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TWO WEIGHT NORM INEQUALITY FOR THE FRACTIONAL MAXIMAL OPERATOR AND THE FRACTIONAL INTEGRAL OPERATOR

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Abstract

New sufficient conditions on the weight functions $u(.)$ and $v(.)$ are given in order that the fractional maximal [resp. integral] operator M_s [resp. I_s], $0 \leq s < n$, [resp. $0 < s < n$] sends the weighted Lebesgue space $L^p(v(x)\,dx)$ into $L^p(u(x)\,dx)$, $1 < p < \infty$. As a consequence a characterization for this estimate is obtained whenever the weight functions are radial monotone.

1. Introduction

The fractional maximal operator M_s of order s, with $0 \leq s < n$, acts on locally integrable function $f(.)$ of $\mathbb{R}^n$ as

$$(M_s f)(x) = \sup \left\{ |Q|^{\frac{s}{n}-1} \int_Q |f(y)|\,dy; \quad Q \text{ a cube with } Q \ni x \right\}.$$

These cubes have sides parallel to the coordinate-axes. Here M_0 is the well known Hardy-Littlewood maximal operator.

The purpose of this paper is to determine weight functions $u(.)$ and $v(.)$ for which M_s is bounded from $L^p_v = L^p(\mathbb{R}^n, v\,dx)$ into L^p_u with $1 < p < \infty$. This means there is $C > 0$ for which

$$\int_{\mathbb{R}^n} (M_s f)^p(x) u(x)\,dx \leq C \int_{\mathbb{R}^n} f^p(x) v(x)\,dx \quad \text{for all } f(.) \geq 0.$$

For convenience such a estimate will be denoted by $M_s : L^p_v \to L^p_u$.

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Sawyer [**Sa1**] proved that $M_s : L^p_v \to L^p_u$ if and only for a constant $S > 0$:

$$\int_Q (M_s v^{-\frac{1}{p-1}} 1\!\!\mathrm{I}_Q)^p(x)u(x)\,dx \le S\int_Q v^{-\frac{1}{p-1}}(x)\,dx < \infty \quad \text{for all cubes } Q.$$

Here $1\!\!\mathrm{I}_Q(.)$ denotes the characteristic function of the cube Q. Unfortunately for given weight functions, this condition is not easy to check since it is expressed in term of the maximal operator M_s itself.

According to Pérez [**Pe**] the above estimate holds whenever for some $t > 1$ and $A > 0$

$$|Q|^{\frac{s}{n}} \left(\frac{1}{|Q|}\int_Q u(y)\,dy\right)^{\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q v^{-\frac{t}{p-1}}(y)\,dy\right)^{\frac{1}{tp'}} \le A \quad \text{for all cubes } Q.$$

It is an “almost necessary condition” in the sense that $M_s : L^p_v \to L^p_u$ implies this last inequality with $t = 1$. Although this Pérez's condition is not expressed in term of M_s, it can be non-satisfactory because of integrations on arbitrary cubes. Take, for instance, $w(x) = |x|^{\frac{1}{2}} \ln^{-1}(e+|x|)[\ln(e+|x|) - |x|(e+|x|)^{-1}]$. For cubes Q non-centered at the origin, a direct computation of $\int_Q w(x)\,dx$ seems to be extremely hard to do. This is not the case in evaluating $\int_{|x|<R} w(x)\,dx$, $R > 0$. Such an observation leads to consider and study again the estimate $M_s : L^p_v \to L^p_u$, which necessarily [see Section 2] implies

$$R^{s-n}\left(\int_{|x|<R} u(x)\,dx\right)^{\frac{1}{p}} \left(\int_{|x|<R} v^{-\frac{1}{p-1}}(x)\,dx\right)^{\frac{1}{p'}} \le A \quad \text{for all } R > 0, \tag{1.1}$$

and

$$\left(\int_{R<|x|} |x|^{(s-n)p}u(x)\,dx\right)^{\frac{1}{p}} \left(\int_{|x|<R} v^{-\frac{1}{p-1}}(x)\,dx\right)^{\frac{1}{p'}} \le A \quad \text{for all } R > 0. \tag{1.2}$$

So the main question, answered in Section 2, is to find a third condition so that the three conditions together are sufficient to derive $M_s : L^p_v \to L^p_u$ [Theorem 2.1]. As a consequence it will be proved [Corollary 2.5] that (1.1) and (1.2) together are necessary and sufficient for this estimate to hold whenever the weight functions are radial monotone.

The corresponding problem and study for $I_s : L_v^p \to L_u^p$, $0 < s < n$, are performed in Section 3, where I_s is the fractional integral operator given by

$$(I_s f)(x) = \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)\,dy.$$

Ideas for the proofs are inspired from a former paper due to B. Muckenhoupt and R. L. Wheeden [**Mu-Wh**].

Throughout this paper, it will be always assumed that:

$$\begin{aligned}
&1 < p < \infty, && p' = \frac{p}{p-1},\\
&0 \le s < n && \text{for the fractional maximal operator } M_s,\\
&0 < s < n && \text{for the fractional integral operator } I_s,\\
&u(.), v(.), v^{-\frac{1}{p-1}}(.) && \text{are nonnegative locally integrable functions.}
\end{aligned}$$

2. Results for the Fractional Maximal Operator

A variant of M_s is the restricted maximal operator $\widetilde{M}_s$ defined by

$$(\widetilde{M}_s f)(x) = \sup_{0<r<\frac{1}{2}|x|} \left\{ r^{s-n} \int_{\{y;|x-y|<r\}} |f(y)|\,dy \right\}.$$

The first main result, which is also the high point of the present paper, asserts that the two weight problem for M_s can be essentially reduced to the corresponding weighted inequality for the restricted operator $\widetilde{M}_s$. Precisely, we have

Theorem 2.1. *The estimate $M_s : L_v^p \to L_u^p$ holds if and only if $\widetilde{M}_s : L_v^p \to L_u^p$ and both the Muckenhoupt condition (1.1) and the Hardy condition (1.2) are satisfied.*

So the remainder of this paragraph will be devoted first to derive sufficient conditions for the estimate of $\widetilde{M}_s$, and then to give applications showing the gain over past results.

Proposition 2.2. *The estimate $\widetilde{M}_s : L_v^p \to L_u^p$ holds whenever*

$$|.|^s \Big(\widetilde{M}_0 v^{-\frac{t}{p-1}}\Big)^{\frac{1}{tp'}}(.)\Big(u(.)\Big)^{\frac{1}{p}} \in L^\infty(\mathbb{R}^n, dx) \quad \textit{for some } t > 1. \tag{2.1}$$

The sufficient condition involved in this result does not require $v^{-\frac{1}{p-1}}(.) \in L^t_{\text{loc}}(\mathbb{R}^n, dx)$, since for $r < \frac{1}{2}|x|$ and $|x-y| < r$ then $\frac{1}{2}|x| < |y| < \frac{3}{2}|x|$.

