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COMPLETE MINIMAL SURFACES IN $\mathbb{R}^3$

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Abstract

In this paper we review some topics on the theory of complete minimal surfaces in three dimensional Euclidean space.

Contents

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List of Figures

1. Introduction

In this paper we review some topics on the theory of minimal surfaces in three dimensional Euclidean space.

The study of minimal surfaces in $\mathbb{R}^3$ started with Lagrange in 1762. He studied the problem of determining a graph over an open set Ω in $\mathbb{R}^2$, with the least possible area among all surfaces that assume given values on $\partial(\Omega)$.

Mathematics soon realized that here was not only a problem of extraordinary difficulty, but also of unlimited possibilities.

In 1776, Meusnier supplied a geometric interpretation of the minimal graph equation (7): the mean curvature H vanishes. On this premise it has become customary to use the term *minimal surface* for any surface satisfying $H = 0$, notwithstanding the fact that such surfaces often do not provide a minimum for the area.

During the nineteenth century, more discoveries and publications appeared, thanks to the works of Catalan, Bonnet, Serret, Riemann, Weierstrass, Enneper, Schwarz, among others. From the point of view of modern theory, Weierstrass and Enneper's works are specially important. They introduced the so called *Enneper-Weierstrass representation* for minimal surfaces, which established a closed relationship between this theory and Complex Analysis.

In the middle of the nineteenth century, Plateau observed that minimal surfaces can be physically realized as soap films. So, the problem of determining a minimal surface with fixed topology and bounded by a prescribed Jordan curve is now usually called *Plateau's problem.* In this field we place emphasis on the works of Courant, Douglas, Morse, Rado, Schiffman, among others. However, this topic is not covered in this survey, and [**17**], [**26**], [**61**], [**71**] can be used as good references.

This paper is devoted to some aspects of the theory of complete minimal surfaces. As we will see, completeness has a strong influence on the topology, conformal structure and other geometrical properties of a minimal surface.

To be more precise, we include a brief study of: the Gauss map of complete minimal surfaces, complete minimal surfaces with bounded coordinate functions, some of the latest achievements about properly embedded minimal surfaces and what is known about complete minimal surfaces with finite total curvature.

The aim of this work is not to carry out an exhaustive exposition of these subjects. We only give a summary of the most relevant results or discuss its main applications and related questions. However, we have tried to compile a complete list of references that could help the interested reader to delve more deeply into these subjects.

In Section 2 we briefly study the construction of minimal surfaces with polygonal boundary and its applications.

Section 3 is devoted to the Gauss map of minimal surfaces, where we deal with Fujimoto's theorem, which asserts that the plane is the only complete minimal surface in $\mathbb{R}^3$ whose Gauss map omits at least five points of the sphere. We include the Osserman-Mo generalization of this theorem. For further treatment on the subject see [**24**].

In Section 4 we study complete minimal surfaces which are bounded as subsets of $\mathbb{R}^3$ (Naridashvili's theorem), and some questions related with Calabi's problem for minimal surfaces.

In Section 5 (the most extensive) we deal with complete minimal surfaces of finite total curvature. We look at Osserman's classical theorems and the formula of Jorge and Meeks, including a complete list of examples of surfaces of this kind. Our interest lies in: existence and uniqueness theorems for surfaces with critical total curvature (from the point of view of the Jorge-Meeks formula), minimal surfaces with high symmetry group, and nonorientable minimal surfaces. The study of embedded minimal surfaces with finite total curvature is included in Section 6.

Finally, in Section 6 we briefly study some of the latest achievements in the theory of properly embedded minimal surfaces in $\mathbb{R}^3$. Since the discovery of Costa's minimal surface, the study of this area gathered new speed. From Collin's theorem, a properly embedded minimal surface has finite topology if and only if it has finite total curvature, and provided that the number of ends is greater or equal to two. In this section, we review the families of surfaces with three or more ends, placing the emphasis the study of the Costa-Hoffman-Meeks and Hoffman-Meeks families of surfaces. We also include a uniqueness theorem for the Hoffman-Meeks family in terms of its symmetry. The last part of Section 6 is devoted to Meeks' conjecture about properly embedded planar domains. So, we prove the López-Ros theorem and include a summary of the Meeks-Pérez-Ros theorem.

Further results about properly embedded minimal surfaces, not included in this survey, can be found in Meeks and Rosenberg works [**65**], [**66**], and in Meeks' survey [**63**].

Acknowledgments. We would like to thank M. Weber for allowing us to include some of his images in this survey. We would also like to thank E. Thayer for allowing us to include some of his pictures, which were produced by using Jim Hoffman's graphics programs. Finally, we are indebted to J. Pérez for helpful criticisms of the paper.

1.1. Preliminaries.

In this section, we review some basic topics about minimal surfaces in $\mathbb{R}^3$, emphasizing the Weierstrass representation. The most part of these results can be found in [**73**], although we have also included additional references for some of them.

Let (M, ds^2) be a connected Riemannian surface. Let K, Δ and dA denote the Gauss curvature, the Laplace operator and the area element associated to ds^2, respectively.

We can associate to (M, ds^2) a conformal structure. This fact is a consequence of the following classical result.

Theorem 1.1. *Let (M, ds^2) be a Riemannian surface. Then, any point $P \in M$ has a neighborhood in which there exists a parametrization of M in terms of isothermal parameters.*

So, any orientable Riemannian surface has an underlying structure of Riemann surface. A proof of this result for can be found in [**44**].

Thus, if M is orientable, it is possible to define harmonic, holomorphic and meromorphic functions and 1-forms on M. We also denote by d the exterior differential on functions and 1-forms, and label $\star$ as the Hodge operator on 1-forms. For details see [**20**].

Let $X : M \to \mathbb{R}^3$ be an orientable isometric immersion of (M, ds^2) in three dimensional Euclidean space, and call $N : M \to \mathbb{S}^2$ its Gauss map. Denote $H : M \to \mathbb{R}$ as the mean curvature function associated to N. Recall that the mean curvature at a point of the surface is defined as half the sum of the principal curvatures at this point. A well-known formula says

$$\Delta X = 2\, H\, N. \tag{1}$$

Definition 1. The immersion X is minimal if and only if $H = 0$.

1.1.1. Weierstrass representation.

As a consequence of (1), X is minimal if and only if $X = (X_1, X_2, X_3)$ is harmonic (this fact only depends on the complex structure associated to (M, ds^2)). In that case, $d\,X_j$, $j = 1, 2, 3$, are harmonic 1-forms on M, and so

$$\Phi_j \stackrel{\text{def}}{=} \partial(X_j) = d\,X_j + i(\star d\,X_j), \quad j = 1, 2, 3,$$

are holomorphic 1-forms on M. Usually, we write $\Phi = (\Phi_1, \Phi_2, \Phi_3)$.

In what follows, we will assume that X is an orientable minimal immersion.

If z is a conformal parameter in M, then it is not hard to deduce that

$$\sum_{j=1}^{3}(\partial(X_j))^2 = 0,$$

i.e.,

$$\Phi_1^2 + \Phi_2^2 + \Phi_3^2 \equiv 0. \tag{2}$$

Moreover, $ds^2 = |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2$, and since X is an immersion,

$$|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 \not\equiv 0. \tag{3}$$

On the other hand, if we consider P_0, P in M and γ any differentiable curve in M starting at P_0 and ending at P, then one has

$$\mathrm{Real}\left(\int_\gamma \Phi\right) = X(P) - X(P_0).$$

Note that the right hand side of the last equality does not depend on the curve γ connecting P_0 and P. In other words, the 1-forms Φ_j, $j = 1, 2, 3$, verify:

$$\mathrm{Real}\int_\gamma \Phi_j = 0, \quad j = 1, 2, 3,$$

for any closed curve γ in M. Usually, we say that Φ_1, Φ_2, and Φ_3 *have no real periods.*

Therefore, we write $\mathrm{Real}\int_{P_0}^{P} \Phi_j$ instead of $\mathrm{Real}\int_\gamma \Phi_j$, $j = 1, 2, 3$.

If we define

$$g = \frac{\Phi_3}{\Phi_1 - i\Phi_2}, \quad \eta = \Phi_1 - i\Phi_2,$$

then equation (2) becomes:

$$\begin{aligned} \Phi_1 &= \frac{1}{2}(1 - g^2)\eta, \\ \Phi_2 &= \frac{i}{2}(1 + g^2)\eta, \\ \Phi_3 &= g\eta, \end{aligned} \tag{4}$$

and so, it is not hard to check that:

$$N(P) = \left(2\frac{\mathrm{Real}(g(P))}{1 + |g(P)|^2}, 2\frac{\mathrm{Im}(g(P))}{1 + |g(P)|^2}, \frac{1 - |g(P)|^2}{1 + |g(P)|^2}\right).$$

This means that the meromorphic function g is the stereographic projection, from the point $(0,0,1)$, of the Gauss map N of X. In particular, N is a conformal map, and this property characterizes minimal surfaces (besides the sphere).

Observe that equation (3) means that η is holomorphic with zeroes precisely at the poles of g, but with twice order.

We define (M,η,g) (or (M,Φ)) as the Weierstrass representation of the immersion X.

Conversely, one can construct minimal surfaces as follows:

Theorem 1.2. *Let M be a Riemann surface, and let η, g denote a holomorphic 1-form and a meromorphic function on M, respectively. Define Φ_j, $j=1,2,3$, as in (4), and suppose that (3) holds.*

Assume also that Φ_j, $j=1,2,3$, have no real periods on M, i.e., for any closed curve γ in M,

$$\mathrm{Real}\int_\gamma \Phi_j = 0, \quad j=1,2,3.$$

Fix a point $P_0 \in M$. Then, the map

$$X : M \longrightarrow \mathbb{R}^3$$
$$X(P) = \mathrm{Real}\int_{P_0}^{P} (\Phi_1,\Phi_2,\Phi_3),$$

is a well defined conformal minimal immersion.

Furthermore, (M,η,g) is the Weierstrass representation of X.

Note that the Weierstrass representation determines the minimal immersion up to translations.

We can write in terms of the Weierstrass representation any geometrical matter. So, if we fix a conformal parameter z on M, and write $\eta = f(z)\,dz$, then, straightforward computations give:

$$ds^2 = \frac{1}{4}|f(z)|^2(1+|g(z)|^2)^2|dz|^2, \tag{5}$$

$$K(z) = -\left(\frac{4|g'(z)|}{|f(z)|(1+|g(z)|^2)^2}\right)^2. \tag{6}$$

1.1.2. Minimal surfaces and symmetries.

Now we are going to give some basic results about isometries of minimal surfaces. Let $X : M \to \mathbb{R}^3$ be a minimal immersion, and label (η, g) its Weierstrass data. Let $A : M \to M$ be a diffeomorphism. We say that A is a symmetry of M if and only if there exists $\mathcal{A} \in \mathcal{O}(3, \mathbb{R})$ and $\vec{v} \in \mathbb{R}^3$ such that $(X \circ A)(P) = \mathcal{A} \cdot {}^t(X(P)) + \vec{v}$, where ${}^t(\cdot)$ means transpose matrix. Denote $\mathrm{Sym}(M)$ as the group of symmetries of M, and write $\mathrm{Iso}(M)$ as the isometry group of M. Then, it is clear that $\mathrm{Sym}(M)$ is a subgroup of $\mathrm{Iso}(M)$. Calabi proved the following:

Theorem 1.3 (Calabi [49]). *Let $X, X' : M \to \mathbb{R}^3$ be two conformal minimal immersions inducing the same Riemannian metric on M. Label Φ, Φ' as their Weierstrass data, respectively.*

Then, there exists $\mathcal{A} \in \mathcal{O}(3, \mathbb{R})$ and $\theta \in \mathbb{C}$, $|\theta| = 1$, such that

$$ {}^t\Phi' = \theta(\mathcal{A} \cdot {}^t\Phi). $$

In particular, if there exists $j \in \{1, 2, 3\}$ such that Φ_j is not exact then $\mathrm{Iso}(M) = \mathrm{Sym}(M)$. Indeed, if F is an isometry of M, then $X' = X \circ F$ is a well defined minimal immersion. Taking into account that the 1-forms Φ'_j, $j = 1, 2, 3$, in the Weierstrass data of X' have no real periods and Theorem 1.3, it is not hard to conclude.

Define $L(M)$ as the group of holomorphic and antiholomorphic diffeomorphisms, α, of M satisfying: ${}^tN \circ \alpha = \mathcal{A} \circ {}^tN$, where $\mathcal{A} \in \mathcal{O}(3, \mathbb{R})$ is a linear isometry of $\mathbb{R}^3$. Hoffman and Meeks have proved essentially the following theorem:

Theorem 1.4 (Hoffman, Meeks). *If $X : M \longrightarrow \mathbb{R}^3$ is a complete minimal immersion with finite total curvature, and there exists $j \in \{1, 2, 3\}$ such that Φ_j is not exact, then:*

$$ L(M) = \mathrm{Iso}(M) = \mathrm{Sym}(M). $$

A complete discussion about this subject can be found in [**32**], [**49**].

