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A STABILITY RESULT ON MUCKENHOUPT'S WEIGHTS

Juha Kinnunen

Abstract

We prove that Muckenhoupt's $\mathcal{A}_1$-weights satisfy a reverse Hölder inequality with an explicit and asymptotically sharp estimate for the exponent. As a by-product we get a new characterization of $\mathcal{A}_1$-weights.

1. Introduction and statement of results

Muckenhoupt's weights are important tools in harmonic analysis, partial differential equations and quasiconformal mappings. The self-improving property of Muckenhoupt's weights is probably one of the most useful results in the field. The surprising fact that the weights are more regular than they seem to be *a priori* was observed already by Muckenhoupt [**16**]. The same phenomenon was studied by Gehring in [**6**] where he introduced the concept of reverse Hölder inequalites and proved that they improve themselves. Later Coifman and Fefferman [**3**] showed that Muckenhoupt's weights are exactly those weights which satisfy a reverse Hölder inequality. Since then reverse Hölder inequalities have had a vast number of applications in modern analysis. An excellent source for all the mentioned results and other properties of Muckenhoupt's weights is the monograph [**7**].

We are interested in a stability question related to Muckenhoupt's $\mathcal{A}_1$-class and reverse Hölder inequalities. Suppose that $w : \mathbf{R}^n \to [0,\infty]$ is a locally integrable function satisfying Muckenhoupt's $\mathcal{A}_1$-condition,

$$\frac{1}{|B|}\int_B w(x)\,dx \le c_w \operatorname*{ess\,inf}_{x\in B} w(x), \tag{1.1}$$

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for all balls $B \subset \mathbf{R}^n$ with the constant $c_w \geq 1$ independent of the ball B. Here $|B|$ is the volume of B. If w belongs to Muckenhoupt's class $\mathcal{A}_1$, we denote $w \in \mathcal{A}_1$; the smallest constant c_w for which (1.1) holds is called the $\mathcal{A}_1$-constant of w.

Condition (1.1) can be expressed in terms of the Hardy-Littlewood maximal function, defined by

$$\mathcal{M}w(x) = \sup_B \frac{1}{|B|} \int_B w(y)\, dy,$$

where the supremum is over all balls $B \subset \mathbf{R}^n$ containing the point x. It is easy to see [**7**, p. 389] that (1.1) is equivalent to the requirement that

$$\mathcal{M}w(x) \leq c_w\, w(x)$$

almost everywhere with exactly the same c_w as in (1.1).

It is clear that (1.1) imposes a serious restriction on the function. If the $\mathcal{A}_1$-constant is one, then

$$0 \leq \frac{1}{|B|} \int_B \bigl(w(y) - \operatorname*{ess\,inf}_{x \in B} w(x)\bigr)\, dy \leq \operatorname*{ess\,inf}_{x \in B} w(x) - \operatorname*{ess\,inf}_{x \in B} w(x) = 0$$

and hence w is constant. We are interested in the regularity of $\mathcal{A}_1$-weights as the constant tends to one. It is well-known that $\mathcal{A}_1$-weights satisfy the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B w(x)^p\, dx\right)^{1/p} \leq c \frac{1}{|B|} \int_B w(x)\, dx, \tag{1.2}$$

for some $p > 1$ and c independent of the ball B. Using (1.2) and (1.1) we see that $w^p \in \mathcal{A}_1$ and w is locally integrable to power p. The question is: how large can p be? If the $\mathcal{A}_1$-constant is one, then the weight is essentially bounded and it seems reasonable to expect that the degree of the local integrablity increases as the $\mathcal{A}_1$-constant tends to one. Questions related to the stability of reverse Hölder inequalities have obtained considerable attention in the last two decades, see [**1**], [**2**], [**9**], [**11**], [**12**], [**13**], [**14**], [**15**], [**17**], [**18**], [**19**], [**20**] and [**21**].

Our contribution is twofold. First, we present a new and a simple method which gives an explicit and asymptotically optimal bound for p. Second, our proof leads to a new characterization of $\mathcal{A}_1$-weights (Corollary 2.11) which may be of independent interest.

Now we are ready to present our main result.

1.3. Theorem. *If $w \in \mathcal{A}_1$ with the constant c_w, then there is a constant ν depending only on the dimension such that w satisfies the reverse Hölder inequality (1.2) whenever*

$$1 \le p < 1 + \frac{\nu}{c_w - 1}. \tag{1.4}$$

In the one-dimensional case we may take $\nu = 1$ in (1.4), see [**2**] and [**11**], but our proof generally yields a small ν. Our method also allows us to replace balls in the $\mathcal{A}_1$-condition by cubes. Observe that the bound (1.4) for the local integrability of the weight is arbitrarily large provided c_w is close enough to one.