In applications, the restricted operator $\widetilde{M}_0$ in (2.1) is not a brake for computations, since trivially $(\widetilde{M}_0 w)(x) \leq \sup_{\frac{1}{4}|x|<|y|<4|x|} w(y)$. Then $\widetilde{M}_s : L^p_v \to L^p_u$ whenever for a constant $C > 0$:

$$|x|^s\Big(\widetilde{\sigma}(x)\Big)^{\frac{1}{p'}}\Big(u(x)\Big)^{\frac{1}{p}} \leq C \tag{2.2}$$

$$\text{a.e. and with } \widetilde{\sigma}(x) = \sup_{\frac{1}{4}|x|<|y|<4|x|} v^{-\frac{1}{p-1}}(y).$$

For a weight function $v(.)$ constant on annuli then $\widetilde{\sigma}(x) \approx v^{-\frac{1}{p-1}}(x)$. In considering the estimate $\widetilde{M}_s : L^p_v \to L^p_u$ for the usual weight functions, it would be helpful to consider the particular properties they have, whose two of them are now recalled.

So the weight $w(.)$ satisfies the growth condition $(\mathcal{H})$ [or $w(.) \in \mathcal{H}$] if

$$\sup_{\{\frac{1}{4}|x|<|y|\leq 4|x|\}} w(y) \leq C\frac{1}{|x|^n}\int_{\{a|x|<|z|\leq b|x|\}} w(z)\,dz$$

for some fixed constants C, a, $b > 0$. If $w(.)$ is given by a real monotone weight function $\omega(.)$ i.e. $w(x) = \omega(|x|)$ then $(\mathcal{H})$ is satisfied, and moreover the corresponding constants do not depend on $\omega(.)$. And $w(.)$ satisfies the reverse doubling RD_ρ, $\rho > 0$, [or merely $w(.) \in RD_\rho$] when there is $C > 0$ for which

$$\int_{Q_1} w(y)\,dy \leq C\left(\frac{|Q_1|}{|Q_2|}\right)^\rho \int_{Q_2} w(y)\,dy \quad \text{for all cubes } Q_1, Q_2 \text{ with } Q_1 \subset Q_2.$$

As a first application of the above results is a sort of "improvement" of a Cordoba-Fefferman's inequality [**Co-Fe**] which states that

$$\int_{\mathbb{R}^n} |(Tf)(x)|^p u(x)\,dx \leq C\int_{\mathbb{R}^n} |f(x)|^p (M_0 u^t)^{\frac{1}{t}}(x)\,dx$$

$$\text{for all } f(.) \in C^\infty_0(\mathbb{R}^n),$$

with the constant $C > 0$ depending only on n and p. Here T is a Calderon-Zygmund operator, i.e. a linear operator taking $C^\infty_0(\mathbb{R}^n)$ into $L^1_{\text{loc}}(\mathbb{R}^n, dx)$, bounded on $L^2(\mathbb{R}^n, dx)$ with $(Tf)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy$ a.e. $x \notin \operatorname{supp} f$ for each $f \in C^\infty_0(\mathbb{R}^n)$. And $K(x,y)$ is a continuous function defined on $\{(x,y);\, x \neq y\}$ and satisfying the standard estimates: $|K(x,y)| \leq C|x-y|^{-n}$ and $|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C\left(\frac{|x-x'|}{|x-y|}\right)^\epsilon |x-y|^{-n}$ whenever $2|x-x'| \leq |x-y|$; $C > 0$ and $\varepsilon \in]0,1]$ do not depend on x, y and x'.

Proposition 2.3. *Suppose $u(.) \in RD_\rho \cap \mathcal{H}$ for a $\rho > 0$. Then for a constant $C > 0$:*

$$\int_{\mathbb{R}^n} |(Tf)(x)|^p u(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p (M_0 u)(x)\,dx \qquad \text{for all } f(.) \in C_0^\infty(\mathbb{R}^n). \tag{2.3}$$

Here C depends on n, p, and on the constants in the conditions RD_ρ and $\mathcal{H}$.

Inequality (2.3) is better than the preceding one since $(M_0 w)(x) \le (M_0 w^t)^{\frac{1}{t}}(x)$.

Now we revert on the weighted inequality for the restricted operator $\widetilde{M}_s$.

Proposition 2.4. *Suppose $u(.), v^{-\frac{1}{p-1}}(.) \in \mathcal{H}$. The estimate $\widetilde{M}_s : L_v^p \to L_u^p$ holds whenever the Muckenhoupt condition (1.1) is satisfied.*

Consequently we obtain

Corollary 2.5. *Suppose $u(.), v^{-\frac{1}{p-1}}(.) \in \mathcal{H}$. The estimate $M_s : L_v^p \to L_u^p$ holds if and only if the Muckenhoupt condition (1.1) and the Hardy condition (1.2) are both satisfied.*

This results yields a complete solution of the two weight inequalities for monotone weight functions as announced in the introduction.

We will end this paragraph by a second example of applications of Theorem 1, which seems difficult to treat by a direct use of the Pérez's result [**Pe**] quoted above.

Corollary 2.6. *Let*

$$0 < \nu < np, \tag{2.4}$$

$$0 \le \rho < \beta < (n-s)p, \tag{2.5}$$

$$0 \le \delta < \frac{1}{p-1}(np - \nu) = \lambda. \tag{2.6}$$

And define

$$u(x) = |x|^{\beta-n} \ln^{-(\rho+1)}(e+|x|)\Big[\beta \ln(e+|x|) - \rho|x|(e+|x|)^{-1}\Big],$$

$$v(x) = |x|^{\nu-n} \ln^{(\delta+1)(p-1)}(e+|x|)\Big[\lambda \ln(e+|x|) - \delta|x|(e+|x|)^{-1}\Big]^{1-p}.$$

Then $M_s : L_v^p \to L_u^p$ if and only if

$$s + \frac{\beta}{p} = \frac{\nu}{p}. \tag{2.7}$$

3. Results for the Fractional Integral Operator

In fact the above results for M_s are inspired from those for I_s, whose proofs seem rather simplistic, but appear significant.

The estimate $I_s : L^p_v \to L^p_u$ was first characterized by Sawyer [**Sa2**] by using both two test conditions expressed in term of I_s and arbitrary (dyadic) cubes. Latter Sawyer and Wheeden [**Sa-Wh**], [see also [**Pe**]] introduced the sufficient condition

$$|Q|^{\frac{s}{n}} \left(\frac{1}{|Q|} \int_Q u^t(y)\, dy \right)^{\frac{1}{tp}} \left(\frac{1}{|Q|} \int_Q v^{-\frac{t}{p-1}}(y)\, dy \right)^{\frac{1}{tp'}} \leq A \quad \text{for all cubes } Q.$$

Here $A > 0$ and $t > 1$ are fixed constants. This test condition requires $t < \frac{n}{s}$ and $u(.)$, $v^{-\frac{1}{p-1}}(.) \in L^t_{\text{loc}}(\mathbb{R}^n, dx)$.