1.1.3. Maximum principle for minimal surfaces.

A minimal surface can, at least locally, be represented in the form $x_3 = u(x_1, x_2)$; the function u satisfies minimal surface equation, a quasilinear elliptic second order partial differential equation:

$$ (1 + u_{x_1}^2) u_{x_1 x_1} - 2u_{x_1} u_{x_2} u_{x_1 x_2} + (1 + u_{x_2}^2) u_{x_2 x_2} = 0. \tag{7} $$

Next, we state two theorems which summarize the well known versions of the maximum principle which we require in this survey. These theorems are a consequence of a deep analysis of the above partial differential equation (see [**25**]).

Theorem 1.5 (Interior maximum principle). *Suppose M_1, M_2 are connected minimal surfaces in $\mathbb{R}^3$. Suppose p is an interior point of both M_1 and M_2, and suppose $T_pM_1 = T_pM_2$. Assume that $T_pM_1 = \{x_3 = 0\}$ so that both M_1, M_2 are given near p as the graphs of two real analytic functions u_1 and u_2, respectively. If $u_1 \geq u_2$ in a neighborhood of p, then $M_1 = M_2$.*

Theorem 1.6 (Maximum Principle at Infinity [48], [68]). *Suppose N is a flat three-dimensional manifold and M_1 and M_2 are disjoint, connected, properly immersed surfaces in N with compact boundary (possibly empty). Then:*

1. *If $\partial(M_1)$ or $\partial(M_2)$ is nonempty, then, after possibly reindexing, there exists a point $x \in \partial(M_1)$ and a point $y \in M_2$, such that $\mathrm{dist}(x, y) = \mathrm{dist}(M_1, M_2)$.*
2. *If $\partial(M_1)$ and $\partial(M_2)$ are empty, then M_1 and M_2 are flat.*

One of the more recent and nicer applications of the maximum principle is the following theorem by Hoffman and Meeks:

Theorem 1.7 (Strong halfspace theorem [31]). *A connected, proper, possibly branched, nonplanar minimal surface M in $\mathbb{R}^3$ is not contained in a halfspace.*

This theorem has been a fundamental tool in obtaining a large number of results in this field.

1.1.4. Nonorientable minimal surfaces.

We now discuss the case of nonorientable minimal surfaces. An immersion $X : M' \to \mathbb{R}^3$ of a nonorientable surface is minimal if and only if the mean curvature of X on any orientable piece of M is zero.

Consider now $X' : M' \longrightarrow \mathbb{R}^3$ a conformal minimal immersion of a nonorientable Riemannian surface M' in $\mathbb{R}^3$. Let $\pi_0 : M \to M'$, $I : M \to M$ denote the conformal oriented two sheeted covering of M' and the antiholomorphic order two deck transformation for this covering, respectively.

If (g, η) represents the Weierstrass data of $X = X' \circ \pi_0$, then it is not hard to deduce that :

$$I^*(\Phi_j) = \overline{\Phi_j}, \quad j = 1, 2, 3. \tag{8}$$

In particular, $g \circ I = I_0 \circ g$, where $I_0(z) = -1/\overline{z}$, and so there is a unique map

$$G : M' \longrightarrow \mathbb{RP}^2 \equiv \overline{\mathbb{C}}/\langle I_0 \rangle$$

satisfying

$$G \circ \pi_0 = g \circ p_0,$$

where $p_0 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}/\langle I_0 \rangle$ is the natural projection. We call G the *generalized Gauss map* of X'.

Conversely, if (M, g, η) is the Weierstrass representation of a minimal immersion X of an orientable surface M in $\mathbb{R}^3$, and $I : M \to M$ is an antiholomorphic involution without fixed points on M satisfying (8), then X induces a minimal immersion X' of $M' = M/\langle I \rangle$ in $\mathbb{R}^3$ such that $X = X' \circ \pi_0$.

By definition, (M, I, g, η) is the Weierstrass representation of the non-orientable minimal immersion X'. For more details see [**62**].

1.1.5. Classical examples.

Finally, we present the Weierstrass representation of some classical examples.

- **The helicoid.** $M = \mathbb{C}$, $g = e^z$, $\Phi_3 = i\,dz$.

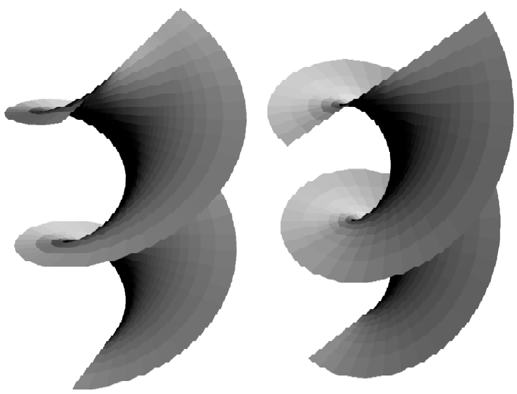

Figure 1. The helicoid.

- **The catenoid.** $M = \mathbb{C} - \{0\}$, $g = z$, $\Phi_3 = \frac{dz}{z}$.

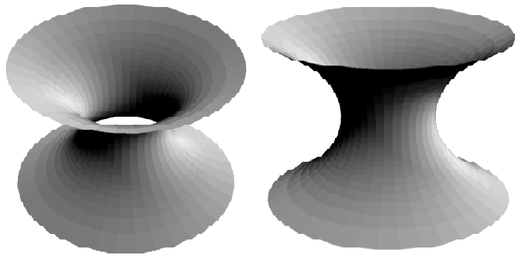

Figure 2. The catenoid.

- **Enneper's surface.** $M = \mathbb{C}$, $g = z$, $\Phi_3 = z\,dz$.

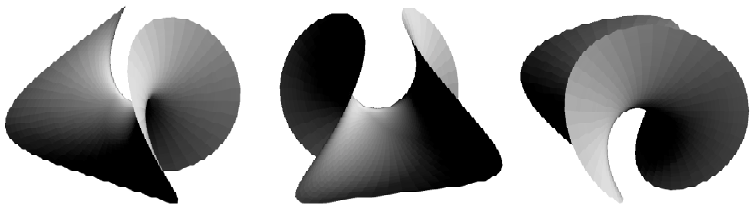

Figure 3. Enneper's surface.

- **Scherk's surfaces.** Let $M' = \mathbb{C} - \{1, -1, i, -i\}$, $g' = z$, $\Phi_3' = \sqrt{(-1)^j}\frac{4z\,dz}{z^4-1}$, where $j \in \{0, 1\}$. The map $X' = \text{Real}\int \Phi'$ is not well defined because the 1-forms Φ_i', $i = 1, 2, 3$, have real periods on M'. To solve this, we define M as the universal covering of M', and take g, Φ_3 as the lifts of g', Φ_3' to M, respectively. Thus, $X = \text{Real}\int \Phi$ is well defined. If $j = 0$, we get Scherk's doubly periodic surface, and in case $j = 1$ we obtain Scherk's singly periodic surface.

- **Henneberg's surface.** $M = \mathbb{C} - \{0\}$, $I(z) = -1/\overline{z}$, $g = z$, $\Phi_3 = 2z(1 - \frac{1}{z^4})\,dz$. These meromorphic data induce a minimal Möbius strip, but unfortunately the immersion is not regular at the points $\{1, -1\}$ and $\{i, -i\}$ (where (3) fails).

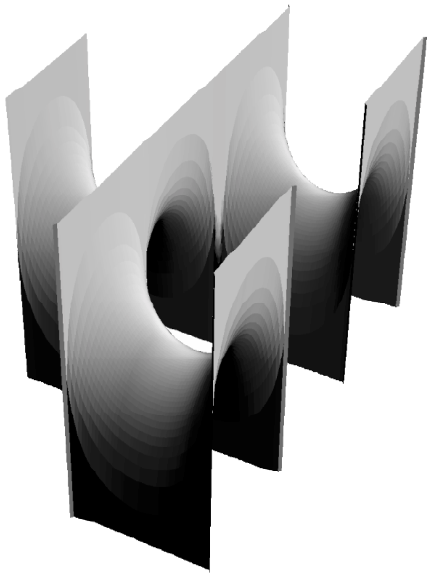

Figure 4. Scherk's doubly periodic surface.

2. Construction of minimal surfaces with polygonal boundary

Although Plateau's problem is one of the classical questions in geometry and analysis, progress in solving it has been very slow. The first satisfactory solution of the Plateau problem for a general contour was given by Douglas and Radó in 1930. The following theorem summarizes several existence results by Douglas, Radó, Osserman, Gulliver, Alt, among others.

Theorem 2.1 (Fundamental existence theorem). *Every closed rectifiable Jordan curve Γ in $\mathbb{R}^3$ bounds an area minimizing surface $X : \overline{\mathbb{D}} \to \mathbb{R}^3$ of the disc type, and all solutions of this type are regular surfaces, i.e., they are free of branch points. If Γ is regular and real analytic, then they have no branch points on $\partial\mathbb{D}$, either.*

We refer to books [**17**], [**26**], [**61**], [**71**], [**73**] for a good setting.

A classical problem considered by Schwarz, Weierstrass and Riemann was to determine minimal surfaces bounded by straight lines. These authors obtained existence results for minimal surfaces with boundary a given polygon, where the sides of the polygon could be of finite or infinite length.

The works of Riemann are especially interesting. Riemann's posthumous paper [**82**] treated minimal surfaces passing through one or several straight lines. In particular, it dealt with the following special boundaries: (i) Two infinitely long, skew straight lines. (ii) Three straight lines, two of which lie in a plane P and intersect; the third lies in a plane P' parallel to P. (iii) Three intersecting straight lines. (iv) A quadrilateral. (v) Two arbitrary circles which lie in parallel planes.

In relation to the last case, Riemann constructed doubly connected minimal surfaces bounded by two parallel and distinct straight lines. We refer to Paragraph 6.2.2.

A comprehensive presentation of the Schwarz-Riemann-Weierstrass approach to the solution of Plateau's problem for polygonal boundaries can be found in Darboux's treatise [**16**, Vol. 1 and 3].

Let $X : M \to \mathbb{R}^3$ be a complete minimal surface, and γ_0 a curve in M. If $\gamma = X(\gamma_0)$ is a straight line, then the *Schwarz reflection principle* (see [**17**], [**39**]) implies that the rotation of 180° about γ is a symmetry of $X(M)$. If γ_0 is a planar geodesic of M (i.e., γ is the orthogonal intersection of $X(M)$ with a plane, Π), then $X(M)$ is symmetric under the reflection through Π.

If we label (g, η) as the Weierstrass representation of X, then $g(\gamma)$ lies, in both cases, in the great circle of $\overline{\mathbb{C}}$ determined, up to composing with the stereographic projection, by:

- the vector plane which is orthogonal to γ, if γ is a straight line, or
- the vector plane Π_0, parallel to Π, if γ is a planar geodesic contained in Π.

Sometimes it is useful to know that the curve γ is a straight line (resp., a planar geodesic) of $X(M)$ if and only if $X^*(\gamma_0)$ is a planar geodesic (resp., a straight line) of $X^*(M)$, where $X^* = \mathrm{Re}(\int i\partial X)$ is the adjoint surface.

We are going to explain a classical method used to construct compact minimal surfaces with polygonal boundary.

A Schwarzian chain is a set $C = \{L_1, \dots, L_r, E_1, \dots, E_s\}$, where $L_1, \dots, L_s$ are straight lines and $E_1, \dots, E_s$ are planes. We say that a compact minimal surface $X : M \to \mathbb{R}^3$ is a solution of the Schwarzian chain problem for chain C if and only if:

- M is simply connected and the Gauss map of X is injective,
- $X(\partial M)$ lies in the union of the straight lines and planes in C, and along its boundary, X is perpendicular to all planar parts of C.

Suppose that X is a solution of the Schwarzian chain problem for C, and let (g, η) be its Weierstrass representation. Since g is a biholomorphism, we can identify M and $g(M) \subset \overline{\mathbb{C}}$. So, up to a Möbius transformation, we can suppose that M is a domain of $\mathbb{C}$ bounded by pieces of great circles, and $g(z) = z$. Write $\eta = f(z)\, dz$, and define

$$q(z) = \int_{z_0}^{z} \sqrt{-f(w)}\, dw, \quad z_0 \in M.$$

If we assume that $q : M \to \Omega = q(M) \subset \mathbb{C}$ is a biholomorphism, then it is not hard to see that:

> *The q-images of the straight lines in ∂M lie on straight lines which intersect the real axis at an angle of 45° or of 135°, whereas the planar geodesics are mapped by q into straight lines which are parallel either to the real axis or to the imaginary axis.*

Conversely, suppose that $M \subset \mathbb{C}$ is a simply connected domain bounded by pieces of great circles, $\Omega \subset \mathbb{C}$ is a polygonal domain bounded by lines as above, and $q : M \to \Omega$ is a biholomorphism. Then, defining $g(z) = z$ and $\eta(z) = -(\frac{dq}{dz})^2$, the Weierstrass data (M, g, η) determine a minimal surface bounded by a Jordan curve which consists of pieces of straight lines and planar geodesic arcs.

