n) = 4^{-n} \text{ for } n = 1, 2, \ldots . \tag{1.2}$$

Then the Hausdorff h-measure Λ_h of $E(\lambda)$ is positive and finite, and moreover there are positive and finite constants c and d such that

$$ch(r) \le \Lambda_h\big(E(\lambda) \cap B(z,r)\big) \le dh(r) \text{ for } z \in E(\lambda),\ 0 < r \le 2, \tag{1.3}$$

see e.g. [**G2**] or [**M**]. Here $B(z,r)$ is the closed disc with centre z and radius r. Moreover, $C_1\big(E(\lambda)\big) > 0$ if and only if $\int_0^1 r^{-2}\, h(r)\, dr < \infty$, see [**G2**]. In this paper we shall prove that if

$$\int_0^1 r^{-3}\, h(r)^2\, dr < \infty$$

then $E(\lambda)$ has positive analytic capacity. I believe that also the converse holds but I have not been able to prove it.

Note that if $h(r) \le r$ for $r > 0$, the case which is really of the main interest for us, then $\int_0^1 r^{-2}\, h(r)\, dr = \int_0^1 r^{-1}\, h(r)\, dr/r < \infty$ implies $\int_0^1 r^{-3}\, h(r)^2\, dr = \int_0^1 \big(r^{-1}\, h(r)\big)^2\, dr/r < \infty$, but not vice versa. So it is easy to construct sequences (λ_n) such that for h related to (λ_n) as above

$$\int_0^1 r^{-3}\, h(r)^2\, dr < \infty \text{ but } \int_0^1 r^{-2}\, h(r)\, dr = \infty.$$

Thus many of the sets $E(\lambda)$ have $\gamma(E) > 0$ but $C_1(E) = 0$.

Garnett considered the analytic capacity of the sets $E(\lambda)$ in [**G2**]. He claimed there that $\gamma\big(E(\lambda)\big) > 0$ if and only if $C_1\big(E(\lambda)\big) > 0$. However, Eiderman, see the Commentary in [**I2**], found a mistake in the proof and the result of this paper shows that also the statement of Garnett's result was false. Earlier in [**G1**] Garnett, and independently Ivanov, see [**I1**], proved for $E(\lambda)$ with $\lambda_n = 1/4$ for all n (which is the case $0 < \Lambda_{h_1}\big(E(\lambda)\big) < \infty$) that $\gamma\big(E(\lambda)\big) = 0$. Using this fact Ivanov showed in [**I2**] that there exists λ for which $\gamma\big(E(\lambda)\big) = 0$ and yet the corresponding h satisfies $\lim_{r \downarrow 0} h(r)/r = 0$.

2. The Menger and Melnikov curvatures. As in [**MV**] and [**MMV**] our method will be based on the so-called Menger curvature $c(x,y,z)$ of the triple x, y, $z \in \mathbb{C}$, and its relation to the Cauchy kernel first discovered by Melnikov in [**Me**]. By definition, $c(x,y,z)$ is the reciprocal of the radius of the circle passing through x, y and z; $c(x,y,z) = 0$ if and only if x, y and z are collinear. By elementary geometry,

$$c(x,y,z) = \frac{2d(z, L_{x,y})}{|x-z|\,|y-z|} \tag{2.1}$$

where $L_{x,y}$ is the line through x and y, and $d(z,A)$ denotes the distance of z from the set A. Melnikov [**Me**] introduced the curvature

$$c^2(\mu) = \iiint c(x,y,z)^2 \, d\mu x \, d\mu y \, d\mu z$$

for any Borel measure μ and found that it is closely related to the Cauchy transform of μ; we return to this in Section 3.

We shall first prove the following general inequality.

2.2. Theorem. *Let $h : [0,\infty) \to [0,\infty)$ be a non-decreasing function with $\int_0^\infty r^{-3} h(r)^2 \, dr < \infty$. If μ is a finite Borel measure on $\mathbb{C}$ such that $\mu B(z,r) \le h(r)$ for $z \in \mathbb{C}$ and $r > 0$, then*

$$c^2(\mu) \le 12\mu(\mathbb{C}) \int_0^\infty r^{-3} \, h(r)^2 \, dr.$$

Proof: Note that $\big(h(r)/r\big)^2 \le 8 \int_r^{2r} t^{-3} \, h(t)^2 \, dt$, whence $\lim_{r \downarrow 0} h(r)/r = 0$. Set

$$A = \big\{(x,y,z) \in \mathbb{C}^3 : |x-y| \le |x-z| \text{ and } |x-y| \le |y-z|\big\}.$$