Since M_s is pointwise dominated by I_s, then each of (1.1) and (1.2) is a necessary condition for $I_s : L^p_v \to L^p_u$ to be held. The dual of the Hardy inequality (1.2):

$$(1.2^*) \qquad \left(\int_{R<|x|} |x|^{(s-n)p'} v^{-\frac{1}{p-1}}(x)\, dx \right)^{\frac{1}{p'}} \left(\int_{|x|<R} u(x)\, dx \right)^{\frac{1}{q}} \leq A \quad \text{for all } R > 0$$

is also a necessary condition for the above estimate. As in the case of the fractional maximal operator, the corresponding restricted operator

$$(\widetilde{I}_s f)(x) = \int_{\{\frac{1}{2}|x|<|y|\leq 2|x|\}} |x-y|^{s-n} f(y)\, dy,$$

will be useful for the sequel.

The main result also states that the two weight problem for I_s is essentially reduced to the corresponding weighted inequality for the restricted operator $\widetilde{I}_s$.

Theorem 3.1. *The estimate $I_s : L^p_v \to L^p_u$ holds if and only if $\widetilde{I}_s : L^p_v \to L^p_u$ and both the Hardy condition (1.2) and its dual version* (1.2*) *are satisfied.*

A characterizing condition for $\widetilde{I}_s : L^p_v \to L^p_u$ is not known, however a sufficient condition [easily verifiable for a large weight functions] can be obtained by elementary arguments.

Proposition 3.2. *The estimate $\widetilde{I}_s : L^p_v \to L^p_u$ holds whenever*

$$\Big(\widetilde{I}_s[u(\widetilde{I}_s v^{-\frac{1}{p-1}})^{p-1}]\Big)(.) \in L^\infty(\mathbb{R}^n, dx). \tag{3.1}$$

Note that Verbitsky and Wheeden [**Ve-Wh**] got $I_s : L^p_v \to L^p_u$ from

$$\Big(I_s[u(I_s v^{-\frac{1}{p-1}})^p]\Big)(.) \le C(I_s v^{-\frac{1}{p-1}})(.)$$

and by assuming $(I_s v^{-\frac{1}{p-1}})(.) \in L^p_{\text{loc}}(\mathbb{R}^n, u(x)\,dx)$. Compared to this last inequality, condition (3.1) is easier to check at least for usual weight functions. Indeed, clearly (3.1) is satisfied if for some constant $C > 0$:

$$|x|^s \Big(\widetilde{\sigma}(x)\Big)^{\frac{1}{p'}} \Big(\widetilde{u}(x)\Big)^{\frac{1}{p}} \le C \quad \text{a.e.} \tag{3.2}$$

where $\widetilde{\sigma}(x) = \sup_{\frac{1}{4}|x|<|y|<4|x|} v^{-\frac{1}{p-1}}(y)$ and $\widetilde{u}(x) = \sup_{\frac{1}{4}|x|<|y|<4|x|} u(y)$.

As a first application of the above results is a sort of “improvement” of a Adam’s inequality [**Ad**] which states that, for $1 < p < \frac{n}{s}$:

$$\int_{\mathbb{R}^n} (I_s f)(x)^p u(x)\,dx \le C \int_{\mathbb{R}^n} f^p(x)(M_{spt} u^t)^{\frac{1}{t}}(x)\,dx \qquad \text{for all } f(.) \ge 0 \text{ and } 1 < t < \frac{n}{sp},$$

with the constant $C > 0$ depending only on s, n and p.

Proposition 3.3. *Let $1 < p < \frac{n}{s}$. Suppose $u(.) \in RD_\rho \cap \mathcal{H}$ for a $\rho > 0$. Then for a constant $C > 0$:*

$$\int_{\mathbb{R}^n} (I_s f)(x)^p u(x)\,dx \le C \int_{\mathbb{R}^n} f^p(x)(M_{sp} u)(x)\,dx \qquad \textit{for all } f(.) \ge 0, \tag{3.3}$$

Here C depends on n, p and on the constants in the conditions RD_ρ and $\mathcal{H}$.

Inequality (3.3) is better than the previous one since $(M_\beta w)(x) \le (M_{\beta t} w^t)^{\frac{1}{t}}(x)$.

Let us consider again the weighted inequality for the restricted operator $\widetilde{I}_s$.

Proposition 3.4. *Suppose* $u(.), v^{-\frac{1}{p-1}}(.) \in \mathcal{H}$. *The estimate* $\widetilde{I}_s : L^p_v \to L^p_u$ *holds whenever the Muckenhoupt condition (1.1) is satisfied.*

Consequently we have

Corollary 3.5. *Suppose* $u(.), v^{-\frac{1}{p-1}}(.) \in \mathcal{H}$. *The estimate* $I_s : L^p_v \to L^p_u$ *holds if and only if both the Muckenhoupt condition (1.1), the Hardy conditions (1.2) and* (1.2^*) *are satisfied.*

As for the case of the fractional maximal operator, we have

Corollary 3.6. *Let* ρ, β, δ, ν, $u(.)$ *and* $v(.)$ *as in Corollary 2.6, and where instead of (2.4):*

$$sp < \nu < np. \tag{3.4}$$

Then $I_s : L^p_v \to L^p_u$ *if and only if condition (2.7) is satisfied.*

Observe that for fixed constants $C, c > 0$ then

$$(\widetilde{I}_s u)(x) \leq C|x|^s \int_{|y|<c|x|} u(y)\, dy$$

whenever $u(.) \in RD_\rho$ with $1 - \frac{s}{n} < \rho$. So we are attempted to state that the Muckenhoupt condition (1.1) implies $I_s : L^p_v \to L^p_u$ whenever both $u(.)$ and $v^{-\frac{1}{p-1}}(.)$ satisfy the reverse doubling condition RD_ρ with $1 - \frac{s}{n} < \rho$. Unfortunately this is not the case, since

Lemma 3.7. *There is no nontrivial weight functions* $u(.)$ *and* $v(.)$ *so that both* $u(.)$ *and* $v^{-\frac{1}{p-1}}(.)$ *satisfy the reverse doubling condition* RD_ρ *with* $1 - \frac{s}{n} < \rho$ *and for which the Muckenhoupt type condition (1.1) is satisfied.*

Theorems 3.1 and 2.1 will be proved in Section 5. Proofs for all propositions and corollaries are presented in the next paragrah.

4. Proofs of Propositions and Corollaries

Proof of Propositions 3.2 and 2.2: To derive $\widetilde{I}_s : L^p_v \to L^p_u$ from condition (3.1), we set $\mathcal{C}(x) = \{y : \frac{1}{2}|x| < |y| \leq 2|x|\}$. By the Hölder inequality

$$(\widetilde{I}_s f)(x) \leq \left(\widetilde{I}_s v^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}}(x) \times \left(\int_{z\in\mathbb{R}^n} |x-z|^{s-n} 1\!\!\mathrm{I}_{\mathcal{C}(x)}(z) f^p(z) v(z)\, dz\right)^{\frac{1}{p}}$$

for $f(.) \geq 0$.