The above reasoning provides a handy method to solve Schwarzian chain problems, and in particular, to construct compact minimal surfaces bounded by straight lines. It can also be used to construct complete periodic minimal surfaces by successive Schwarz reflections about straight lines and planar geodesic arcs. For details, we refer to Karcher's survey [**39**] and Dierkes et al. [**17**].

Regarding minimal surfaces with non compact polygonal boundary, in 1966, Jenkins and Serrin in [**35**] proved an existence and uniqueness theorem for minimal graphs bounded by straight lines. They obtained simple, necessary and sufficient conditions to solve the Dirichlet problem in a compact convex domain bounded by a polygon assuming values $+\infty$, $-\infty$ and continuous data on different straight segments in the boundary.

To be more precise, they prove:

Theorem 2.2 (Jenkins, Serrin). *Let D be a bounded convex domain whose boundary contains two sets of open straight segments $A_1, \dots, A_k$ and $B_1, \dots, B_l$, with the property that no two segments A_i and no two segments B_i have a common endpoint. The remaining portion of the boundary consists of endpoints of the segments A_i and B_i, and open arcs $C_1, \dots, C_m$. Consider the Dirichlet problem:*

> *Determine a minimal graph in D which assumes the value $+\infty$ on each A_i, $-\infty$ on each B_i and assigned continuous data on each of the open arcs C_i.*

Let $\mathcal{P}$ denote a simple closed polygon whose vertices are chosen from among the end points of the segments A_i and B_j. Let α, β be, respectively, the total length of the segments A_i and B_j which are part of $\mathcal{P}$. Finally, let γ denote the perimeter of $\mathcal{P}$.

Then, if the family of arcs $\{C_i\}$ is not empty, the Dirichlet problem stated above is solvable if and only if

$$2\alpha < \gamma \quad \text{and} \quad 2\beta < \gamma.$$

Furthermore, the solution is unique if it exists.

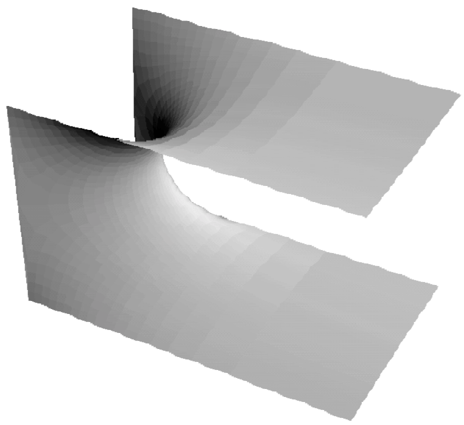

Figure 5. A Jenkins-Serrin graph.

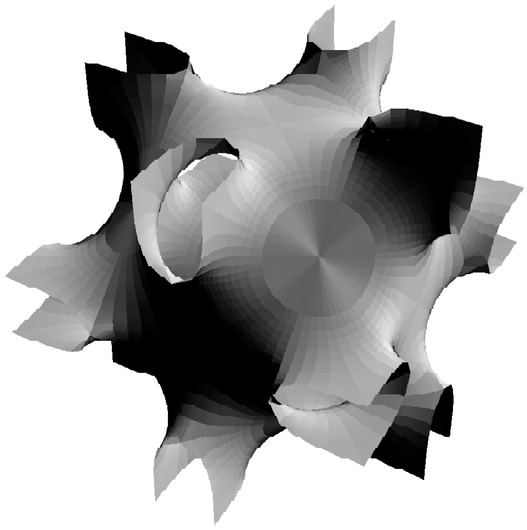

Figure 6. A fundamental piece of the Neovius surface.

Finally, we review some complete surfaces that can be constructed by using the above methods. The main idea is to give a fundamental piece of the surface bounded by straight lines or planar geodesic arcs, and so, in successive steps, to use the Schwarz reflection principle for constructing a complete example. Those constructions which lead to embedded examples are of special interest. For more details about this subject, we refer to the excellent survey [**39**].

Schwarz's surface. This triply periodic surface $X : M \to \mathbb{R}^3$ contains a disc-type fundamental piece which is bounded by a nonplanar quadrilateral. Its Weierstrass data are:

$$M = \overline{\mathbb{C}}, \quad g = z, \quad \eta = \frac{dz}{\sqrt{1 - 14z^4 + z^8}}.$$

Jenkins-Serrin surfaces. In Figure 5, we illustrate a particular Jenkins-Serrin graph. In this case, the polygon is a rectangle and the data on the four edges are $+\infty$, 0, $+\infty$ and 0, respectively. The doubly-periodic minimal surface obtained by successive Schwarz reflections is embedded, and its Weierstrass data are:

$$M = \mathbb{C} - \{0\}, \quad g = z, \quad \eta g = \frac{dz}{\sqrt{z^4 + 2rz^2 + 1}},$$

where $r \in]-1, 1[$.

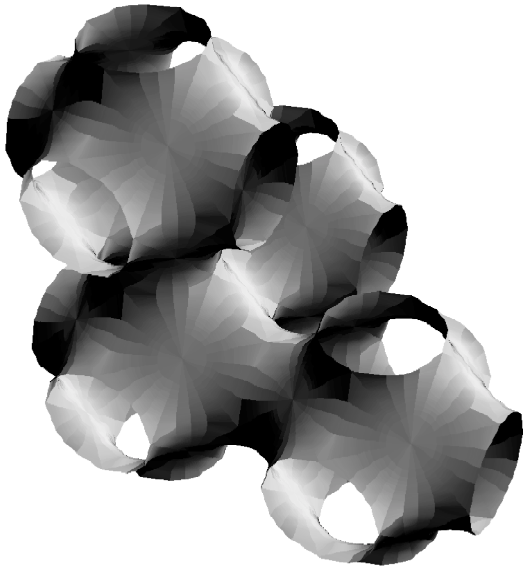

Figure 7. Schwarz's surface.

Scherk's doubly and singly periodic surfaces. These surfaces were described in Paragraph 1.1.5.

3. Gauss map of minimal surfaces

O. Bonnet (1860) proved that the Gauss map $N : M \to \mathbb{S}^2$ of a minimal surface $X : M \to \mathbb{R}^3$ is conformal, and E. B. Christoffel that this property characterizes minimal surfaces, besides the round sphere.

Furthermore, the area of the spherical image of M, counting multiplicities, can be computed as follows

$$A(N(M)) = -\int_M K\, dA. \tag{9}$$

The last integral is known as the total curvature $\mathcal{C}(M)$ of the immersion X.

One of the fundamental problems in classical theory of minimal surfaces is to obtain Liouville type results for complete minimal surfaces. R. Osserman was who started the systematic development of this theory, and so, in 1961 he proved that the Gauss map of a complete nonflat orientable minimal surface misses at most a set of logarithmic capacity zero. In 1981 F. Xavier [**96**] proved that the image of the Gauss map covers the sphere except at most six values, and finally in 1988 H. Fujimoto [**22**], [**23**] obtained the best possible theorem, and proved that the number of exceptional values of the Gauss map is at most four. An interesting extension of Fujimoto's results was proved in 1990 by X. Mo and R. Osserman [**69**]. They showed that if the Gauss map of a complete orientable minimal surface takes on five distinct values only a finite number of times, then the surface has finite total curvature.

There are many kinds of complete orientable minimal surfaces whose Gauss map omits four points of the sphere. Among these examples we emphasize the classical Scherk's doubly periodic surface and those described by K. Voss in [**91**] (see also [**73**]). The first author of this paper in [**53**] constructs orientable examples with nontrivial finite topology.

Under the additional hypothesis of finite total curvature, R. Osserman [**72**] proved that the number of exceptional values is at most three.

In the nonorientable case, the Gauss map of the two sheeted orientable covering surface induces, in a natural way, a *generalized Gauss map* from the nonorientable surface on the projective plane. Very recently [**56**], the authors of this survey have found complete nonorientable minimal surfaces in $\mathbb{R}^3$ whose generalized Gauss map omits two points of $\mathbb{RP}^2$. This result proves that Fujimoto's theorem is sharp for this kind of surfaces.

In this section we shall give a brief outline of some of the above results.

We start with a classical theorem by Osserman. For a good understanding of the theorem, it is advisable to read the preliminaries of Section 5.

Theorem 3.1 (Osserman [72]). *The Gauss map of a complete orientable nonflat minimal surface with finite total curvature omits at most three points of* $\mathbb{S}^2$.

Proof: Let $X : M \to \mathbb{R}^3$ be a complete conformal nonflat minimal immersion with finite total curvature, and label (η, g) its Weierstrass data. Recall that the meromorphic function g is the stereographic projection of the Gauss map N of X.

Since $\mathcal{C}(M)$ is finite, we may assume that M is conformally equivalent to a compact Riemann surface M' minus a finite set of points $\{P_1, \dots, P_r\}$, and that the Weierstrass data extend meromorphically to M' (see Theorems 5.1 and 5.2).

The zeroes and poles of g correspond to points of M' with vertical normal vector. As g has only a finite set of branch points, we can make a rotation of coordinates in $\mathbb{R}^3$ in such a way that:

- g has simple poles on M'.
- g assumes finite values at $\{P_1, \dots, P_r\}$.

On the other hand, given $P \in M'$ and z a conformal parameter on M' centered at P, we define the branching number $n(P)$ of g at P as the order of the zero of dg/dz at 0. Since g is a branched covering of $\mathbb{S}^2$ with a finite number $\deg(g)$ of sheets, then there are only a finite set of points in M' where $n(P) \neq 0$. Hence, we can define the total order of branching of g as follows:

$$n = \sum_{P \in M'} n(p).$$

If $\gamma = \text{genus}(M')$, then by Riemann's relation (see [**20**]):

$$n = 2(\deg(g) + \gamma - 1). \tag{10}$$

Recall that the 1-form η has double zeros exactly at the poles of g. Moreover, if z is a conformal parameter centered at P_j on M', then

$$\sum_{j=1}^{3} |\Phi_j|^2 \sim c/|z|^{2m_j},$$

where $m_j \geq 2$. (See the comments previous to Theorem 5.3.)

By Riemann's relation once again

$$2n - \sum_{j=1}^{r} m_j = 2\gamma - 2,$$

and so it is easy to deduce that

$$r + \gamma - 1 \leq \deg(g). \tag{11}$$

Suppose now that $g|_M$ omits k points $q_1, \dots, q_k \in \mathbb{S}^2$. Then $g^{-1}(\{q_1, \dots, q_k\}) \subset \{P_1, \dots, P_r\}$, and counting multiplicities, each q_j has exactly $\deg(g)$ preimages. Thus,

$$k \deg(g) \leq \sum_{j=1}^{r} (1 + n(p_j)) = r + \sum_{j=1}^{r} n(p_j) \leq r + n.$$

Last inequality and (10) imply that

$$k \deg(g) - r \leq 2(\deg(g) + \gamma - 1).$$

Adding (11) gives

$$1 - \gamma \leq (3 - k)\deg(g),$$

and taking into account (11), we infer $r \leq (4-k)\deg(g)$. Since M is not compact, $r > 0$, and so $k < 4$. ■

Enneper's surface and the catenoid are examples of finite total curvature whose Gauss map omits one and two points of $\mathbb{S}^2$, respectively.

A. Weitsman and F. Xavier in [**94**] and Y. Fang in [**19**] have obtained nonexistence results for complete, nonflat, orientable minimal surfaces in $\mathbb{R}^3$ whose Gauss map omits three points of $\mathbb{S}^2$, provided that the absolute value of the total curvature is less than or equal to 16π and 20π, respectively.

Therefore, it left open the following questions:

1. *Are there complete nonflat orientable minimal surfaces in* $\mathbb{R}^3$ *with finite total curvature whose Gauss map omits three points of* $\mathbb{S}^2$*?*
2. *Are there complete nonorientable minimal surfaces in* $\mathbb{R}^3$ *with finite total curvature whose generalized Gauss map omits one point of* $\mathbb{RP}^2$*?*

Now, we deal with the general problem for complete, orientable, minimal surfaces. Mo and Osserman proved the following extension of Fujimoto's theorem:

Theorem 3.2 (Mo, Osserman [69]). *If the Gauss map of a complete orientable minimal surface takes on five distinct values of* $\mathbb{S}^2$ *only a finite number of times, then the surface has finite total curvature.*

Proof: We will need the following function-theoretic lemma:

Lemma 3.3 (Fujimoto [22]). *Let* $h(w)$ *be analytic in* $|w| < R$ *and omits the points* α_1, α_2, α_3, α_4. *Let* ϵ, ϵ' *satisfy* $0 < 4\epsilon' < \epsilon < 1$.