We remark that using factorization results of [**10**] and [**4**], our method gives similar estimates for Muckenhoupt's $\mathcal{A}_p$-weights as well. In the one-dimensional case this has been studied by Neugebauer [**17**].

2. Characterization of $\mathcal{A}_1$-weights

We begin by showing that every $\mathcal{A}_1$-weight can be approximated by smooth $\mathcal{A}_1$-weights.

2.1. Lemma. *Suppose that $w \in \mathcal{A}_1$ with the constant c_w and let $\varphi \in C_0^\infty(\mathbf{R}^n)$, $\varphi \ge 0$ with $\int_{\mathbf{R}^n} \varphi\, dx = 1$. Then $w * \varphi \in \mathcal{A}_1$ with the constant c_w.*

Proof: A direct calculation gives

$$\begin{aligned}
\frac{1}{B(x,r)} \int_{B(x,r)} w * \varphi(y)\, dy &= \frac{1}{B(x,r)} \int_{B(x,r)} \int_{\mathbf{R}^n} w(y-z)\varphi(z)\, dz\, dy \\
&= \int_{\mathbf{R}^n} \varphi(z) \frac{1}{B(x-z,r)} \int_{B(x-z,r)} w(y)\, dy\, dz \\
&\le c_w \int_{\mathbf{R}^n} \varphi(z) \operatorname*{ess\,inf}_{y \in B(x-z,r)} w(y)\, dz \\
&= c_w \int_{\mathbf{R}^n} \varphi(z) \operatorname*{ess\,inf}_{y \in B(x,r)} w(y-z)\, dz \\
&\le c_w \operatorname*{ess\,inf}_{y \in B(x,r)} \int_{\mathbf{R}^n} w(y-z)\varphi(z)\, dz \\
&= c_w \operatorname*{ess\,inf}_{y \in B(x,r)} w * \varphi(y).
\end{aligned} \tag{2.2}$$

This completes the proof. ■

We record a well-known covering theorem.

2.3. Besicovitch's covering Theorem. *Suppose that E is a bounded subset of $\mathbf{R}^n$ and that $\mathcal{B}$ is a collection of balls such that each point of E is a center of some ball in $\mathcal{B}$. Then there exists an integer $N \geq 2$ (depending only on the dimension) and subcollections $\mathcal{B}_1, \dots, \mathcal{B}_N \subset \mathcal{B}$ of at most countably many balls such that the balls $B_{i,j}$, $j = 1, 2, \dots$, in each family $\mathcal{B}_i$, $i = 1, 2, \dots, N$, are pairwise disjoint and*

$$E \subset \bigcup_{i=1}^{N} \bigcup_{j=1}^{\infty} B_{i,j}.$$

For the proof of Besicovitch's covering Theorem we refer to [**5**, Theorem 1.1]. Some estimates for the constant N are obtained in [**8**].

Now we show that $\mathcal{A}_1$-weights satisfy a reverse Chebyshev inequality. This observation is a crucial ingredient in the proof of Theorem 1.3. For short we denote

$$E_\lambda = \{x \in \mathbf{R}^n : w(x) > \lambda\}, \quad \lambda > 0,$$

throughout the paper.

2.4. Lemma. *Let $B \subset \mathbf{R}^n$ be a ball and suppose that $w : \mathbf{R}^n \to [0, \infty]$ is an $\mathcal{A}_1$-weight with the constant c_w. Then there is a constant η, depending only on the dimension, so that*

$$\int_{E_\lambda \cap B} w(x)\, dx \leq \big(c_w + \eta(c_w - 1)\big)\lambda |E_\lambda \cap B|, \tag{2.5}$$

whenever $\operatorname{ess\,inf}_{x \in B} w(x) \leq \lambda < \infty$.

Proof: Fix a ball $B \subset \mathbf{R}^n$. Suppose first that w is a continuous $\mathcal{A}_1$-weight with the constant c_w and that $\lambda \geq \inf_{x \in B} w(x)$. Then E_λ is open and for every $x \in E_\lambda$ we take the ball $B(x, r_x)$ where r_x is the distance from x to the boundary of E_λ. Let $\mathcal{B} = \{B(x, r_x) : x \in E_\lambda \cap B\}$. The radii of the balls in $\mathcal{B}$ are bounded, because $\overline{B} \setminus E_\lambda \neq \emptyset$. By Besicovitch's covering Theorem, there are families $\mathcal{B}_i = \{B_{i,j} : j = 1, 2, \dots\}$, $i = 1, 2, \dots, N$, of countably many balls, chosen from $\mathcal{B}$, such that