Then by (2.1) and the fact $d(z, L_{x,y}) \le |x-z|$,

$$\begin{aligned}
c^2(\mu) &\le 3 \iiint_A c(x,y,z)^2 \, d\mu x \, d\mu y \, d\mu z \\
&\le 12 \iiint_A \left(\frac{d(z, L_{x,y})}{|x-z| \, |y-z|} \right)^2 d\mu x \, d\mu y \, d\mu z \\
&\le 12 \iint \int_{B(y,|y-z|)} |y-z|^{-2} \, d\mu x \, d\mu y \, d\mu z \\
&= 12 \iint \frac{\mu B(y,|y-z|)}{|y-z|^2} \, d\mu z \, d\mu y \\
&= 12 \int \int_0^\infty \frac{\mu_y(r)}{r^2} \, d\mu_y(r) \, d\mu y
\end{aligned}$$

where $\mu_y(r) = \mu B(y,r)$ and we are using Riemann-Stieltjes integration. Integrating by parts and using the facts $\mu_y(r)/r \le h(r)/r \to 0$ as $r \to 0$ and $\mu_y(r)/r \le \mu(\mathbb{C})/r \to 0$ as $r \to \infty$, we have

$$c^2(\mu) \le 12 \int \int_0^\infty \frac{\mu_y(r)^2}{r^3} \, dr \, d\mu y \le 12\mu(\mathbb{C}) \int_0^\infty \frac{h(r)^2}{r^3} \, dr. \quad \blacksquare$$

The argument in the above proof is rather precise except that the converse of the inequality $d(z, L_{x,y}) \le |x - z|$ may fail, even with constants, very badly. But if μ is at no scales concentrated near lines, then such a converse inequality holds for many triples (x, y, z), and we can obtain a converse for Theorem 2.2. We do this now for the Cantor sets $E(\lambda)$. We denote by $\mu \llcorner A$ the restriction of the measure μ to the set A.

2.3. Theorem. *Let λ_n be a non-increasing sequence with $0 < \lambda_n < 1/2$ and let h be related to it as in (1.1) and (1.2). Then*

$$c^2\big(\Lambda_h \llcorner E(\lambda)\big) = \iiint\limits_{E(\lambda)^3} c(x, y, z)^2 \, d\Lambda_h x \, d\Lambda_h y \, d\Lambda_h z < \infty$$

if and only if $\int_0^1 r^{-3} h(r)^2 \, dr < \infty$.

Proof: That $\int_0^1 r^{-3} h(r)^2 \, dr < \infty$ implies $c^2\big(\Lambda_h \llcorner E(\lambda)\big) < \infty$ follows immediately from Theorem 2.2 and (1.3). To prove the converse, let for y, $z \in E(\lambda)$,

$$C(y, z) = \big\{x \in E(\lambda) : d(z, L_{x,y}) \ge |x - z|/4\big\}.$$

Note that by the construction of $E(\lambda)$ there is a constant c_1 such that for all y, $z \in E(\lambda)$ with $y \ne z$ and for all $0 < r \le |y - z|/2$

$$\mu\big(C(y, z) \cap B(y, 2r) \setminus B(y, r)\big) \ge c_1 \mu\big(B(y, 2r) \setminus B(y, r)\big)$$

with μ denoting the restriction of Λ_h to $E(\lambda)$. Hence

$$\begin{aligned}
c^2(\mu) &= \iiint \left(\frac{2d(z, L_{x,y})}{|x - z| \, |y - z|}\right)^2 d\mu x \, d\mu y \, d\mu z \\
&\ge \sum_{i=1}^{\infty} \iiint\limits_{C(y,z) \cap B(y, 2^{1-i}|y-z|) \setminus B(y, 2^{-i}|y-z|)} \left(\frac{2d(z, L_{x,y})}{|x - z| \, |y - z|)}\right)^2 d\mu x \, d\mu y \, d\mu z \\
&\ge \frac{1}{4} \sum_{i=1}^{\infty} \iint \frac{\mu\big(C(y, z) \cap B(y, 2^{1-i}|y - z|) \setminus B(y, 2^{-i}|y - z|)\big)}{|y - z|^2} \, d\mu y \, d\mu z \\
&\ge \frac{c_1}{4} \iint \frac{\mu B(y, |y - z|)}{|y - z|^2} \, d\mu y \, d\mu z.
\end{aligned}$$