Then there is a positive constant B *depending only on* α_1, α_2, α_3, α_4, ϵ, ϵ', *such that*

$$\frac{(1 + |h(w)|^2)^{\frac{3-\epsilon}{2}} |h'(w)|}{\prod_{j=1}^4 |h(w) - \alpha_j|^{1-\epsilon'}} \leq B \frac{2R}{R^2 - |w|^2}.$$

We omit the proof of this lemma.

Let $X : S \to \mathbb{R}^3$ be a complete nonflat minimal surface whose Gauss map omits the points α_1, α_2, α_3, α_4, α_5. We denote (η, g) as the Weierstrass representation of X.

Up to a rigid motion of $\mathbb{R}^3$, we can assume $\alpha_5 = \infty$. Then the hypothesis of the theorem implies the existence of a compact set, D, such that $g|_D$ is holomorphic and omits the values α_1, α_2, α_3, α_4.

Define

$$S' = \{P \in S - D / g' \neq 0 \text{ at } P\}, \quad S'' = S' \cup D.$$

We consider the new metric

$$ds_1^2 = \left| \frac{f(\xi)^{\frac{1}{1-p}} \prod_{j=1}^4 (g(\xi) - \alpha_j)^{\frac{p(1-\epsilon')}{1-p}}}{g'(\xi)^{\frac{p}{1-p}}} \right| |d\xi|^2,$$

where $0 < 4\epsilon' < \epsilon < 1$, $p = 2/(3-\epsilon)$, and $\eta = f(\xi)\, d\xi$. Then it is easy to check that the above expression is independent of both the choice of local parameter ξ and the indeterminacy arising from fractional exponents. As f and g are holomorphic, then ds_1 is flat on S', and it extends smoothly over S''.

Our purpose is to prove that (S'', ds_1^2) is complete.

We proceed by contradiction. Then, there is a divergent path $\gamma(t) : [0,1[\to S''$ with finite length. Without loss of generality, we can suppose that there is a positive distance d between γ and D. It is clear that either $\gamma(t)$ is divergent on S or $\gamma(t)$ tends to a point where $g' = 0$.

If we put $g(\xi) \sim c(\xi - \xi_0)^m$, $m \geq 1$, then $ds_1 \sim c'|\xi - \xi_0|^{\frac{2m}{\epsilon - 1}} > c'|\xi - \xi_0|^{-2}$. Thus, the length of γ in (S'', ds_1^2) is infinite, which is absurd.

So, we can assume that γ is divergent in S. Choose t_0 such that $\int_{t_0}^1 ds_1 < d/3$, i.e., the length of $\gamma([t_0, 1[)$ is less than $d/3$. Consider a small geodesic disk Δ centered at $\gamma(t_0)$. As ds_1^2 is flat around Δ, then there exists $r > 0$ such that the exponential map $F : D(0,r) \to \Delta$ is an isometry, where $D(0,r) = \{z \in \mathbb{C} / |z| < r\}$ and $F(0) = \gamma(t_0)$. We can extend F in S' as a local isometry to the largest disk possible $D(0,R)$. Since γ is divergent on S and the length of $\gamma([t_0, 1[)$ is less than $d/3$, then $R \leq d/3$, and so the distance between $F(D(0,R))$ and D must be at least $2d/3$. As R is the largest possible and the points $g' = 0$ are infinitely far away, there is a point $w_0 \in \partial D(0,R)$ such that the image under F of the segment joining 0 and w_0 is a divergent curve Γ on S.

To get a contradiction, it suffices to prove that Γ has finite length in the original metric ds^2 in S (recall that this metric is complete). Let $h = g \circ F$ be the Gauss map pulled back to the disk $D(0,R)$. Since $F(D(0,R)) \subset S'$, h omits the values $\alpha_1, \dots, \alpha_5 = \infty$. As F is a local isometry, one has $ds_1^2 = |dw|^2$, i.e., ds_1^2 is the Euclidean metric in $D(0,R)$.

Using w as local parameter, then we obtain

$$\left|\frac{f(w)\prod_{j=1}^4(h(w)-\alpha_j)^{p(1-\epsilon')}}{h'(w)^p}\right| = 1.$$

Therefore, if C is the segment in $D(0,R)$ corresponding to the curve Γ and L is its length, we easily deduce from Fujimoto's Lemma that

$$\begin{aligned} 2L &= \int_C |f(w)|(1+|h(w)|^2)|dw| \\ &\le B^p \int_C \left(\frac{2R}{R^2-|w|^2}\right)^p |dw| = \frac{(2B)^p}{R^{p-1}}\int_0^1 \frac{dt}{(1-t^2)^p}. \end{aligned}$$

As $p<1$, L is finite, which is absurd.

This contradiction proves that (S'', ds_1^2) is complete. Since the metric ds_1^2 is flat outside a compact subset of S'', then it has finite total curvature. By Theorem 5.1, S'' is finitely connected. In particular, g' has a finite number of zeros and S is finitely connected. Furthermore, (see [**73**]) S is conformally equivalent to a compact Riemann surface $\overline{S}$ punctured in a finite number of points. Since g is holomorphic and omits four values in $\mathbb{C}$, Picard's theorem implies that g has a meromorphic extension to $\overline{S}$. If we call m the degree of g as holomorphic function between compact Riemann surfaces, one has (see equality (9))

$$\int_S K\, dA = -4\pi m,$$

and so (S, ds^2) has finite total curvature. This proves the theorem. ■

As a consequence of Theorems 3.1 and 3.2, we can obtain Fujimoto's theorem:

Theorem 3.4 (Fujimoto [22], [23]). *The plane is the only complete orientable minimal surface in $\mathbb{R}^3$ whose Gauss map omits at least five points of the sphere.*

As we have mentioned at the beginning of this section, Fujimoto's theorem is sharp for orientable complete minimal surfaces. The same holds in the nonorientable case (i.e., for the orientable two sheeted cover of a complete nonorientable minimal surface), as a consequence of the following result:

Theorem 3.5 ([56]). *There are complete nonorientable minimal surfaces in $\mathbb{R}^3$ whose Gauss map omits two points of the projective plane.*

We omit the proof.

4. Complete minimal surfaces with bounded coordinate functions

Calabi asked if it is possible to have a complete minimal surface in $\mathbb{R}^3$ entirely contained in a halfspace. Jorge and Xavier [**37**] showed complete nonflat minimal surfaces contained in slabs of $\mathbb{R}^3$. The proof is based in a ingenious idea of using Runge's theorem.

Very recently, Nadirashvili in [**70**] have used Runge's theorem in a more elaborate way to produce complete bounded minimal surfaces in $\mathbb{R}^3$. In this section we summarize Nadirashvili's techniques (see also [**10**]) to obtain a complete minimal disc inside a ball in $\mathbb{R}^3$.

As consequence of the strong halfspace theorem (Theorem 1.7), none of these examples is properly immersed.

Let $\mathbb{D}$ and $|dz|^2$ be the unit disc in $\mathbb{C}$ and the Euclidean metric on the disc, respectively. The space of harmonic maps from $\mathbb{D}$ in $\mathbb{R}^3$ is denoted by $\mathrm{Har}(\mathbb{D}, \mathbb{R}^3)$. Moreover, for $r \in \mathbb{R}^+$, denote: $D_r = \{z \in \mathbb{C} / |z| < r\}$ and $B_r = \{x \in \mathbb{R}^3 / \|x\| < r\}$.

Given $X : \mathbb{D} \to \mathbb{R}^3$ a conformal immersion and $U, V \subset \mathbb{D}$, we write:

- $ds_X^2 = \lambda_X^2 \cdot |dz|^2$ as the metric in $\mathbb{D}$ induced by X,
- K_X as the Gauss curvature of X,
- $\mathrm{dist}_X(U, V)$ instead of $\mathrm{dist}_{ds_X^2}(U, V)$.

Next lemma will be very important during the proof of the Main Theorem of this section.

Lemma 4.1. *Let $X : \mathbb{D} \longrightarrow \mathbb{R}^3$ be a complete minimal immersion satisfying:*

(i) $(\mathbb{D}, ds_X^2)$ *is a geodesic disc of radius* $d > 0$*,*
(ii) $X(\mathbb{D}) \subset B_r$*,* $r > 0$*,*
(iii) $X(0) = (0, 0, 0)$ *and* $K_X(z) > 0$*,* $\forall\, z \in \mathbb{D}$*.*

Then, $\forall\, s, \epsilon > 0$*, there exists a complete minimal immersion* $Y : \mathbb{D} \to \mathbb{R}^3$ *such that:*

1. $(\mathbb{D}, ds_Y^2)$ *is a geodesic disc of radius* $d + s$*,*
2. $\|X(z) - Y(z)\| < \epsilon$*,* $\forall\, z \in D_{1-\epsilon}$*,*
3. $Y(\mathbb{D}) \subset B_R$*, where* $R = \sqrt{r^2 + s^2} + \epsilon$*,*
4. $Y(0) = (0, 0, 0)$ *and* $K_Y(z) > 0$*,* $\forall\, z \in \mathbb{D}$*.*

The original proof of this lemma can be found in [**70**]. For a more detailed exposition, see [**10**].

Theorem 4.2 (Nadirashvili [70]). *There exists a complete minimal immersion $X : \mathbb{D} \to \mathbb{R}^3$ satisfying:*

1. $X(\mathbb{D})$ *is bounded,*
2. $K_X(z) < 0$, $\forall\, z \in \mathbb{D}$.

Proof: We consider the sequence:

$$\alpha_1 = \frac{1}{2} e^{\frac{1}{2}}, \quad \alpha_n = e^{-\frac{1}{2^n}},\; n \geq 2.$$

It is straightforward to check that $\frac{1}{2} < \alpha_n < 1$ and $\displaystyle\prod_{n=1}^{\infty} \alpha_n = \frac{1}{2}$.

The first step in the proof of this theorem is to construct a sequence $\{X_n\}_{n\in\mathbb{N}}$ of minimal immersions from $\overline{\mathbb{D}}$ in $\mathbb{R}^3$ and a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$, $a_n \in]0,1[$, $\forall\, n \in \mathbb{N}$, satisfying:

(i) $a_n < a_{n+1} < 1$, and $a_n \geq 1 - \frac{1}{n}$,

(ii) $\operatorname{dist}_{X_n}(0, \partial(D_{a_n})) \geq \frac{2}{3} d_n$, where $d_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$,

(iii) $\|X_{n+1}(z) - X_n(z)\| \leq \frac{1}{(n+1)^2}$, $\forall\, z \in \overline{D_{a_n}}$,

(iv) $\lambda_{X_{n+1}}(z) \geq \alpha_n\, \lambda_{X_n}(z)$, $\forall\, z \in \overline{D_{a_n}}$,

(v) $K_{X_{n+1}}(z) \leq \alpha_n\, K_{X_n}(z)$, $\forall\, z \in \overline{D_{a_n}}$.

We choose as the first term (X_1, a_1), where $X_1 : \overline{\mathbb{D}} \to \mathbb{R}^3$ is a minimal immersion verifying:

- $(\mathbb{D}, ds^2_{X_1})$ is a geodesic disc of radius 1,
- $X_1(\overline{\mathbb{D}}) \subset B_1$,
- $X_1(0) = (0,0,0)$ and $K_{X_1} < 0$;

and a_1 is a real number, $0 < a_1 < 1$ such that

$$\operatorname{dist}_{X_1}(0, \partial(D_{a_1})) \geq \frac{2}{3} = \frac{2}{3} d_1.$$

Assume that we have constructed $(X_1, a_1), \ldots, (X_n, a_n)$. Let $s_{n+1} = 1/(n+1)$ and consider $\{\epsilon_k\}_{k\in\mathbb{N}}$ a sequence in $\mathbb{R}^+$ which converges to 0. We apply Lemma 4.1 to $(X_n, s_{n+1}, \epsilon_k)$, $k \in \mathbb{N}$, and so we obtain a sequence of minimal immersions $\{Y_k\}_{k\in\mathbb{N}}$ satisfying Statements 1–4 in the lemma. From Statement 2 it is obvious that $\{Y_k\}_{k\in\mathbb{N}} \to X_n$ uniformly on $\overline{D_{a_n}}$. Even more, as $Y_k \in \operatorname{Har}(\mathbb{D}, \mathbb{R}^3)$, then for any multi-index, α, the sequence $\{D^\alpha Y_k\}_{k\in\mathbb{N}} \to D^\alpha X_n$ uniformly on $\overline{D_{a_n}}$. So, taking into account the above assertion, it is clear that $\{\lambda_{Y_k}\}_{k\in\mathbb{N}} \to \lambda_{X_n}$ and $\{K_{Y_k}\}_{k\in\mathbb{N}} \to K_{X_n}$.