Consequently

$$\int_{x\in\mathbb{R}^n} (\widetilde{I}_s f)^p(x)u(x)\,dx$$

$$\leq \int_{x\in\mathbb{R}^n} \left[\int_{z\in\mathbb{R}^n} |x-z|^{s-n}\mathbb{1}_{\mathcal{C}(x)}(z)f^p(z)v(z)\,dz\right](\widetilde{I}_s v^{-\frac{1}{p-1}})^{p-1}(x)u(x)\,dx$$

$$= \int_{z\in\mathbb{R}^n} f^p(z)v(z)\left[\int_{x\in\mathcal{C}(z)} |z-x|^{s-n}(\widetilde{I}_s v^{-\frac{1}{p-1}})^{p-1}(x)u(x)\,dx\right]dz$$

by: $z\in\mathcal{C}(x)$ iff $x\in\mathcal{C}(z)$

$$= \int_{z\in\mathbb{R}^n} f^p(z)v(z)\Big(\widetilde{I}_s[u(\widetilde{I}_s v^{-\frac{1}{p-1}})^{p-1}]\Big)(z)\,dz \leq C\int_{z\in\mathbb{R}^n} f^p(z)v(z)\,dz$$

by condition (3.1).

Next our purpose is to get $\widetilde{M}_s : L^p_v \to L^p_u$ from condition (2.1). Also by the Hölder inequality, for each $t>1$:

$$(\widetilde{M}_s f)(x) = \sup_{0<r<\frac{1}{2}|x|}\left\{r^{s-n}\int_{B(x,r)}|f(y)|\,dy\right\}$$

where $B(x,r)=\{y;|x-y|<r\}$

$$\leq c|x|^s$$

$$\times \sup_{0<r<\frac{1}{2}|x|}\left\{\left(r^{-n}\int_{B(x,r)} v^{-\frac{t}{p-1}}(z)\,dz\right)^{\frac{1}{tp'}}\left(r^{-n}\int_{B(x,r)}(fv^{\frac{1}{p}})^{(tp')'}(y)\,dy\right)^{\frac{1}{(tp')'}}\right\}$$

$$\leq c|x|^s\Big(\widetilde{M}_0 v^{-\frac{t}{p-1}}\Big)^{\frac{1}{tp'}}(x)\times\Big(\widetilde{M}_0[fv^{\frac{1}{p}}]^{(tp')'}\Big)^{\frac{1}{(tp')'}}(x).$$

With this last inequality we can conclude as follows

$$\int_{x\in\mathbb{R}^n}(\widetilde{M}_s f)^p(x)u(x)\,dx$$

$$\leq c\int_{x\in\mathbb{R}^n}\Big(\widetilde{M}_0[fv^{\frac{1}{p}}]^{(tp')'}\Big)^{\frac{p}{(tp')'}}(x)\times|x|^{sp}\Big(\widetilde{M}_0 v^{-\frac{t}{p-1}}\Big)^{\frac{p}{tp'}}(x)u(x)\,dx$$

$$\leq cC^p\int_{x\in\mathbb{R}^n}\Big(M_0[fv^{\frac{1}{p}}]^{(tp')'}\Big)^{\frac{p}{(tp')'}}(x)\,dx \quad \text{by condition (2.1)}$$

$$\leq c_1C^p\int_{z\in\mathbb{R}^n} f^p(x)v(x)\,dx.$$

This last inequality is a consequence of the well-known maximal theorem which asserts that $M_0 : L_1^{\frac{p}{(tp')'}} \to L_1^{\frac{p}{(tp')'}}$ whenever $1 < \frac{p}{(tp')'}$ (which is true since $p' < tp'$ and $t > 1$). ■

Proof of Propositions 2.3 and 3.3: Since these results are proved in details in [**Ra**], we only outline the main points.

By duality, the estimate (2.3) it is equivalent $T^* : L^{p'}_{u^{1-p'}} \to L^{p'}_{(M_0 u)^{1-p'}}$, where T^* is the dual of the operator T. Note that $(M_0 u)^{1-p'}(.) = \Big[(M_0 u)^{\frac{p'-1}{t-1}}(.)\Big]^{1-t}$, with $0 < \frac{p'-1}{t-1} < 1$ for some $t > p'$, so $(M_0 u)^{1-p'}(.) \in A_\infty$ (see [**Ga-Rb**]) and then by Coifman [**Co**]:

$$\int_{\mathbb{R}^n} |(T^* f)(x)|^{p'} (M_0 u)^{1-p'}(x)\, dx \le C \int_{\mathbb{R}^n} |(M_0 f)(x)|^{p'} (M_0 u)^{1-p'}(x)\, dx$$

for a fixed constant $C > 0$. Consequently (2.3) is reduced to $M_0 : L^{p'}_{v_1} \to L^{p'}_{u_1}$ with $v_1(.) = u^{1-p'}(.)$, $u_1(.) = (M_0 u)^{1-p'}(.)$. From $u \in \mathcal{H}$ the pointwise inequality (2.2) holds [with $s = 0$ and respectively p, p', $v(.)$, $u(.)$ are changed into p', p, $v_1(.)$, $u_1(.)$]. For such $v_1(.)$ and $u_1(.)$ the corresponding Muckenhoupt condition (1.1) is satisfied. The associated Hardy condition (1.2) is implied by (1.1), since $\sigma_1(.) = v_1^{1-p}(.) = u(.) \in RD_\rho$ [see [**Sa-Wh**] for a proof of this implication].

For Proposition 3.3, it remains to prove $I_s : L^p_v \to L^p_u$ with $v(.) = (M_{sp} u)(.)$. Since $u(.) \in \mathcal{H}$ then condition (3.2) is satisfied. Here $u(.) \in RD_\rho$, $\sigma(.) = v^{-\frac{1}{p-1}}(.) \in RD_{\rho'}$ for some $\rho, \rho' > 0$; thus the Hardy conditions (1.2) and (1.2*) are implied by the Muckenhoupt condition (1.1), which is also satisfied. Here $\sigma(.) = (M_{sp} u)^{1-p'}(.) \in RD_{\rho'}$ since $(M_{sp} u)^{1-p'}(.) \in A_{p'} \subset D_\infty \subset \bigcup_\rho RD_\rho$. ■

Proof of Propositions 3.4 and 2.4: To prove $\widetilde{I}_s : L^p_v \to L^p_u$ it remains to get (3.2). Thus using $u(.), v^{-\frac{1}{p-1}}(.) \in \mathcal{H}$ and the Muckenhoupt condition (1.1), then

$$|x|^{sp}\Big(\sup_{\{\frac14|x|<|z|\le 4|x|\}} u(z)\Big)\Big(\sup_{\{\frac14|x|<|y|\le 4|x|\}} v^{-\frac{1}{p-1}}(y)\Big)^{p-1}$$

$$\le c_1 |x|^{sp}\left(\frac{1}{|x|^n}\int_{\{a|x|<|z|\le b|x|\}} u(z)\, dz\right)\left(\frac{1}{|x|^n}\int_{\{a|x|<|z|\le b|x|\}} v^{-\frac{1}{p-1}}(y)\, dy\right)^{p-1}$$

$$\le c_2 (b|x|)^{sp}\left(\frac{1}{(b|x|)^n}\int_{|z|<(b|x|)} u(z)\, dz\right)\left(\frac{1}{(b|x|)^n}\int_{|z|<(b|x|)} v^{-\frac{1}{p-1}}(y)\, dy\right)^{p-1}$$

$\le c_2 A^p$ by the Muckenhoupt condition (1.1).