If $\liminf_{r \downarrow 0} h(r)/r = 0$, we have as in the proof of Theorem 2.2, recalling also (1.3), that

$$c^2(\mu) \ge \frac{c_1}{4} \int \int_0^{\infty} \frac{\big(\mu B(y, r)\big)^2}{r^3} \, dr \, d\mu y \ge \frac{c^2 c_1}{4} \Lambda_h\big(E(\lambda)\big) \int_0^1 \frac{h(r)^2}{r^3} \, dr,$$

which proves the theorem in this case. If $\liminf_{r\downarrow 0} h(r)/r > 0$ there is $c_2 > 0$ such that $h(r) \geq c_2 r$ for $0 < r \leq 2$. Then the above estimate yields by (1.3) and the change of variable $r = 1/t$,

$$\begin{aligned}
c^2(\mu) &\geq \frac{c_1}{4} \iint \frac{\mu B(y, |y-z|)}{|y-z|^2}\, d\mu y\, d\mu z \\
&\geq \frac{c_1 c_2 c}{4} \iint \frac{1}{|y-z|}\, d\mu y\, d\mu z \\
&= (c_1 c_2 c/4) \int \int_0^\infty \mu\{y : |y-z|^{-1} > t\}\, dt\, d\mu z \\
&= (c_1 c_2 c/4) \int_0^\infty r^{-2} \mu B(z, r)\, dr\, d\mu z \\
&\geq (c_1 c_2 c^2/4) \int_0^1 r^{-2} h(r)\, dr\, \Lambda_h\big(E(\lambda)\big) = \infty,
\end{aligned}$$

and the theorem follows also in this case. ■

2.4. Remark. Theorem 2.2 yields also immediately that

$$\iiint_{(E(\lambda)\cap B(a,r))^3} c(x,y,z)^2\, d\Lambda_h x\, d\Lambda_h y\, d\Lambda_h z \leq 12 d h(r) \int_0^\infty t^{-3}\, h(t)^2\, dt$$

for $a \in E(\lambda)$ and $0 < r \leq 1$. Moreover, an inspection of the proofs of Theorems 2.2 and 2.3 shows that this triple integral is comparable to

$$h(r) \int_0^r t^{-3}\, h(t)^2\, dt.$$

3. Cauchy integral and analytic capacity. The following identity relating the Menger curvature to the Cauchy kernel $1/z$ was found by Melnikov in [**Me**], see also [**MV**]: for z_1, z_2, $z_3 \in \mathbb{C}$, $z_i \neq z_j$ for $i \neq j$,

$$c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{\big(z_{\sigma(1)} - z_{\sigma(3)}\big)\,\overline{\big(z_{\sigma(2)} - z_{\sigma(3)}\big)}} \tag{3.1}$$

where the sum is over all six permutations σ of $\{1, 2, 3\}$.

Let μ be a finite Borel measure on $\mathbb{C}$. For a disc Δ and $\varepsilon > 0$, let

$$T_\varepsilon(\Delta) = \big\{(z_1, z_2, z_3) \in \Delta^3 : |z_1 - z_3| > \varepsilon, |z_2 - z_3| > \varepsilon, |z_1 - z_2| \le \varepsilon\big\}.$$

Using (3.1) and Fubini's theorem, we have, as in [**MV**] or [**MMV**], that

$$\int_\Delta \left| \int_{\Delta\setminus B(z,\varepsilon)} \frac{1}{\zeta - z}\, d\mu\zeta \right|^2 d\mu z \le \iiint_{\Delta^3} c(z_1, z_2, z_3)^2\, d\mu z_1\, d\mu z_2\, d\mu z_3 + \iiint_{T_\varepsilon(\Delta)} \frac{1}{(z_1 - z_3)\,\overline{(z_2 - z_3)}}\, d\mu z_1\, d\mu z_2\, d\mu_3.$$

Assuming that μ satisfies

$$\mu B(z, r) \le r \text{ for } z \in \mathbb{C},\ r > 0,$$

it is easy to see that the last term is bounded by $c_1\mu(\Delta)$ for some constant c_1 independent of Δ and ε. Thus

$$\int_\Delta \left| \int_{\Delta\setminus B(z,\varepsilon)} (\zeta - z)^{-1}\, d\mu\zeta \right|^2 d\mu z \le \iiint_{\Delta^3} c(z_1, z_2, z_3)^2\, d\mu z_1\, d\mu z_2\, d\mu z_3 + c_1\mu(\Delta). \tag{3.2}$$