These facts imply the existence of $k_0 \in \mathbb{N}$, large enough, such that:

- $\|Y_{k_0}(z) - X_n(z)\| \leq \frac{1}{(n+1)^2}$, $\forall\, z \in \overline{D_{a_n}}$,
- $\lambda_{Y_{k_0}}(z) \geq \alpha_n \, \lambda_{X_n}(z)$, $\forall\, z \in \overline{D_{a_n}}$,
- $K_{Y_{k_0}}(z) \leq \alpha_n \, K_{X_n}(z)$, $\forall\, z \in \overline{D_{a_n}}$.

Hence, we take $X_{n+1} \stackrel{\text{def}}{=} Y_{k_0}$. On the other hand, from Lemma 4.1 we obtain $\mathrm{dist}_{X_{n+1}}(0, \partial(D_{a_n})) = d_n + \frac{1}{n+1} = d_{n+1}$. Thus, we can choose a_{n+1} satisfying: $a_n < a_{n+1} < 1$, $\mathrm{dist}_{X_{n+1}}(0, \partial(D_{a_{n+1}})) \geq \frac{2}{3} d_{n+1}$, and $a_{n+1} \geq 1 - \frac{1}{n+1}$. So, the pair (X_{n+1}, a_{n+1}) satisfies conditions (i)–(v).

Observe that the choice of $\{a_n\}_{n\in\mathbb{N}}$ guarantees that

$$\bigcup_{n\in\mathbb{N}} D_{a_n} = \mathbb{D}. \tag{12}$$

If $\mathcal{K} \subset \mathbb{D}$ is a compact set, then, using (12), there exists $n_0 \in \mathbb{N}$ such that $\mathcal{K} \subset D_{a_n}$, $\forall\, n \geq n_0$. Thus, from (ii), $\{X_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence on $\mathcal{K}$.

Using Harnack's theorem (see [**11**]) we deduce that $\{X_n\}_{n\in\mathbb{N}}$ converges in the space $\mathrm{Har}(\mathbb{D}, \mathbb{R}^3)$ to an harmonic map X. We are going to see that X is the minimal immersion that we are looking for.

Claim 1. *$X : \mathbb{D} \longrightarrow \mathbb{R}^3$ is an immersion and $K_X < 0$.*

Indeed, it is easy from (12) to see that for any $z \in \mathbb{D}$, there exists $n_z \in \mathbb{N}$ such that $z \in D_{a_n}$, $\forall\, n \geq n_z$. So, taking (iii) and (iv) into account we have

$$\lambda_{X_n}(z) \geq \alpha_{n-1} \cdots \alpha_{n_z} \, \lambda_{X_{n_z}}(z) \geq \frac{1}{2} \lambda_{X_{n_z}}(z), \quad \forall\, n \geq n_z, \tag{13}$$

$$K_{X_n}(z) \leq \alpha_{n-1} \cdots \alpha_{n_z} \, K_{X_{n_z}}(z) \leq \frac{1}{2} K_{X_{n_z}}(z), \quad \forall\, n \geq n_z, \tag{14}$$

where we are using that $\displaystyle\prod_{n=1}^{\infty} \alpha_n = \frac{1}{2}$. Taking limits in (13) and (14) we obtain $\lambda_X(z) \geq \frac{1}{2} \lambda_{X_{n_z}}(z) > 0$, and $K_X(z) \leq \frac{1}{2} K_{X_{n_z}}(z) < 0$, for any $z \in \mathbb{D}$. This concludes the proof of this claim.

Claim 2. $X(\mathbb{D}) \subset B_3$.

From Lemma 4.1, one has $X_n(\mathbb{D}) \subset B_{R_n}$, where $R_1 = 1$ and $R_n = \sqrt{R_{n-1}^2 + s_n^2} + \epsilon_n$. Hence, it is straightforward to check that

$$R_n \leq \sqrt{R_{n-1}^2 + \frac{1}{n^2}} + \frac{1}{n^2} \leq R_{n-1} + \frac{2}{n^2}.$$

Applying successively the above inequality we obtain

$$R_n \leq 1 + \sum_{k=2}^{n} \frac{2}{k^2} \leq 1 + \sum_{k=2}^{+\infty} \frac{2}{k^2} < 3.$$

Therefore, $X_n(\mathbb{D}) \subset B_3 \; \forall\, n \in \mathbb{N}$, and so the statement of the claim holds.

Claim 3. *The disc* $(\mathbb{D}, ds_X^2)$ *is complete.*

Reasoning as in (13), it is possible to prove that

$$\operatorname{dist}_X(0, \partial(D_{a_n})) \geq \frac{1}{2} \operatorname{dist}_{X_n}(0, \partial(D_{a_n})) \geq \frac{1}{3} d_n, \quad \forall\, n \in \mathbb{N}.$$

Therefore, $\operatorname{dist}_X(0, \partial(\mathbb{D})) \geq \frac{1}{3} d_n$, $\forall\, n \in \mathbb{N}$. As $\lim_{n \to +\infty} d_n = +\infty$, then we conclude the proof. ■

One of the most interesting open questions as regards to complete bounded minimal surfaces in $\mathbb{R}^3$ is to construct, if possible, complete minimal surfaces inside a ball of $\mathbb{R}^3$ with arbitrary genus. This problem has been solved in [**53**] for minimal surfaces in a slab.

5. Complete minimal surfaces with finite total curvature

The study of complete minimal surfaces of finite total curvature began with Huber and Osserman's theorems (Theorems 5.1 and 5.2). Minimal surfaces of this kind have some special properties that are not shared by general minimal surfaces. For being more precise, complete minimal surfaces with finite total curvature have a quite controlled asymptotic behavior at infinity. Furthermore, this asymptotic behavior is surprisingly related with the topology of the surface (Theorem 5.3).

Let $X : M \longrightarrow \mathbb{R}^3$ be an isometric minimal immersion of an orientable Riemannian surface (M, ds^2) in three dimensional Euclidean space. Write $\mathcal{C}(M)$ the total curvature of X: $\int_M K\, dA$. As we mentioned in Subsection 1.1, M has a conformal structure in a natural way, and we label (g, η) the Weierstrass data of X. In the remaining part of this section we suppose M is complete and $\mathcal{C}(M) > -\infty$. Under these assumptions, A. Huber proved:

Theorem 5.1 (Huber [34]). *The Riemann surface M is conformally diffeomorphic to a compact Riemann surface $\overline{M}$ punctured in a finite number of points.*

Hence, we can write $M = \overline{M} - \{P_1, \ldots, P_n\}$, and we refer to the points $\{P_1, \ldots, P_n\}$ as the *ends* of M. We define $\mathrm{genus}(M) \stackrel{\mathrm{def}}{=} \mathrm{genus}(\overline{M})$.

By using Huber's theorem, Osserman showed:

Theorem 5.2 (Osserman [72], [73]). *The Weierstrass data g and η extend meromorphically to $\overline{M}$.*

Therefore, g has well defined degree and, from (9), it is not hard to prove that $\mathcal{C}(M) = -4\pi \deg(g)$. Furthermore, we can define the normal vector of X at an end P_i, $i \in \{1, \ldots, n\}$, as the unique vector in $\mathbb{S}^2$ whose stereographic projection is equal to $g(P_i)$.

The geometry of the ends $\{P_1, \ldots, P_n\}$ is strongly controlled by the order of the poles of Φ at these points.

W. H. Meeks and L. P. Jorge showed how these singularities determine the asymptotic behavior of the minimal surface around each end, giving geometric meaning to the numbers:

$$\nu_i = (\mathrm{Maximum}\{\mathrm{ord}(\Phi_j, P_i), \quad j = 1, 2, 3\}) - 1$$

where $\mathrm{ord}(\Phi_j, P_i)$ is the order of the pole of Φ_j at P_i, $i = 1, \ldots, n$, $j = 1, 2, 3$. We call ν_i as the *weight* of the end P_i. Since Φ has no real periods, then $\mathrm{Residue}(\Phi, P_i) \in \mathbb{R}^3$, and so $\nu_i \geq 1$.

Theorem 5.3 (Jorge, Meeks [36]). *If $X : M \to \mathbb{R}^3$ is minimal, complete and of finite total curvature, then the immersion X is proper.*

Moreover, if $\Upsilon_r \stackrel{\mathrm{def}}{=} X(M) \cap \mathbb{S}^2(r)$, then Υ_r / r consists of n closed curves $\Gamma_1, \ldots, \Gamma_n$ in $\mathbb{S}^2(1)$ which converge C^1 to closed geodesics $\gamma_1, \ldots, \gamma_n$ of $\mathbb{S}^2(1)$, with multiplicities $\nu_1, \ldots, \nu_n$, as r goes to infinity. Moreover,

$$2\deg(g) = -\chi(\overline{M}) + \sum_{i=1}^{r} (\nu_i + 1) \geq n - \chi(\overline{M}),$$

and equality holds if and only if each end is embedded.

The last expression in Theorem 5.3 is called in mathematical literature the formula of Jorge and Meeks.

We say that the end P_i, $i \in \{1, \dots, n\}$ is embedded if and only if there exists a neighborhood $D_i \subset \overline{M}$ of this point such that $X|_{D_i - \{P_i\}} : D_i - \{P_i\} \to \mathbb{R}^3$ is an embedding. This is equivalent to the fact $\nu_i = 1$. In this case we have the following result:

Theorem 5.4 (Schoen [87]). *Suppose that P_i, $i \in \{1, \dots, n\}$ is embedded, and assume $N(P_i) = (0, 0, \pm 1)$. Then, outside of a compact set, $X(D_i - \{P_i\})$ is a graph over the exterior of a bounded domain in the (x_1, x_2)-plane with the following series expansion:*

$$x_3(x_1, x_2) = a_i \log(r) + b_i + \frac{c_i x_1 + d_i x_2}{r^2} + O\left(\frac{1}{r^2}\right), \tag{15}$$

where $r = \sqrt{x_1^2 + x_2^2}$. Furthermore, Φ_1, Φ_2 have poles of order two at P_i and have no residues, while Φ_3 is either regular ($\Leftrightarrow a_i = 0$) or has a simple pole at this point.

Remark 1. There are no complete minimal surfaces with finite total curvature contained in a halfspace (see also Theorem 1.7). Indeed, from Theorem 5.4 such a surface has a coordinate function bounded either from above or from below. Since M is parabolic (see Theorem 5.1), this coordinate function is constant, and so, the surface is a plane. Then, it is clear that the only complete minimal surface with finite total curvature and one embedded end is the plane.

From (15), it is clear that $a_i \neq 0$ if and only if $X(D_i - \{P_i\})$ is asymptotic to a half catenoid, and $a_i = 0$ if and only if $X(D_i - \{P_i\})$ is asymptotic to a plane.

Definition 2. We say that an embedded end P_i, $i \in \{1, \dots, n\}$, is:

- A catenoid end iff $a_i \neq 0$.
- A planar end iff $a_i = 0$.

If P_i, $i \in \{1, \dots, n\}$ is a catenoid end, then we call a_i the *logarithmic growth* associated to P_i.

If the number ν_i is equal to 3 and the Gauss map is regular at P_i (i.e., P_i is not a ramification point of g) then it is not hard to prove that $X(M)$ is asymptotic to the Enneper's surface around P_i. In general, we say that P_i is an *Enneper end* iff $\nu_i = 3$. Other values for ν_i and the ramification number of g at P_i yield different asymptotic behaviors of X around the end P_i.

Any properly immersed minimal surface satisfies the monotonicity formula (see [**45**]):

Theorem 5.5 (Monotonicity formula). *Let $X : M \to \mathbb{R}^3$ be a properly immersed connected minimal surface. Consider $A(r)$ the area of the part of $X(M)$ inside a ball of radius $r > 0$ centered at $p \in \mathbb{R}^3$. Then,*

$$\frac{A(r)}{\pi r^2}$$

is a nondecreasing function of r.

In case of finite total curvature, we can obtain the following easy consequence of the preceding result and Theorem 5.3:

Theorem 5.6. *Let $X : M \to \mathbb{R}^3$ be a connected complete minimal immersion with finite total curvature. Following Theorem 5.3, define*

$$n(M) = \sum_{i=1}^{n} \nu_i.$$

Then, for any $p \in \mathbb{R}^3$, the cardinal number of $X^{-1}(p)$ is at most $n(M) - 1$, with the sole exception of the case $X(M)$ is a plane.