The argument for the estimate $\widetilde{M}_s : L^p_v \to L^p_u$ follows after checking condition (2.2) [see the remark after Proposition 2.2]. This condition is satisfied since with $u(.), v^{-\frac{1}{p-1}}(.) \in \mathcal{H}$ then (1.1) $\Longrightarrow$ (3.2) and also trivially (3.2) $\Longrightarrow$ (2.2). The first implication is just proved above. ■

Proof of Lemma 3.7: The Muckenhoupt condition (1.1) can be written as $\left(R^{s-n} \int_{|x|<R} v^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1} \left(R^{s-n} \int_{|x|<R} u(y)\,dy\right) \leq C$ for all $R > 0$. Thus the contradiction will appear once we get

$$\lim_{R\to\infty} R^{s-n} \int_{|x|<R} v^{-\frac{1}{p-1}}(y)\,dy = \lim_{R\to\infty} R^{s-n} \int_{|x|<R} u(y)\,dy = \infty.$$

The proof can be limited for the second identity. Since $u(.)$ is not a trivial weight function, then $0 < \int_{|y|<c} u(y)\,dy < \infty$ for a constant $c > 0$. Without loss of generality it can be assumed that $c = 1$ and $R > 1$. From $u(.) \in RD_\rho$ then $\int_{|x|<1} u(y)\,dy \leq c_0 \left(\frac{|Q_0|}{|Q_1|}\right)^\rho \int_{Q_1} u(y)\,dy \leq c_1 R^{-n\rho} \int_{Q_1} u(y)\,dy$ for some fixed constants c_0 and $c_1 > 0$, here Q_0 [resp. Q_1] is the smallest cube containing the ball $B(0,1) = \{z;\ |z| < 1\}$ [resp. the largest cube contained in the ball $B(0,R) = \{z;\ |z| < R\}$]. Consequently $\left(c_2 \int_{|x|<1} u(y)\,dy\right) \times R^{n[\frac{s}{n}-1+\rho]} \leq R^{s-n} \int_{|x|<R} u(y)\,dy$ and then $\lim_{R\to\infty} R^{s-n} \int_{|x|<R} u(y)\,dy = \infty$ since $0 < \frac{s}{n} - 1 + \rho$. ■

Proof of Corollaries 2.6 and 3.6: These results are just based on the following lemmas.

Lemma 4.1. *Define*

$$\phi(r) = \phi_{\eta,\mu}(r) = r^{\eta-1} \ln^{-(\mu+1)}(e+r)\Big[\eta \ln(e+r) - \mu r(e+r)^{-1}\Big] \quad \textit{for all } r > 0,$$

and where $0 \leq \mu < \eta$. *Then*

$$\phi(r) = \frac{d}{dr}[\psi_{\eta,\mu}(r)] > 0 \quad \textit{for } r > 0 \textit{ and with } \psi_{\eta,\mu}(r) = r^\eta \ln^{-\mu}(e+r); \tag{4.1}$$

$$\phi(r) \approx r^{\eta-1} \quad \textit{for } 0 < r \leq e \textit{ and } \phi(r) \approx r^{\eta-1} \ln^{-\mu} r \textit{ for } r > e. \tag{4.2}$$

Lemma 4.2. *Define*

$$\begin{aligned} w(x) &= |x|^{1-n} \phi_{\eta,\mu}(|x|) \\ &= |x|^{\eta-n} \ln^{-(\mu+1)}(e+|x|)\Big[\eta \ln(e+|x|) - \mu|x|(e+|x|)^{-1}\Big] \end{aligned}$$

with $0 \le \mu < \eta$*. Then* $w(.) \in \mathcal{H}$*, and for a fixed constant* $C > 0$

$$\int_{|x|<R} w(x)\,dx \approx R^{\eta} \quad \text{for } 0 < R \le e \tag{4.3}$$

and

$$\int_{|x|<R} w(x)\,dx \approx R^{\eta} \ln^{-\mu} R \quad \text{for } R > e. \tag{4.4}$$

Moreover if $\eta < (n-s)q$ *then*

$$\int_{R<|x|} |x|^{(s-n)q} w(x)\,dx \le C R^{(s-n)q+\eta} \quad \text{for } 0 < R \le e \tag{4.5}$$

and

$$\int_{R<|x|} |x|^{(s-n)q} w(x)\,dx \le C R^{(s-n)q+\eta} \ln^{-\mu} R \quad \text{for } R > e. \tag{4.6}$$

Hypothesis (2.5) and inequalities (4.5) and (4.6), with $p = q$, imply $\int_{R<|x|} |x|^{(s-n)p} u(x)\,dx \le C R^{(s-n)p+\beta}$ for $0 < R \le e$ and $\int_{R<|x|} |x|^{(s-n)p} u(x)\,dx \le C R^{(s-n)p+\beta} \ln^{-\rho} R$ for $R > e$. Note that $\sigma(x) = v^{-\frac{1}{p-1}}(x) = |x|^{\lambda-n} \ln^{-(\delta+1)}(e+|x|)\Big[\lambda \ln(e+|x|) - \delta|x|(e+|x|)^{-1}\Big]$ with λ defined as in (2.6). Using (2.4), (2.6), (4.3) and (4.4) then $\int_{|x|<R} \sigma(x)\,dx \approx R^{\lambda}$ for $0 < R \le e$, and $\int_{|x|<R} \sigma(x)\,dx \le C R^{\lambda} \ln^{-\delta} R$ for $R > e$. Let $\mathcal{A}(R) = R^{s-n}\Big(\int_{|x|<R} u(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_{|x|<R} \sigma(x)\,dx\Big)^{\frac{1}{p'}}$ and $\mathcal{H}(R) = \Big(\int_{R<|x|} |x|^{(s-n)p} u(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_{|x|<R} \sigma(x)\,dx\Big)^{\frac{1}{p'}}$.

If $M_s : L^p_v \to L^p_u$ then in particular there is $C > 0$ so that

$$\mathcal{A}(R) \approx R^{s+\frac{\beta}{p}-\frac{\nu}{p}} \times \ln^{-(\frac{\rho}{p}+\frac{\delta}{p'})}(e+R) < C \quad \text{for all } R > 0.$$

Letting $R \to 0$ this forces that $0 \le s + \frac{\beta}{p} - \frac{\nu}{p}$, and really we have $0 = s + \frac{\beta}{p} - \frac{\nu}{p}$ else a contradiction appears by taking $R \to \infty$. Therefore (2.7) is a necessary condition for the above estimate.

Conversely, using (2.7) and these above computations, then $\mathcal{A}(R) < C$ and $\mathcal{H}(R) < C$ for all $R > 0$ and for a fixed constant $C > 0$. Thus by Corollary 2.5 then $M_s : L^p_v \to L^p_u$.