$$E_\lambda \cap B = \bigcup_{i=1}^{N} \bigcup_{j=1}^{\infty} B_{i,j} \cap B$$

and the balls in every $\mathcal{B}_i$, $i = 1, 2, \dots, N$, are pairwise disjoint. We denote the union of the pairwise disjoint balls by

$$E_\lambda^i = \bigcup_{j=1}^{\infty} B_{i,j}, \quad i = 1, 2, \dots, N.$$

The balls $B_{i,j}$ touch the boundary of E_λ and, since w is continuous, using the $\mathcal{A}_1$-condition we get

$$(2.6)\qquad \frac{1}{|B_{i,j}|}\int_{B_{i,j}} w(x)\,dx \le c \inf_{x\in B_{i,j}} w(x) \le c\lambda, \qquad i=1,2,\dots,N,\ j=1,2,\dots$$

The balls $B_{i,j}$ are not, in general, contained in B, but there is a constant $\gamma>0$, depending only on the dimension, so that

$$|B_{i,j}\setminus B| \le \gamma |B_{i,j}\cap B|, \quad i=1,2,\dots,N,\ j=1,2,\dots$$

To see this, let $B_{i,j}$ be the ball $B(x,r_x)\subset E_\lambda$ with $x\in E_\lambda\cap B$. Then by geometry, there is a ball $B(y,r_x/2)\subset B(x,r_x)\cap B$. This gives us the estimate

$$|B(x,r_x)\setminus B| \le |B(x,r_x)| = 2^n|B(y,r_x/2)| \le 2^n|B(x,r_x)\cap B|.$$

Hence we may take $\gamma = 2^n$.

By observing that $w(x)>\lambda$ for every $x\in B_{i,j}$ and recalling (2.6) we see that

$$\begin{aligned}\int_{B_{i,j}\cap B} w(x)\,dx &\le c_w\lambda|B_{i,j}\cap B| + c_w\lambda|B_{i,j}\setminus B| - \int_{B_{i,j}\setminus B} w(x)\,dx\\ &\le c_w\lambda|B_{i,j}\cap B| + (c_w-1)\lambda|B_{i,j}\setminus B|\\ &\le \big(c_w+\gamma(c_w-1)\big)\lambda|B_{i,j}\cap B|,\\ &\qquad\qquad i=1,2,\dots,N,\ j=1,2,\dots\end{aligned}$$

Since the balls in each $\mathcal{B}_i$, $i=1,2,\dots,N$, are pairwise disjoint, we arrive at

$$(2.7)\qquad \begin{aligned}\int_{E_\lambda^i\cap B} w(x)\,dx &= \sum_{j=1}^{\infty}\int_{B_{i,j}\cap B} w(x)\,dx\\ &\le \big(c_w+\gamma(c_w-1)\big)\lambda|E_\lambda^i\cap B|, \quad i=1,2,\dots,N.\end{aligned}$$

Let μ be a measure. Then we use the elementary inequality

$$(2.8)\qquad \mu(E_\lambda\cap B) = \sum_{i=1}^{N}\mu(E_\lambda^i\cap B) - \sum_{k=2}^{N}\mu(F_\lambda^k\cap B),$$

where

$$F_\lambda^k = \bigcup_{\{l_1,\dots,l_k\}\subset\{1,\dots,N\}} \big(E_\lambda^{l_1}\cap\dots\cap E_\lambda^{l_k}\big), \quad k=2,3,\dots,N.$$

A simple computation using (2.8), (2.7) and the fact that $w(x)>\lambda$ in $F_\lambda^k\cap B$, $k=2,\dots,N$, gives

$$\begin{aligned}
\int_{E_\lambda\cap B} w(x)\,dx &= \sum_{i=1}^N \int_{E_\lambda^i\cap B} w(x)\,dx - \sum_{k=2}^N \int_{F_\lambda^k\cap B} w(x)\,dx \\
&\le \big(c_w+\gamma(c_w-1)\big)\lambda \sum_{i=1}^N |E_\lambda^i\cap B| - \lambda\sum_{k=2}^N |F_\lambda^k\cap B| \\
&= \big(c_w+\gamma(c_w-1)\big)\lambda|E_\lambda\cap B| + \lambda(1+\gamma)(c_w-1)\sum_{k=2}^N |F_\lambda^k\cap B| \\
&\le \big(c_w+\gamma(c_w-1)\big)\lambda|E_\lambda\cap B| + (N-1)(1+\gamma)(c_w-1)\lambda|E_\lambda\cap B| \\
&= \big(c_w+\eta(c_w-1)\big)\lambda|E_\lambda\cap B|,
\end{aligned} \tag{2.9}$$

where $\eta = N\gamma+N-1$ and $\lambda\ge\inf_{x\in B} w(x)$.