During the last century or more, the only known complete orientable minimal surfaces with finite total curvature were the plane, the catenoid and Enneper's surface, and Henneberg's example the only nonorientable one. At the beginning of the 80's a large quantity of new examples started to appear thanks to the works of Chen, Gackstatter, Costa, Hoffman, Meeks and Karcher, among others. In the next three subsections we give a brief description of the most remarkable ones. All the examples will be constructed by using the Weierstrass representation, i.e., Theorem 1.2.

5.1. Existence of minimal surfaces of least total curvature.

Let $X : M \to \mathbb{R}^3$ be a complete minimal surface with finite total curvature. We say that X has critical total curvature iff $|\mathcal{C}(M)| \leq |\mathcal{C}(M')|$, where $X' : M' \to \mathbb{R}^3$ is any complete minimal surface with the same genus as M. Looking at the formula of Jorge and Meeks, this means that the degree of the Gauss map is the least possible among the surfaces with the same genus.

Using Theorem 5.1, write $M = \overline{M} - \{P_1, \dots, P_n\}$, and recall that Theorem 5.3 implies $\nu_i \geq 1$, $i = 1, \dots, n$. Taking into account Remark 1 and the formula of Jorge and Meeks, it is not hard to deduce that $|\mathcal{C}(M)|$ is critical if and only if $\mathcal{C}(M) = -4\pi(\text{genus}(M) + 1)$ (i.e., degree$(g) =$ genus$(M) + 1$), and so either $n = 1$ and $\nu_1 = 3$ or $n = 2$ and $\nu_1 = \nu_2 = 1$. If the genus of M is not zero, the second case cannot occur (see Theorem 5.16 in Subsection 5.4), and so only the first one holds. Thus, surfaces of this kind have only one end of Enneper type.

All these facts were observed by D. Hoffman, who conjectured that there should be such examples of every genus. In the following paragraphs, we summarize some results that give an affirmative answer to the question.

5.1.1. Chen and Gackstatter's surface of genus one.

The following is an example given by Chen and Gackstatter in [**8**]. Let $\overline{M}$ be the algebraic curve of genus one

$$\overline{M} = \left\{ (z, w) \in \overline{\mathbb{C}}^2 : w^2 = z(z^2 - 1) \right\},$$

with the natural complex structure (see [**88**]). Define

$$M = \overline{M} - \{(\infty, \infty)\}, \quad g = A\frac{w}{z}, \quad \eta = B\frac{z}{w}\, dz,$$

where $A \in \mathbb{R} - \{0\}$ and $B \in \mathbb{C}$, $|B| = 1$. If we define Φ_j as in (4), then equation (3) holds on M.

Following Theorem 1.2, we have to prove that it is possible to find A and B in such a way that the 1-forms

$$\begin{aligned}
\Phi_1 &= \frac{B}{2A}\left(\frac{z}{w} - A^2\frac{w}{z}\right) dz, \\
\Phi_2 &= \frac{B\,i}{2A}\left(\frac{z}{w} + A^2\frac{w}{z}\right) dz, \\
\Phi_3 &= A\,B\,dz,
\end{aligned}$$

have no real periods.

The existence of real periods must be searched among the cycles that generate the first homology group of M. This group is generated by the curves γ_1, γ_2 and β defined as follows:

- γ_i is a lift to M of the simple closed curve c_i in the z-plane illustrated in Figure 8, $i = 1, 2$.

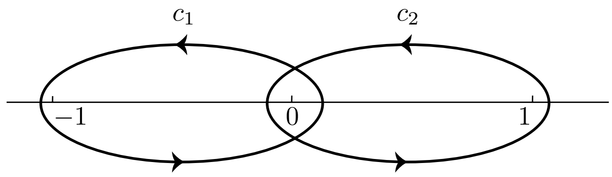

Figure 8. The curves c_1 and c_2.

- β is the boundary of a conformal disc around the point (∞, ∞).

The period problem is equivalent to solve the following system of equations:

$$\text{Real} \int_{\gamma_j} \Phi_k = 0, \quad j = 1, 2, \quad k = 1, 2, 3$$

$$\text{Real} \int_{\beta} \Phi_k = 0, \qquad\qquad k = 1, 2, 3.$$

The 1-form Φ_3 is exact, and so all its periods vanish. Moreover, since the 1-forms Φ_k are meromorphic on $\overline{M}$ and their only pole is the end (∞, ∞), then their residues at this point vanish too. This means that $\int_\beta \Phi_k = 0$, $k = 1, 2, 3$.

Therefore, it suffices to solve

$$\text{Real} \int_{\gamma_j} \Phi_k = 0, \quad j = 1, 2, \quad k = 1, 2. \tag{16}$$

If we label $f_j = \int_{\gamma_j} \frac{z}{w}\, dz$, and $g_j = \int_{\gamma_j} \frac{w}{z}\, dz$, $j = 1, 2$, then elementary algebraic arguments imply that (16) is equivalent to:

$$f_1 \overline{g_2} - f_2 \overline{g_1} = 0, \quad \text{and} \quad A^2 = B^2 f_1 / \overline{g_1}. \tag{17}$$

Note that we have to solve only the first equation of (17), and then choose A and B satisfying the second one.

To do this, consider the holomorphic automorphism $J((z, w)) = (-z, iw)$. Without loss of generality, we can suppose that $\gamma_2 = J_*(\gamma_1)$. Observe also that $J^*(\frac{z}{w}\, dz) = -i \frac{z}{w}\, dz$ and $J^*(\frac{w}{z}\, dz) = i \frac{w}{z}\, dz$. Therefore, $f_2 = -i f_1$ and $g_2 = i g_1$, and so the first equation in (17) holds.

In fact, it is not hard to see that $f_j/\overline{g_j} \in \mathbb{R}^+$, and so $B = \pm 1$, $A = \pm\sqrt{f_j/\overline{g_j}}$. The different choices of the sign produce, up to a rigid motion, the same surface.

The arising surface has the following properties:

- $\deg(g) = 2$, and so $\mathcal{C}(M) = -8\pi$.
- M has only one end of Enneper type (i.e., its weight is 3), and M is asymptotic to Enneper's surface.
- The conformal transformations on M: $J((z,w)) = (-z, iw)$ and $S((z,w)) = (\overline{z}, \overline{w})$, induce on $X(M)$ a rotation about the x_3-axis by angle $\frac{\pi}{2}$ followed by a symmetry with respect to the plane $x_3 = 0$ and a symmetry with respect to the plane $x_2 = 0$, respectively. Following Theorem 1.4, these transformations generate the symmetry group of the surface, which contains 8 elements. Therefore, Chen-Gackstatter's surface of genus one has the same symmetries as Enneper's surface.

5.1.2. Chen and Gackstatter's surface of genus two.

The example that will be described is also due to Chen and Gackstatter [**8**].

Let $\overline{M}_a$, $a \in]1, +\infty[$, be the compact Riemann surface:

$$\overline{M}_a = \left\{ (t,w) \in \overline{\mathbb{C}}^2 : w^2 = \frac{t(t^2 - a^2)}{t^2 - 1} \right\}$$

and label $\infty = (\infty, \infty)$, $0 = (0,0)$, $\pm 1 = (\pm 1, \infty)$, $\pm a = (\pm a, 0)$.

Consider the following Weierstrass data:

$$M_a = \overline{M}_a - \{\infty\}, \quad g = Aw, \quad \eta g = B\,dt, \quad A \in \mathbb{R}, \quad B \in \mathbb{C}, \quad |B| = 1$$

on $\overline{M}_a$. Then, defining Φ_j, $j = 1, 2, 3$ as in (4), the inequality (3) holds. Therefore, from Theorem 1.2, if Φ_j, $j = 1, 2, 3$, have no real periods, we get a minimal immersion $X : M_a \to \mathbb{R}^3$.

The main achievement of this paragraph is to show that there exists $a_0 \in]1, +\infty[$ such that $X : M_{a_0} \to \mathbb{R}^3$ is well-defined for a suitable choice of the constants A, B.

First, define the following mappings:

$$J, S : \overline{M}_a \longrightarrow \overline{M}_a$$
$$J(t,w) = (-t, iw) \quad S(t,w) = (\overline{t}, \overline{w}).$$

Note that J is holomorphic and has order 4, and S is an antiholomorphic involution. So, they generate a group with 8 elements which is isomorphic to $\mathcal{D}(4)$. Moreover, J and S fix 0, ∞, and J^2 fixes 1, -1, a and $-a$.

To write the period problem easily, we need to do a complete description of the first homology group of the surface. Let $\alpha_j(s)$, $\beta_j(s)$, $j=1,2$, be the oriented simple closed curves in the t-plane illustrated in Figure 9. We assume that $\alpha_1(0)\in\mathbb{R}$, $\alpha_1(0)>a$, $\alpha_2(0)\in\mathbb{R}$, $1>\alpha_2(0)>0$, $\beta_1(0)\in\mathbb{R}$, $0>\beta_1(0)>-1$, $\beta_2(0)\in\mathbb{R}$, $a>\beta_2(0)>1$. Let $a_j(s)$ be the unique lift of $\alpha_j(s)$ to $\overline{M}_{k\,a}$ satisfying $w(a_j(0))\in\mathbb{R}_+$, $j=1,2$. Denote in the same way as $b_j(s)$, the corresponding lifts of $\beta_j(s)$ with initial conditions $w(b_j(0))\in i\mathbb{R}_+$, $j=1,2$.

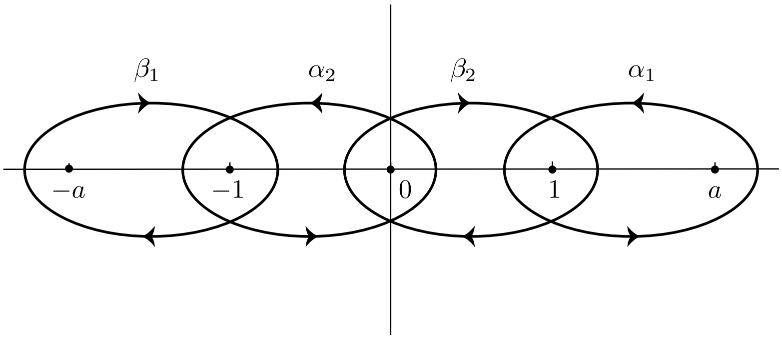

Figure 9. The curves α_i and β_i, $i=1,2$.

Then observe that

$$J_*(b_i)=a_i,\quad i=1,2. \tag{18}$$

Elementary topological arguments give that the set $\mathcal{B}=\{a_i,\, b_i,\, i=1,2\}$ is a homology basis on $\overline{M}_a$. To solve the period problem, we have to prove that Φ has no real periods on the curves in $\mathcal{B}$.

Let τ_1, τ_2 be the following 1-forms on $\overline{M}_a$

$$\tau_1=\frac{dt}{w},\quad \tau_2=w\,dt.$$

Observe

$$\Phi_1=\frac{B}{2A}(\tau 1-A^2\tau 2)$$

$$\Phi_2=\frac{i\,B}{2A}(\tau 1+A^2\tau 2)$$

and Φ_3 is exact.

So the period problem associated to (Φ_1, Φ_2, Φ_3) deals with the following functions on $]1, +\infty[$

$$f_1(a) = \frac{1}{2}\int_{b_1} \tau_1, \quad f_2(a) = \frac{1}{2}\int_{b_2} \tau_1,$$

$$g_1(a) = \frac{1}{2}\int_{b_1} \tau_2, \quad g_2(a) = \frac{1}{2}\int_{b_2} \tau_2.$$

It is not hard to see that

$$J^*(\tau_1) = i\tau_1, \quad J^*(\tau_2) = -i\tau_2. \tag{19}$$

Observe that $f_i(a)$, $g_i(a) > 0$ $i = 1, 2$. From (18)

$$\int_{a_i} \tau_1 = 2if_i(a), \quad \int_{a_i} \tau_2 = -2ig_i(a), \quad i = 1, 2.$$

We need the following technical lemma.

Lemma 5.7. *The asymptotic behavior of f_i, g_i, $i = 1, 2$ at 1, ∞ is given as follows:*

(i)
$$\lim_{a\to 1} \frac{f_1(a)}{a-1} = \frac{\pi}{2}, \quad \lim_{a\to\infty} f_1(a)a^{-\frac{1}{2}} = \frac{1}{2}\mathfrak{B}\left(\frac{1}{2}, \frac{3}{4}\right),$$

$$\lim_{a\to 1} f_2(a) = 2, \quad \lim_{a\to\infty} f_2(a)a = \frac{1}{2}\mathfrak{B}\left(\frac{3}{2}, \frac{1}{4}\right),$$

(ii)
$$\lim_{a\to 1} \frac{g_1(a)}{a-1} = \frac{\pi}{2}, \quad \lim_{a\to\infty} g_1(a)a^{-\frac{3}{2}} = \frac{1}{2}\mathfrak{B}\left(\frac{3}{2}, \frac{1}{4}\right),$$

$$\lim_{a\to 1} g_2(a) = \frac{2}{3}, \quad \lim_{a\to\infty} g_2(a)a^{-1} = \frac{1}{2}\mathfrak{B}\left(\frac{1}{2}, \frac{3}{4}\right),$$

where $\mathfrak{B}$ is the classical Beta function.