For the case of I_s then (4.5) and (4.6) [with $q = p'$] and condition (3.4) [which is stronger than (2.4)] are used to get

$$\int_{R<|x|} |x|^{(s-n)p'} \sigma(x)\, dx \le C R^{(s-n)p'+\lambda} \quad \text{for } 0 < R \le e$$

and

$$\int_{R<|x|} |x|^{(s-n)p'} \sigma(x)\, dx \le C R^{(s-n)p'+\lambda} \ln^{-\delta} R \quad \text{for } R > e,$$

and an estimate of

$$\mathcal{H}^*(R) = \Big(\int_{R<|x|} |x|^{(s-n)p'} \sigma(x)\, dx\Big)^{\frac{1}{p'}} \Big(\int_{|x|<R} u(x)\, dx\Big)^{\frac{1}{p}}.$$

Thus $I_s : L^p_v \to L^p_u$ if and only condition (2.7) is satisfied.

Now the above two lemmas remain to be proved.

Proof of Lemma 4.1: Clearly $\phi(r) = \frac{d}{dr}[r^\eta \ln^{-\mu}(e+r)]$ for $r > 0$ and since $\eta \ln(e+r) - \mu r(e+r)^{-1} > \eta - \mu > 0$ then $\phi(r) > 0$.

For $0 < r \le e$ then $1 < \ln(e+r) \le (1+\ln 2)$ and $\eta - \mu < \eta \ln(e+r) - \mu r(e+r)^{-1} \le (1+\ln 2)\eta$, so $\phi(r) \approx r^{\eta-1}$. For $r > e$ then $\ln r < \ln(e+r) \le (1+\ln 2)\ln r$ and $\eta \ln(e+r) - \mu r(e+r)^{-1} > \eta \ln r - \mu > (\eta - \mu)\ln r$ so $\phi(r) \approx r^{\eta-1} \ln^{-\mu} r$. ■

Proof of Lemma 4.2: Observe that by (4.1):

$$\begin{aligned}\int_{|x|<R} w(x)\, dx &\approx \int_0^R r^{\eta-1} \ln^{-(\mu+1)}(e+r)\Big[\eta \ln(e+r) - \mu r(e+r)^{-1}\Big]\, dr \\ &= \int_0^R \phi_{\eta,\mu}(r)\, dr = R^\eta \ln^{-\mu}(e+R).\end{aligned}$$

Consequently (4.3) and (4.4) are satisfied since as above $\ln^{-\mu}(e+R) \approx 1$ for $0 < R \le e$ and $\ln^{-\mu}(e+R) \approx \ln^{-\mu} R$ for $R > e$.

Inequality (4.5) appears after using (4.2). Indeed for $0 < R \le e$ then

$$\begin{aligned}\int_{R<|x|} |x|^{(s-n)q} w(x)\, dx &\approx \int_e^\infty r^{(s-n)q+\eta} \ln^{-\mu} r \frac{dr}{r} + \int_R^e r^{(s-n)q+\eta} \frac{dr}{r} \\ &\le c_1 + c_2 R^{(s-n)q+\eta} \le c_3 R^{(s-n)q+\eta}.\end{aligned}$$

For $R > e$ then (4.6) can be also deduced from (4.2) and by using the fact that $t \to \ln^{-\mu} t$ is a nonincreasing function and $(s-n)q + \eta < 0$.

To get $w(.) \in \mathcal{H}$ remind that, by (4.2): $w(x) \approx |x|^{\eta-n}$ for $0 < |x| \le e$ and $w(x) \approx |x|^{\eta-n} \ln^{-\mu} |x|$ for $|x| > e$. For R small i.e. $0 < R \le \frac{1}{16}e$ and $R < |x| < 16R$ then

$$w(x) \approx |x|^{\eta-n} \le c_1 R^{-n} \times R^{\eta-1} \times \int_{\frac{1}{2}R}^{R} dr \le c_2 R^{-n} \int_{\frac{1}{2}R}^{R} r^{\eta-1}\, dr$$

$$\le c_3 R^{-n} \int_{\frac{1}{2}R}^{R} \phi_{\eta,\mu}(r)\, dr \le c_4 R^{-n} \int_{\frac{1}{2}R<|x|<R} w(x)\, dx.$$

For R big i.e. $32e < R$ and $R < |x| < 16R$ then

$$\begin{aligned} w(x) \approx |x|^{\eta-n} \ln^{-\mu} |x| &\le c_1 R^{-n} \times R^{\eta-1} \ln^{-\mu} R \times \int_{\frac{1}{2}R}^{R} dr \\ &\le c_2 R^{-n} \int_{\frac{1}{2}R}^{R} r^{\eta-1} \ln^{-\mu} r\, dr \\ &\le c_3 R^{-n} \int_{\frac{1}{2}R}^{R} \phi_{\eta,\mu}(r)\, dr \\ &\le c_4 R^{-n} \int_{\frac{1}{2}R<|x|<R} w(x)\, dx \quad \text{since } e < \frac{1}{2}R. \end{aligned}$$

Finally assume $R \approx 1$ i.e. $\frac{1}{16}e < R \le 32e$. For $e < R < |x| < 16R$ or $R \le e < |x| < 16R$ then

$$\begin{aligned} w(x) \approx |x|^{\eta-n} \ln^{-\mu} |x| &\le c_1 R^{-n} \times R^{\eta-1} \times \int_{\frac{1}{64}R}^{\frac{1}{32}R} dr \\ &\le c_2 R^{-n} \int_{\frac{1}{64}R}^{\frac{1}{32}R} \phi_{\eta,\mu}(r)\, dr \\ &\le c_3 R^{-n} \int_{\frac{1}{64}R<|x|<\frac{1}{32}R} w(x)\, dx \quad \text{since } \frac{1}{32}R < e. \end{aligned}$$

For $R < |x| < e < 16R$ then

$$\begin{aligned} w(x) \approx |x|^{\eta-n} &\le c_1 R^{-n} \times R^{\eta-1} \times \int_{\frac{1}{2}R}^{R} dr \\ &\le c_2 R^{-n} \int_{\frac{1}{2}R}^{R} \phi_{\eta,\mu}(r)\, dr \\ &\le c_3 R^{-n} \int_{\frac{1}{2}R<|x|<R} w(x)\, dx \quad \text{since } R < e. \ \blacksquare \end{aligned}$$

5. Proof of Theorems 3.1 and 2.1

The main key for these results is the well-known weighted inequalities for the Hardy operator [**Mu**], which in our context is stated as follows

Lemma. *There is $C > 0$ such that*

$$\int_{\mathbb{R}^n} \left[\int_{|y|\leq|x|} f(y)\,dy \right]^p w(x)\,dx \leq C^p \int_{\mathbb{R}^n} f^p(x)v(x)\,dx \quad \textit{for all } f(.) \geq 0$$

if and only for a constant $A > 0$

$$\text{(5.1)}\quad \left(\int_{R<|y|} w(y)\,dy \right)^{\frac{1}{p}} \left(\int_{|y|<R} v^{-\frac{1}{p-1}}(y)\,dy \right)^{\frac{1}{p'}} \leq A \quad \textit{for all } R > 0.$$

Also if $w^{-\frac{1}{p-1}}(.) \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$ then

$$\int_{\mathbb{R}^n} \left[\int_{|x|<|y|} g(y)\,dy \right]^p u(x)\,dx \leq C^p \int_{\mathbb{R}^n} g^p(x)w(x)\,dx \quad \textit{for all } g(.) \geq 0$$

if and only if

$$\text{(5.1}^*\text{)}\quad \left(\int_{|y|<R} u(y)\,dy \right)^{\frac{1}{p}} \left(\int_{R<|y|} w^{-\frac{1}{p-1}}(y)\,dy \right)^{\frac{1}{p'}} \leq A \quad \textit{for all } R > 0.$$

The constants A and C are related by the relation $c_1 A \leq C \leq c_2 A$ with c_1, $c_2 > 0$ depending on n and p.