The general case follows from a standard approximation argument using Lemma 2.1. Suppose that $w\in\mathcal{A}_1$ with the constant c_w. Let $\varphi\in C_0^\infty(\mathbf{R}^n)$, $\varphi\ge 0$ with $\int_{\mathbf{R}^n}\varphi\,dx=1$. We define $w_\varepsilon = w*\varphi_\varepsilon$, where $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$ and $\varepsilon>0$. Lemma 2.1 shows that w_ε is a continuous $\mathcal{A}_1$-weight with the constant c_w for every $\varepsilon>0$. Using (2.9) we see that

$$\int_{\{w_\varepsilon>\lambda\}\cap B} w_\varepsilon(x)\,dx \le \big(c_w+\eta(c_w-1)\big)\lambda|\{w_\varepsilon>\lambda\}\cap B|, \qquad \inf_{x\in B} w_\varepsilon(x)\le\lambda<\infty.$$

Letting $\varepsilon\to 0$ we obtain (2.5). This completes the proof. ■

2.10. Remark. (1) Observe that the constant on the right side of (2.5) tends to one as c_w tends to one. On the other hand, it blows up as c_w increases.

(2) We also remark that inequalities of type (2.5) appear already in the proof of Theorem 4 in [**3**]. However, their approach does not seem to give the correct behaviour as c_w tends to one.

We observe that (2.5) gives a characterization of $\mathcal{A}_1$-weights.

2.11. Corollary. *Suppose that $w : \mathbf{R}^n \to [0,\infty]$ is a measurable function. Then $w \in \mathcal{A}_1$ if and only if there is a constant c, independent of the ball B, so that*

$$\int_{E_\lambda \cap B} w(x)\,dx \le c\,\lambda |E_\lambda \cap B|, \quad \operatorname*{ess\,inf}_{x\in B} w(x) \le \lambda < \infty, \tag{2.12}$$

for every ball $B \subset \mathbf{R}^n$.

Proof: Lemma 2.4 shows that every $\mathcal{A}_1$-weight satisfies (2.12).

To see the reverse implication suppose that (2.12) holds and let B be a ball in $\mathbf{R}^n$. Then

$$\begin{aligned}\int_B w(x)\,dx &= \int_{B\setminus E_\lambda} w(x)\,dx + \int_{E_\lambda\cap B} w(x)\,dx \\ &\le \lambda|B\setminus E_\lambda| + c\,\lambda|B\cap E_\lambda| \\ &\le c\,\lambda|B|, \quad \operatorname*{ess\,inf}_{x\in B} w(x) \le \lambda < \infty.\end{aligned}$$

By inserting $\lambda = \operatorname{ess\,inf}_{x\in B} w(x)$ we get

$$\frac{1}{|B|}\int_B w(x)\,dx \le c \operatorname*{ess\,inf}_{x\in B} w(x),$$

where the constant is independent of the ball and hence $w \in \mathcal{A}_1$. ■

2.13. Remark. In the one-dimensional case we may take the constant in (2.12) equal to the $\mathcal{A}_1$-constant of w, see [**11**].

Lemma 2.4 shows that w satisfies the assumptions of the following sharp version Muckenhoupt's Lemma 4 in [**16**]. See also Lemma 2 in [**2**]. The proof of the following lemma can be found in [**11**], but we present it here for the sake of completeness.

2.14. Lemma. *Suppose that $w : \mathbf{R}^n \to [0,\infty]$ is a measurable function and let $B \subset \mathbf{R}^n$ be a ball. If there are $\alpha \ge 0$ and $c > 1$ such that*

$$\int_{E_\lambda \cap B} w(x)\,dx \le c\,\lambda |E_\lambda \cap B|, \quad \alpha \le \lambda < \infty, \tag{2.15}$$

then for every p, $1 < p < c/(c-1)$, we have

$$\int_{E_\alpha \cap B} w(x)^p\,dx \le \frac{c}{c - p\,(c-1)}\alpha^p |E_\alpha \cap B|. \tag{2.16}$$

Proof: Let $\beta > \alpha$ and denote $w_\beta = \min(w, \beta)$. Then

$$\int_{\{w_\beta > \lambda\} \cap B} w(x)\, dx \le c\,\lambda |\{w_\beta > \lambda\} \cap B|, \quad \alpha \le \lambda < \infty.$$