Recall that the Beta function is defined for $m, n \in \mathbb{C}$, $\mathrm{Re}(m) > 0$, $\mathrm{Re}(n) > 0$, as follows:

$$\mathfrak{B}\,(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt.$$

This is related to the Gamma function according to

$$\mathfrak{B}\,(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Proof: From the definition of f_1 it follows that

$$f_1(a)=\int_1^a\sqrt{\frac{t^2-1}{t(a^2-t^2)}}\,dt.$$

Several changes of variables give

$$\begin{aligned}f_1(a)&=\int_1^a\sqrt{\frac{t^2-1}{t(a^2-t^2)}}\,dt\\&=\frac{1}{2}\int_1^{a^2}u^{-\frac{3}{4}}\sqrt{\frac{u-1}{a^2-u}}\,du\\&=\frac{1}{k}(a^2-1)\int_0^1\frac{((a^2-1)s+1)^{-\frac{3}{4}}\sqrt{s}}{\sqrt{1-s}}\,ds.\end{aligned}$$

Hence,

$$\lim_{a\to 1}\frac{f_1(a)}{a-1}=\int_0^1\sqrt{\frac{s}{1-s}}\,ds=\frac{\pi}{2}.$$

Using that $\lim_{a\to\infty}f_1(a)=\lim_{b\to 0}f_1\left(\frac{1}{b}\right)$ we obtain

$$\begin{aligned}\lim_{a\to\infty}f_1(a)a^{\frac{-1}{2}}&=\lim_{b\to 0}\frac{1-b^2}{2}\int_0^1\frac{((1-b^2)s+b^2)^{-\frac{3}{4}}\sqrt{s}}{\sqrt{1-s}}\,ds\\&=\frac{1}{2}\int_0^1 s^{-\frac{1}{4}}(1-s)^{-\frac{1}{2}}\,ds=\frac{1}{2}\mathfrak{B}\left(\frac{1}{2},\frac{3}{4}\right).\end{aligned}$$

Similar arguments and changes of variables complete the above assertions for g_i, $i=1,2$. ■

Let us define $\varphi:\mathbb{R}_+-\{1\}\longrightarrow\mathbb{R}$,

$$\varphi(a)=\frac{f_2(a)}{f_1(a)}-\frac{g_2(a)}{g_1(a)}.$$

As a consequence of the preceeding analysis we can state the following lemma:

Lemma 5.8. *The function φ vanishes at a point $a_0\in]1,+\infty[$.*

Proof: From Lemma 5.7 we deduce that:

$$\lim_{a\to 1^+} \varphi(a) = \frac{8}{3\pi} \lim_{a\to 1^+} \frac{1}{a-1} = +\infty$$

$$\lim_{a\to+\infty} \varphi(a)a^{\frac{1}{2}} = \lim_{a\to+\infty} \left(\frac{1}{a} \frac{f_2(a)a}{f_1(a)a^{-\frac{1}{2}}} - \frac{g_2(a)a^{-1}}{g_1(a)a^{-\frac{3}{2}}} \right) = -\frac{\mathfrak{B}\left(\frac{1}{2},\frac{3}{4}\right)}{\mathfrak{B}\left(\frac{3}{2},\frac{1}{4}\right)} < 0.$$

An intermediate value argument completes the proof. ■

Now, we are able to solve the period problem. The immersion X is well defined if and only if Real $\left(\int_d \Phi_j\right) = 0$, for every closed curve d in M_a and $j \in \{1,2,3\}$. As Φ_j has only one singularity at ∞, then Residue$(\Phi_j, \infty) = 0$, $j = 1,2,3$. So, it suffices to prove:

$$\text{Real}\left(\int_d \Phi_j\right) = 0, \quad j = 1,2,3$$

for any closed curve, d, lying in the homology basis $\mathcal{B}$ of $\overline{M}_{k\,a}$ defined at the beginning of this section.

It is clear that $J^*({}^t\Phi) = \mathcal{R}\cdot({}^t\Phi)$, where $\mathcal{R} \in \mathcal{O}(3,\mathbb{R})$ is the matrix

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence using the last equality and (18), Real $\left(\int_d \Phi\right) = \vec{0}$, $\forall\, d \in \mathcal{B}$ if and only if:

$$\text{Real}\left(\int_{b_1} \Phi\right) = \text{Real}\left(\int_{b_2} \Phi\right) = \vec{0}.$$

Using the definitions of f_i, $i = 1,2$, the last equations hold if and only if $B^2 = 1$ and

$$f_1(a) = A^2 g_1(a)$$
$$f_2(a) = A^2 g_2(a)$$

for some $A \in \mathbb{R}$, $a \in]1,+\infty[$. As $f_i(a)$, $g_i(a) \in \mathbb{R}^*$, $\forall\, a \in]1,+\infty[$, then the existence of a, A satisfying the former is equivalent to solving the following equation

$$f_1(a)g_2(a) - f_2(a)g_1(a) = 0 \tag{20}$$

and putting $A^2 = \dfrac{f_1(a)}{g_1(a)} > 0$. Recalling the definition of the function φ, this means that $\varphi(a) = 0$. Using Lemma 5.8 we conclude the period problem.

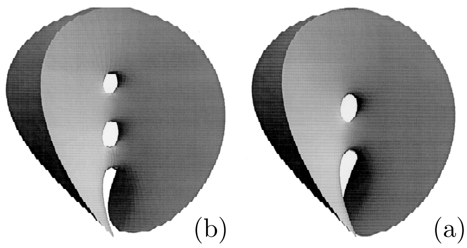

Figure 10. (a) Chen and Gackstatter's surface of genus one.
(b) Chen and Gackstatter's surface of genus two.

The arising surface has the following properties:

- $\deg(g) = 3$, and so $\mathcal{C}(M) = -12\pi$.
- M has only one end of Enneper type (i.e., its weight is 3), asymptotic to Enneper's surface.
- The conformal transformations on M: $J((t,w)) = (-t, iw)$ and $S((t,w)) = (\overline{t}, \overline{w})$, induce on $X(M)$ a rotation about the x_3-axis by angle $\frac{\pi}{2}$ followed by a symmetry with respect to the plane $x_3 = 0$ and a symmetry with respect to the plane $x_2 = 0$, respectively. Following Theorem 1.4, these transformations generate the symmetry group of the surface, which contains 8 elements. Therefore, Chen-Gackstatter genus two minimal surface has the same symmetries as Enneper's surface.

5.1.3. The surfaces of Espirito-Santo, Thayer and Sato.

These examples are due to Espirito-Santo [**18**] (genus three), Thayer [**89**] (genus less than or equal to 35) and Sato [**86**] (arbitrary genus).

The first two authors solved the period problem by giving numerical arguments. However, Sato used a homotopy argument, which can be thought of as an intermediate value theorem of several variables.

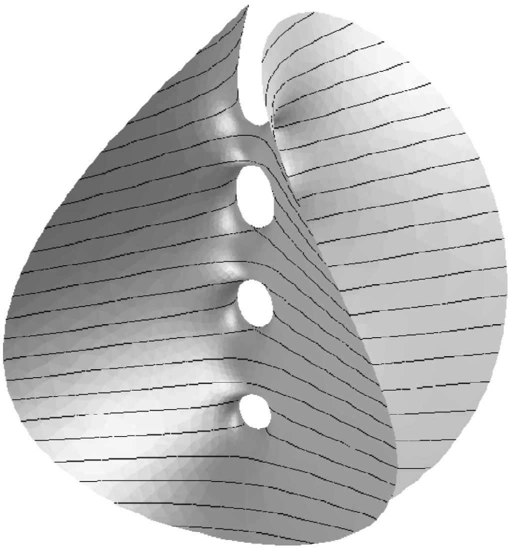

Figure 11. The surface of Espirito-Santo.

We need to introduce the following notation.

Let

$$F_j(z, a_2, \dots, a_j) = z\Pi_{m=1}^{j/2}(z^2 - a_{2m}^2)\Pi_{n=1}^{(j+1)/2}(z^2 - a_{2n-1}^2)^{-1},$$

where $j, k, m, n \in \mathbb{N}$, $j \geq 3$, $a_1, \dots, a_j \in \mathbb{R}$, $1 = a_1 < a_2 < \dots < a_j$. Then, put

$$\overline{M} = \{(z, w) \in \overline{\mathbb{C}}^2 : w^2 = F_j(z, a_1, \dots, a_j)\},$$

with the natural complex structure, and observe that this surface has genus j. Write

$$M_j = \overline{M}_j - \{(\infty, \infty)\}, \quad \text{even } j,$$
$$M_j = \overline{M}_j - \{(\infty, 0)\}, \quad \text{odd } j.$$

Finally, define the following meromorphic data on M_j:

$$g = c_j w, \quad \Phi_3 = \eta g = dz,$$

where $c_j \in \mathbb{R}$. Note that g has degree $j + 1$.

As above, the existence of real periods must be searched among the cycles that generate the first homology group of M_j. Hence, we need a homology basis of M_j. A natural homology basis consists of the curves $\beta_1, \dots, \beta_{2j}$ obtained, respectively, as the lifts to M_j of the segments

$$[-a_j, -a_{j-1}], [-a_{j-1}, -a_{j-2}], \dots, [-a_1, 0], [0, a_1], \dots$$
$$\dots, [a_{j-2}, a_{j-1}], [a_{j-1}, a_j]$$

in the z-plane, together with a Jordan curve γ lying in a conformal disc centered at the only end of M_j (and containing no points in $\{z^{-1}(\pm a_i), i = 1, \dots, j\} \cup \{(0,0)\}$).

Since M_j has only one end, the residue of Φ_i at this end is 0, $i = 1, 2, 3$, and so γ does not generate any real period.

On the other hand, the holomorphic transformations

$$J(z, w) = (-z, iw), \quad S(z, w) = (\overline{z}, \overline{w})$$

are well defined on M_j. Taking into account their behavior on the curves β_k, $k = 1, \dots, 2j$, and the 1-forms Φ_i, $i = 1, 2, 3$, it is not hard to deduce that Φ has no real periods on M if and only if it has no real periods on the curves $\alpha_i \stackrel{\text{def}}{=} \beta_{2i-1}$, $i = 1, \dots, j$.

Note that Φ_3 is exact, and define

$$A_l = \int_{\alpha_l} \frac{dz}{w}, \quad B_l = \int_{\alpha_l} w\, dz, \quad l = 1, \dots, j.$$

Observe that the quotient $\frac{A_l}{B_l}$ is well defined and positive, $l = 1, \dots, j$.

Reasoning as in the above paragraphs, the immersion $X = \operatorname{Real} \int \Phi$ is well defined if and only if

$$c_j^2 = \frac{A_1}{B_1} = \dots = \frac{A_j}{B_j}. \tag{21}$$

Furthermore, denote

$$\varphi_{j\,l}(a_2, \dots, a_j) \stackrel{\text{def}}{=} \frac{A_{l+1}}{B_{l+1}} - \frac{A_l}{B_l}, \quad l = 1, \dots, j-1.$$

With this new notation, (21) becomes

$$\varphi_{j\,l}(a_2, \dots, a_j) = 0, \quad l = 1, \dots, j-1, \tag{22}$$

for a suitable choice of c_j. If $S_j = \{(a_2, \dots, a_j) \in \mathbb{R}^{j-1} : 1 < a_2 < \cdots < a_j\}$, we define

$$\mathcal{P} : S_j \longrightarrow \mathbb{R}^{j-1}$$

$$\mathcal{P}(a_2, \dots, a_j) = (\varphi_{j\,1}(a_2, \dots, a_j), \dots, \varphi_{j\,j-1}(a_2, \dots, a_j)) .$$

So, it suffices to prove that $\mathcal{P}$ vanishes at least once.

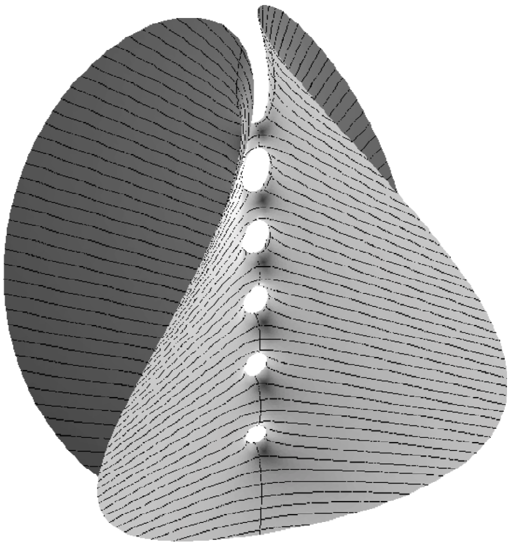

Figure 12. The Thayer-Sato surface of genus five.