We first begin with the proof for the fractional integral operator I_s, $0 < s < n$, which is curiously easier to handle than the case of M_s.

The Fractional Integral Operator.

First suppose $I_s : L^p_v \to L^p_u$. Then $\widetilde{I}_s : L^p_v \to L^p_u$, since $(\widetilde{I}_s f)(.) \leq (I_s f)(.)$. And $\int_{\mathbb{R}^n} \left[\int_{|y|<|x|} f(y)\,dy \right]^p |x|^{(s-n)p} u(x)\,dx \leq C \int_{\mathbb{R}^n} f^p(y)v(y)\,dy$ since $|x|^{s-n} \int_{|y|<|x|} f(y)\,dy \leq c(I_s f)(x)$. So the Hardy condition (1.2) appears in virtue of the above lemma [with $w(x) = |x|^{(s-n)p}u(x)$]. On the other hand $\int_{\mathbb{R}^n} \left[\int_{|y|<|x|} |y|^{s-n} f(y)\,dy \right]^p u(x)\,dx \leq C \int_{\mathbb{R}^n} f^p(y)v(y)\,dy$ since $\int_{|x|<|y|} |y|^{s-n} f(y)\,dy \leq c(I_s f)(x)$, so condition (1.2*) also holds by using (5.1*) in the lemma.

For the converse, it is assumed that $\widetilde{I}_s : L^p_v \to L^p_u$ and both the conditions (1.2) and (1.2*) are satisfied. Our purpose is to get $I_s : L^p_v \to L^p_u$. Since

$$(I_s f)(x) = A_1(x) + A_2(x) + A_3(x)$$

with

$$\begin{aligned}
A_1(x) &= \int_{|y| \le \frac{1}{2}|x|} |x-y|^{s-n} f(y)\,dy,\\
A_2(x) &= \int_{\frac{1}{2}|x| < |y| < 2|x|} |x-y|^{s-n} f(y)\,dy,\\
A_3(x) &= \int_{2|x| \le |y|} |x-y|^{s-n} f(y)\,dy,
\end{aligned}$$

then it is sufficient to estimate each of $\int_{\mathbb{R}^n} A_i^p(x) u(x)\,dx$, $i \in \{1,2,3\}$, by $C \int_{\mathbb{R}^n} f^p(x) v(x)\,dx$, where $C > 0$ is a fixed constant.

Clearly

$$\int_{\mathbb{R}^n} A_2^p(x) u(x)\,dx = \int_{\mathbb{R}^n} (\widetilde{I}_s f)^p(x) u(x)\,dx \le C \int_{\mathbb{R}^n} f^p(x) v(x)\,dx.$$

Observe that $A_1(x) \le c|x|^{s-n} \int_{|y|<|x|} f(y)\,dy$, since $\frac{1}{2}|x| < |x-y|$ whenever $|y| < \frac{1}{2}|x|$. By the Hardy condition (1.2), [which is (5.1) with $w(x) = |x|^{(s-n)p}$], then

$$\begin{aligned}
\int_{\mathbb{R}^n} A_1^p(x) u(x)\,dx &\le c \int_{\mathbb{R}^n} \left[\int_{|y|<|x|} f(y)\,dy \right]^p |x|^{(s-n)p} u(x)\,dx\\
&\le A^p \int_{\mathbb{R}^n} f^p(x) v(x)\,dx.
\end{aligned}$$

Note that $A_3(x) \le c \int_{|x|<|y|} |y|^{s-n} f(y)\,dy$ since $\frac{1}{2}|y| \le |x-y|$ whenever $2|x| < |y|$. By the Hardy condition (1.2*), [which is (5.1*) with $w(x) = |x|^{(n-s)} v(x)$], then

$$\begin{aligned}
\int_{\mathbb{R}^n} A_3^p(x) u(x)\,dx &\le c \int_{\mathbb{R}^n} \left[\int_{|x|<|y|} |y|^{s-n} f(y)\,dy \right]^p u(x)\,dx\\
&\le A^p \int_{\mathbb{R}^n} f^p(x) v(x)\,dx.
\end{aligned}$$

Now we give the proof for M_s, $0 \le s < n$.

The Fractional Maximal Operator.

First suppose $M_s : L^p_v \to L^p_u$. Then $\widetilde{M}_s : L^p_v \to L^p_u$, since $(\widetilde{M}_s f)(.) \le (M_s f)(.)$. For all $f(.) \ge 0$ and x with $|x| < R > 0$ then $R^{s-n}\int_{|y|<R} f(y)\,dy \le R^{s-n}\int_{|x-y|<2R} f(y)\,dy \le c(M_s f)(x)$. On the other hand $|x|^{s-n}\int_{|y|<|x|} f(y)\,dy \le |x|^{s-n}\int_{|x-y|<2|x|} f(y)\,dy \le c(M_s f)(x)$. Thus the assumed estimate implies both (1.1) and (1.2).

For the converse assume $\widetilde{M}_s : L^p_v \to L^p_u$ and both (1.1) and (1.2) are satisfied. Our purpose is now to get $M_s : L^p_v \to L^p_u$. Since

$$(M_s f)(x) \le c \sup_{0<t}\left\{ t^{s-n}\int_{B(x,t)} f(y)\,dy \right\} \le c\Big(A_1(x)+A_2(x)+A_3(x)+A_4(x)\Big)$$

with

$$A_1(x) = \sup_{0<t}\left\{ t^{s-n}\int_{B(x,t)\cap\{|y|<\frac{1}{2}|x|\}} f(y)\,dy \right\},$$

$$A_2(x) = \sup_{0<t<\frac{1}{2}|x|}\left\{ t^{s-n}\int_{B(x,t)\cap\{\frac{1}{2}|x|<|y|\le 2|x|\}} f(y)\,dy \right\},$$

$$A_3(x) = \sup_{\frac{1}{2}|x|\le t}\left\{ t^{s-n}\int_{B(x,t)\cap\{\frac{1}{2}|x|<|y|\le 2|x|\}} f(y)\,dy \right\},$$

$$A_4(x) = \sup_{|x|\le t}\left\{ t^{s-n}\int_{B(x,t)\cap\{2|x|<|y|\}} f(y)\,dy \right\},$$

then it is sufficient to estimate each of $\int_{\mathbb{R}^n} A_i^p(x)u(x)\,dx$, $i \in \{1,2,3,4\}$, by $C\int_{\mathbb{R}^n} f^p(x)v(x)\,dx$.