We multiply both sides by λ^{p-2} and integrate from α to ∞. This implies

$$\int_\alpha^\infty \lambda^{p-2} \int_{\{w_\beta > \lambda\} \cap B} w(x)\, dx\, d\lambda \le c \int_\alpha^\infty \lambda^{p-1} |\{w_\beta > \lambda\} \cap B|\, d\lambda.$$

Then we use the equality

$$\int_{E_\alpha \cap B} w(x)^p\, d\mu = p \int_\alpha^\infty \lambda^{p-1} \mu(E_\lambda \cap B)\, d\lambda + \alpha^p \mu(E_\alpha \cap B), \tag{2.17}$$

where $0 < p < \infty$, with μ replaced by $w\, d\mu$ and p replaced by $p-1$, to get

$$\begin{aligned} \int_{E_\alpha \cap B} w_\beta(x)^p\, dx &\le \int_{E_\alpha \cap B} w_\beta(x)^{p-1} w(x)\, dx \\ &= (p-1) \int_\alpha^\infty \lambda^{p-2} \int_{\{w_\beta > \lambda\} \cap B} w(x)\, dx\, d\lambda + \alpha^{p-1} \int_{E_\alpha \cap B} w(x)\, dx \\ &\le c(p-1) \int_\alpha^\infty \lambda^{p-1} |\{w_\beta > \lambda\} \cap B|\, d\lambda + c\,\alpha^p |E_\alpha \cap B|. \end{aligned}$$

Next we estimate the first integral on the right side using (2.17) and find

$$\int_\alpha^\infty \lambda^{p-1} |\{w_\beta > \lambda\}|\, d\lambda = \frac{1}{p} \left(\int_{E_\alpha \cap B} w_\beta(x)^p\, dx - \alpha^p |E_\alpha \cap B| \right).$$

Hence we obtain

$$\int_{E_\alpha \cap B} w_\beta(x)^p\, dx \le c\, \frac{p-1}{p} \int_{E_\alpha \cap B} w_\beta(x)^p\, dx + \frac{c}{p} \alpha^p |E_\alpha \cap B|.$$

Choosing $p > 1$ such that $c\,(p-1)/p < 1$ and using the fact that all terms in the previous inequality are finite, we conclude

$$\int_{E_\alpha \cap B} w_\beta(x)^p\, dx \le \frac{c}{c - p\,(c-1)} \alpha^p |E_\alpha \cap B|.$$

Finally, as $\beta \to \infty$, the monotone convergence theorem gives (2.16). This proves the lemma. ■

2.18. Remark. Both the bound for p and the constant in (2.16) are the best possible as is easily seen by taking B to be the unit ball and $w : \mathbf{R}^n \to [0, \infty]$, $w(x) = |x|^{n(1/c-1)}$.

3. Proof of Theorem 1.3

Let B be a ball in $\mathbf{R}^n$ and suppose that $w \in \mathcal{A}_1$ with the constant c_w. Using (2.5) we see that

$$\int_{E_\lambda \cap B} w(x)\,dx \le \big(c_w + \eta(c_w - 1)\big)\lambda |E_\lambda \cap B|, \quad \operatorname*{ess\,inf}_{x\in B} w(x) \le \lambda < \infty,$$

where η is the constant given by Lemma 2.4. This shows that w fulfills the assumptions of Lemma 2.14 and from (2.16) we conclude that

$$\begin{aligned}\int_B w(x)^p\,dx &= \int_{B\setminus E_\alpha} w(x)^p\,dx + \int_{B\cap E_\alpha} w(x)^p\,dx\\ &\le \alpha^p |B\setminus E_\alpha| + c\,\alpha^p |B\cap E_\alpha|\\ &\le c\,\alpha^p|B|,\end{aligned}$$

whenever $\operatorname{ess\,inf}_{x\in B} w(x) \le \alpha < \infty$ and

$$1 \le p < 1 + \frac{1}{(\eta+1)(c_w - 1)}.$$

In particular, we get

$$\left(\frac{1}{|B|}\int_B w(x)^p\,dx\right)^{1/p} \le c\,\frac{1}{|B|}\int_B w(x)\,dx.$$

The constant c does not depend on B and hence we may repeat the same reasoning in every ball B and we see that w satisfies the reverse Hölder inequality for every $p > 1$ such that (1.4) holds if we take $\nu = (\eta+1)^{-1}$. This completes the proof of Theorem 1.3. ■

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Department of Mathematics
P.O.Box 4
University of Helsinki
FIN-00014
FINLAND

e-mail: Juha.Kinnunen@Helsinki.Fi

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