We now assume that $h : [0, \infty) \to [0, \infty)$ is a non-decreasing function such that

$$\int_0^\infty r^{-3}\, h(r)^2\, dr < \infty, \tag{3.3}$$

and that μ satisfies

$$\mu B(z, r) \le h(r) \text{ for } z \in \mathbb{C},\ r > 0, \tag{3.4}$$

and the doubling condition with some $c_2 < \infty$,

$$\mu B(z, 2r) \le c_2 \mu B(z, r) \text{ for } z \in \operatorname{spt}\mu,\ r > 0. \tag{3.5}$$

Then we have by (3.2) and Theorem 2.2

$$\int_{\Delta}\left|\int_{\Delta\setminus B(z,\varepsilon)}(\zeta-z)^{-1}\,d\mu\zeta\right|^2 d\mu z\le c_3\mu\Delta \tag{3.6}$$

for all discs Δ and all $\varepsilon>0$, where c_3 is a constant independent of Δ and ε. This means that the operators

$$C_{\mu,\varepsilon}:g\mapsto\int_{|\zeta-z|>\varepsilon}\frac{g(\zeta)}{\zeta-z}\,d\mu\zeta$$

are bounded in $L^2(\mu)$, uniformly with respect to ε. As in [**MV**] one can see this in two ways: either by observing that (3.6) implies that $C_{\mu,\varepsilon}(1)\in\mathrm{BMO}$, uniformly in ε, and appealing to the $T(1)$-theorem. Here BMO means the BMO with respect to the "dyadic" cube system constructed by Christ in [**C2**] as a generalization of the system of David, see [**D**, pp. 91–96]. Or one can first check that the estimates above imply that $C_{\mu,\varepsilon}$ maps boundedly, and uniformly in ε, $L^\infty(\mu)$ to $\mathrm{BMO}(\mu)$ and the atomic $H^1(\mu)$ to $L^1(\mu)$, and then use interpolation. For the validity of the $T(1)$-theorem in this setting, see [**C1**, p. 94] or [**D**, pp. 47–48]. The interpolation can be proven by the method of [**J**, Section 3.III]. Thus we have the following theorem.

3.7. Theorem. *Suppose that h and μ satisfy the conditions (3.3)-(3.5). Then the operators $C_{\mu,\varepsilon}:L^2(\mu)\to L^2(\mu)$ are bounded, uniformly with respect to ε.*

From this we could deduce with standard methods as in [**C1**] that $\gamma(E)>0$ if $E\subset\mathbb{C}$ is a compact set supporting a non-zero Borel measure μ for which (3.3)-(3.5) hold. However, using a recent result of Melnikov we can get this without the doubling condition (3.5), and also without using the L^2-boundedness. Namely, Melnikov proved in [**Me**, Theorem 3, p. 829] that $\gamma(E)>0$ provided E supports μ such that $\mu B(z,r)\le r$ for all $z\in\mathbb{C}$, $r>0$ and $c^2(\mu)<\infty$. Combining this with Theorem 2.2 we have

3.8. Theorem. *Let $E\subset\mathbb{C}$ be a compact set such that there are a non-zero Borel measure μ on E and a non-decreasing function $h:[0,\infty)\to[0,\infty)$ satisfying (3.3) and (3.4). Then $\gamma(E)>0$.*

We now return to the Cantor sets $E(\lambda)$ for non-increasing sequences $\lambda=(\lambda_n)$, $0<\lambda_n<1/2$. We may assume $\lambda_n\ge 1/4$ for all n because otherwise $\gamma\big(E(\lambda)\big)=0$. Letting h be as in (1.1) and (1.2) and $\mu=\Lambda_h\llcorner E(\lambda)$, the condition (3.4) is satisfied. Thus we have by Theorem 3.8.

3.9. Theorem. *If the function h related to the sequence λ by (1.1) and (1.2) satisfies $\int_0^1 r^{-3}\, h(r)^2\, dr < \infty$, then $\gamma\big(E(\lambda)\big) > 0$.*

3.10. Question. Is the converse valid, that is, does $\gamma\big(E(\lambda)\big) > 0$ imply $\int_0^1 r^{-3}\, h(r)^2\, dr < \infty$? I believe that there would be good chances to prove this if the following were true: if $f : \mathbb{C} \setminus E(\lambda) \to \mathbb{C}$ is bounded and analytic with $f(\infty) = 0$, then there is a bounded Borel function $\varphi : E(\lambda) \to \mathbb{C}$ such that

$$f(z) = \int_{E(\lambda)} \frac{\varphi(\zeta)}{\zeta - z}\, d\Lambda_h \zeta, \quad z \in \mathbb{C} \setminus E(\lambda).$$

This is well-known if $h(r) = r$ for $r > 0$, but in the general case I don't even know whether f can be represented as a Cauchy transform of some complex measure.

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