Clearly

$$\int_{\mathbb{R}^n} A_2^p(x)u(x)\,dx = \int_{\mathbb{R}^n} (\widetilde{M}_s f)^p(x)u(x)\,dx \le C\int_{\mathbb{R}^n} f^p(x)v(x)\,dx.$$

Observe that $A_1(x) \le c|x|^{s-n}\int_{|y|<|x|} f(y)\,dy$, since $\frac{1}{2}|x| < t$ whenever $|x-y| < t$ and $|y| < \frac{1}{2}|x|$. Using the lemma as above, then

$$\int_{\mathbb{R}^n} A_1^p(x)u(x)\,dx \le c\int_{\mathbb{R}^n}\left[\int_{|y|<|x|} f(y)\,dy\right]^p |x|^{(s-n)p}u(x)\,dx \le A^p\int_{\mathbb{R}^n} f^p(x)v(x)\,dx.$$

Since $A_3(x) \leq c|x|^{s-n} \int_{\{\frac{1}{2}|x|<|y|\leq 2|x|\}} f(y)\, dy$ then the inequality

$$\int_{\mathbb{R}^n} A_3^p(x) u(x)\, dx \leq A^p \int_{\mathbb{R}^n} f^p(x) v(x)\, dx$$

is reduced to

$$\int_{\mathbb{R}^n} \left[\int_{\{\frac{1}{2}|x|<|y|\leq 2|x|\}} f(y)\, dy \right]^p |x|^{(s-n)p} u(x)\, dx \leq cA^p \int_{\mathbb{R}^n} f^p(x) v(x)\, dx.$$

For this last inequality the Muckenhoupt condition (1.1) is needed as follows

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left[\int_{\{\frac{1}{2}|x|<|y|\leq 2|x|\}} f(y)\, dy \right]^p |x|^{(s-n)p} u(x)\, dx \\
&\quad = \sum_k \int_{2^k<|x|\leq 2^{k+1}} \left[\int_{\{\frac{1}{2}|x|<|y|\leq 2|x|\}} f(y)\, dy \right]^p |x|^{(s-n)p} u(x)\, dx \\
&\quad \leq c_1 \sum_k 2^{k(s-n)p} \left[\int_{2^{k-1}<|y|\leq 8 2^{k-1}} f(y)\, dy \right]^p \left(\int_{2^k<|x|\leq 2^{k+1}} u(x)\, dx \right) \\
&\quad \leq c_2 \sum_k \left[2^{(k+2)(s-n)} \left(\int_{|x|<2^{k+2}} v^{-\frac{1}{p-1}}(x)\, dx \right)^{\frac{1}{p'}} \left(\int_{|x|\leq 2^{k+2}} u(x)\, dx \right)^{\frac{1}{p}} \right]^p \\
&\quad \times \int_{2^{k-1}<|y|\leq 8 2^{k-1}} f^p(y) v(y)\, dy \\
&\quad \leq c_2 A^p \sum_k \int_{2^{k-1}<|y|\leq 8 2^{k-1}} f^p(y) v(y)\, dy \\
&\quad \leq 3c_2 A^p \sum_h \int_{2^h<|y|\leq 2^{h+1}} f^p(y) v(y)\, dy \\
&\quad = 3c_2 A^p \int_{\mathbb{R}^n} f^p(y) v(y)\, dy.
\end{aligned}$$

The main key to get

$$\int_{\mathbb{R}^n} A_4^p(x) u(x)\, dx \leq A^p \int_{\mathbb{R}^n} f^p(x) v(x)\, dx$$

is the pointwise inequality

$$A_4(x) \le c \sup_{j\in\mathbb{N}^*} \left\{ (2^j|x|)^{s-n} \int_{\{2^j|x|<|y|\le 2^{j+1}|x|\}} f(y)\,dy \right\}. \tag{5.2}$$

Indeed with (5.2) then

$$\begin{aligned}
&\int_{\mathbb{R}^n} A_4^p(x)u(x)\,dx \\
&\le c_0 \sum_k \int_{2^k<|x|\le 2^{k+1}} \left[\sup_{j\in\mathbb{N}^*} (2^j|x|)^{s-n} \int_{\{2^j|x|<|y|\le 2^{j+1}|x|\}} f(y)\,dy \right]^p u(x)\,dx \\
&\le c_1 \sum_k \left[\sup_{j\in\mathbb{N}^*} 2^{(j+k)(s-n)} \int_{\{2^{(j+k)}<|y|\le 2^{(j+k+2)}\}} f(y)\,dy \right]^p \left(\int_{2^k<|x|\le 2^{k+1}} u(x)\,dx \right) \\
&\le c_1 \sum_k \sum_{j=1}^{\infty} \left[2^{(j+k)(s-n)} \int_{\{2^{(j+k)}<|y|\le 2^{(j+k+2)}\}} f(y)\,dy \right]^p \left(\int_{2^k<|x|\le 2^{k+1}} u(x)\,dx \right) \\
&= c_1 \sum_m \left[2^{m(s-n)} \int_{\{2^m<|y|\le 2^{m+2}\}} f(y)\,dy \right]^p \sum_{k=-\infty}^{m-1} \left(\int_{2^k<|x|\le 2^{k+1}} u(x)\,dx \right) \\
&\le c_1 \sum_m \left[2^{m(s-n)} \int_{\{2^m<|y|\le 2^{m+2}\}} f(y)\,dy \right]^p \left(\int_{|x|\le 2^m} u(x)\,dx \right) \\
&\le 2c_2 A^p \int_{\mathbb{R}^n} f^p(y)v(y)\,dy.
\end{aligned}$$

This last inequality can be obtained as above, by using the Hölder inequality and (1.1).

Finally to prove (5.2), let

$$\mathcal{S} = \sup_{j\in\mathbb{N}^*} \left\{ (2^j|x|)^{s-n} \int_{\{2^j|x|<|y|\le 2^{j+1}|x|\}} f(y)\,dy \right\}.$$

The conclusion will follow once

$$\int_{B(x,t)\cap\{2|x|<|y|\}} f(y)\,dy \le c\mathcal{S}t^{n-s}$$

for a fixed constant $c>0$ and for all x and t with $B(x,t)\cap\{2|x|<|y|\}\neq\emptyset$. Let x and t satisfying this property. Then $B(x,t)\cap\{2^N|x| < |y| \le$

$2^{N+1}|x|\} \neq \emptyset$ and $B(x,t)\cap\{2^{N+1}|x| < |y| \leq 2^{N+2}|x|\} = \emptyset$, for a nonnegative integer N depending on x, t and $f(.)$. In particular: $2^{N-1}|x| < t$. So the conclusion appears since

$$\begin{aligned}\int_{B(x,t)\cap\{2|x|<|y|\}} f(y)\,dy &= \sum_{k=1}^{N}\int_{B(x,t)\cap\{2^k|x|<|y|\leq 2^{k+1}|x|\}} f(y)\,dy \\ &\leq \sum_{k=1}^{N}\int_{\{2^k|x|<|y|\leq 2^{k+1}|x|\}} f(y)\,dy \\ &\leq \mathcal{S}\sum_{k=1}^{N}(2^k|x|)^{n-s} = \mathcal{S}|x|^{n-s}\sum_{k=1}^{N}(2^k)^{n-s} \\ &\leq c_1\mathcal{S}|x|^{n-s}(2^N)^{n-s} \leq c_2\mathcal{S}t^{n-s}.\end{aligned}$$

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