The proof of this fact is quite technical, and it can be found in [**86**]. We are only going to give a brief sketch of it. The main idea consists of finding a compact polyhedral domain $K \subset S_j$, K homeomorphic to the unit ball

$$B^{j-1} = \{(a_2, \dots, a_j) \in \mathbb{R}^{j-1} : ||(a_2, \dots, a_j)|| \leq 1\},$$

such that:

- $0 \notin \mathcal{P}(\partial(K))$,
- $r \circ (\mathcal{P}|_{\partial(K)}) : \partial(K) \to \partial(B^{j-1})$ has nonzero topological degree, where $r(x) = \frac{x}{||x||}$.

Then, a homotopy argument leads to the existence of a point in $K - \partial(K)$ where $\mathcal{P}$ vanishes.

The arising surface has the following properties:

- $\deg(g) = j + 1$, and so $\mathcal{C}(M) = -4(j+1)\pi$.
- M has only one end of Enneper type (i.e., its weight is 3), asymptotic to Enneper's surface.
- The conformal transformations on M: $J((z,w)) = (-z, iw)$ and $S((z,w)) = (\overline{z}, \overline{w})$, induce on $X(M)$ a rotation about the x_3-axis by angle $\frac{\pi}{2}$ followed by a symmetry with respect to the plane $x_3 = 0$ and a symmetry with respect to the plane $x_2 = 0$, respectively. Following Theorem 1.4, these transformations generate the symmetry group of the surface, which contains 8 elements. Therefore, the surface has the same symmetries as Enneper's surface.

Remark 2. Weber and Wolf in [**92**] have also constructed minimal surfaces with arbitrary genus and critical total curvature. They develop Teichmüller theoretical methods to produce minimal surfaces which have a low degree Gauss map for their genus.

Uniqueness theorems for surfaces of this kind are known only when the genus is zero or one (see Subsection 5.4). Therefore, although it is expected, we cannot assert whether the Weber and Wolf surfaces coincide with Sato's ones or not.

5.2. New families of examples.

This subsection is devoted to review some families of orientable minimal surfaces with finite total curvature which are interesting from different points of view. So, we describe surfaces with arbitrary genus and high symmetry group. We also state a general existence theorem for nonrigid minimal surfaces.

As we will see, the period problem for highly symmetric minimal surfaces becomes quite easy. This is due to the fact that the rotational symmetry acts as a cyclic group on the generators of the first homology group of the surface, thereby reducing the period problem.

Examples derived from Chen-Gackstatter genus one surface. Karcher in [**39**] generalized the Cheng-Gackstatter genus one surface by increasing the order of the normal rotational symmetry from 2 to $k \in \mathbb{N}$. This also increases the genus from 1 to $k-1$, and the weight of the end from 3 to $2k-1$. This technique was introduced before by Hoffman and Meeks in [**32**]. To be more precise, for each $k \in \mathbb{N}$, $k > 2$, consider the following Weierstrass data:

$$\overline{M}_k = \left\{ (z,w) \in \overline{\mathbb{C}}^2 : w^k = \frac{z^2-1}{z} \right\}$$

$$M_k = \overline{M}_k - \{(\infty,\infty)\}, \quad g = Aw^{k-1}, \quad \eta g = dz,$$

where $A \in \mathbb{R} - \{0\}$. To solve the period problem, define the conformal transformations on M_k:

$$J(z,w) = (-z, e^{\frac{\pi i}{k}} w), \quad S(z,w) = (\overline{z}, \overline{w}),$$

and observe that:

$$J^*({}^t\Phi) = \mathcal{R} \cdot ({}^t\Phi), \quad S^*({}^t\Phi) = \mathcal{S} \cdot ({}^t\Phi), \tag{23}$$

where:

$$\mathcal{R} = \begin{pmatrix} \cos\frac{\pi}{k} & \sin\frac{\pi}{k} & 0 \\ -\sin\frac{\pi}{k} & \cos\frac{\pi}{k} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathcal{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We are looking for a homology basis of $\overline{M}_k$. Let $\alpha_1(s)$, $\beta_1(s)$ be the oriented simple closed curves in the t-plane illustrated in Figure 13. We assume that $\alpha_1(0) \in \mathbb{R}$, $\alpha_1(0) > 1$, $\beta_1(0) \in \mathbb{R}$, $0 < \beta_1(0) < 1$. Let $a_1(s)$ be the unique lift of $\alpha_1(s)$ to $\overline{M}_k$ satisfying $w(a_1(0)) \in \mathbb{R}_+$. Denote in the same way as $b_1(s)$, the corresponding lift of $\beta_1(s)$ with initial condition $\arg(w(b_1(0))) = \frac{\pi}{k}$.

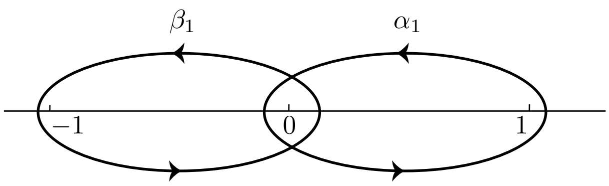

Figure 13. The curves α_1 and β_1.

Then observe that

$$J_*(a_1) = b_1, \quad S_*(a_1) = -a_1. \tag{24}$$

Define also c as the boundary of a closed conformal disc around the end (∞,∞).

If we label $a_j = (J^j)_*(a_1)$, $(j = 0, \ldots, 2k-1)$, then

$$\mathcal{B} = \{a_j, \quad j = 0, \ldots, 2k-1\} \cup \{c\}$$

generates the homology of M_k. Since the residue at the end of Φ vanishes and Φ_3 is exact, it suffices to check that Φ_1 and Φ_2 have no real periods along a_j, $j = 0, \ldots, 2k-1$. Taking (23) into account, it is not hard to see that this period problem reduces to:

$$\int_{a_1} \eta = \overline{\int_{a_1} \eta g^2}.$$

This equality easily holds for a suitable choice of A.

The arising surface has the following properties:

- genus$(M) = k-1$, $\deg(g) = 2(k-1)$, and so $\mathcal{C}(M) = -8(k-1)\pi$.
- M has only one end and its weight is $2k-1$.
- The conformal transformations on M: J and S, induce on $X(M_k)$ a rotation about the x_3-axis by angle $-\frac{\pi}{k}$ followed by a symmetry with respect to the plane $x_3 = 0$ and a symmetry with respect to the plane $x_2 = 0$, respectively. Following Theorem 1.4, these transformations generate the symmetry group of the surface, which contains $4k$ elements.

Thayer and Sato families. Combining an extension of Sato's idea with Karcher's generalization to allow for higher weight on the ends, Thayer, and independently Sato, produced the Weierstrass data for a countable collection of surfaces $M_{p\,k}$ for $p, k \in \mathbb{N}$, $p \geq 0$, $k \geq 2$. These Weierstrass representation are:

$$\overline{M}_{p\,k} = \left\{(z,w)/w^k = F_p(z, q_1, q_2, \ldots, q_p)\right\},$$
$$M_{p\,k} = \overline{M}_{p\,k} - \{(\infty,\infty)\}, \quad g = A\, w^{k-1}, \quad \eta\, g = dz,$$

where $A, q_j \in \mathbb{R}$, with $A > 0$, $1 = q_1 < q_2 < \cdots < q_p$, and

$$F_p(z, q_1, q_2, \ldots, q_p) = \begin{cases} z \prod_{l=1}^m \frac{z^2 - q_{2l}^2}{z^2 - q_{2l-1}^2}, & \text{if } p = 2m, \\ \frac{z}{z^2 - q_p^2} \prod_{l=1}^m \frac{z^2 - q_{2l}^2}{z^2 - q_{2l-1}^2}, & \text{if } p = 2m+1. \end{cases}$$

In [**89**], Thayer presented numerical results suggesting that the period problems were solvable for $p \leq 34$, $k \leq 9$. Later, Sato [**86**] obtained a rigorous proof for the existence of these examples, which is similar in style to that described in paragraph 5.1.3.

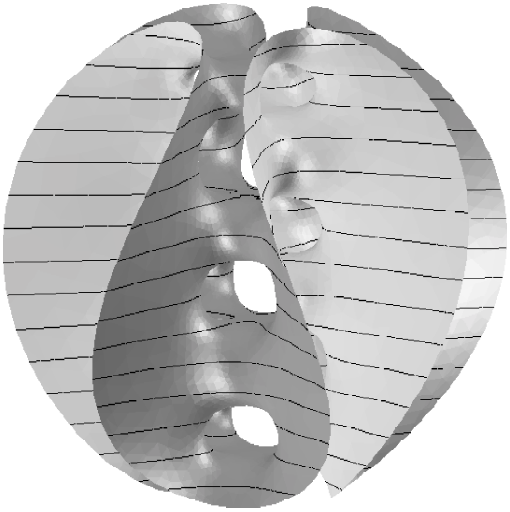

Figure 14. Thayer's surface for $p = 3$ and $k = 4$.

The surface $M_{p\,k}$ has genus $p(k-1)$, one end of weight $2k-1$ and the symmetry of the Karcher's generalized Enneper surface with the same weight at the end.

Examples derived from the Chen-Gackstatter genus two surface. The authors of this survey and D. Rodríguez [**57**] exhibited a family of complete minimal surfaces $X : M_k \to \mathbb{R}^3$ of genus k, $k \geq 2$, k even, that generalizes the Chen-Gackstatter genus two example. Except in the case of $k = 2$ which corresponds to Chen and Gackstatter's example, the examples do not lie in any of the families of surfaces discovered by Thayer, Sato and Weber-Wolf.

Summarizing, the surfaces M_k have the following properties:

(i) *The Weierstrass data* $(M_k, g, \eta g)$ *of* X *are:*

$$M_k = \overline{M}_k - \{(\infty, \infty)\},$$

where

$$\overline{M}_k = \left\{ (t,w) \in (\mathbb{C} \cup \infty)^2 : w^2 = \frac{t(t^k - a^k)}{t^k - 1} \right\}, \quad a \in]1, +\infty[,$$

and

$$g = A\,t^{k/2-1}w, \quad \eta g = t^{k/2-1}\,dt, \quad A \in \mathbb{R} - \{0\}.$$

(ii) $\mathcal{C}(M_k) = -4(2k-1)\pi$.
(iii) $X(M_k)$ *has* $4k$ *symmetries.*
(iv) $X(M_k)$ *intersects the* (x_1, x_2)*-plane in* k *straight lines meeting at equal angles at the origin. Moreover the symmetry group* $\mathrm{Sym}(M_k)$ *is generated by a rotation by angle* π/k *around the* x_3*-axis followed by a symmetry with respect to the* (x_1, x_2)*-plane and a symmetry with respect to the* (x_1, x_3)*-plane.*
(v) $X(M_2)$ *is the genus two Chen-Gackstatter example.*

In [**57**] the authors proved that there exists a unique $a \in]1, +\infty[$ solving the period problem. The ideas are similar to those in paragraph 5.1.2.

Nonrigid minimal surfaces: Pirola's surfaces. Using methods from Algebraic Geometry, Pirola have proved:

Theorem 5.9 (Pirola [80]). *Let* $\overline{M}$ *be a compact connected Riemann surface and* Z *be a nonempty finite subset of* $\overline{M}$*. Then, there is a complete nonrigid minimal immersion* $X : \overline{M} - Z \to \mathbb{R}^3$ *with finite total curvature.*

Looking at Theorem 1.3, the immersion X is nonrigid if and only if the 1-forms Φ_j, $j = 1, 2, 3$, in the Weierstrass representation are exact.

Minimal surfaces with catenoid ends. In this paragraph we follow the notation introduced by Cosin and Ros in [**12**]. A properly immersed minimal surface with r embedded ends will be called an r-oid. Among these kinds of surfaces, we emphasize the following examples:

- The Jorge-Meeks r-oid with symmetry group $\mathcal{D}_r \times \mathbb{Z}_2$ [**36**].
- The r-oids with high genus by Berglund and Rossman [**1**], [**85**].
- The genus zero Platonoids with symmetry groups isomorphic to the symmetry group of the Platonic solids [**99**], [**40**], [**90**].

It is natural to state the following *Plateau problem at infinity*:

> *Given a balanced finite system of planes and halfcatenoids in* $\mathbb{R}^3$ *and a nonnegative integer* g*, find a* r*-oid of genus* g *whose ends are asymptotic, up to translations, to the given data.*

By definition, we say that a system of planes and halfcatenoids is balanced if and only if the sum of the flux vectors (see Paragraph 6.2.1) is zero. Classical theory of compact Riemann surfaces says that any minimal surface with finite total curvature and embedded ends is balanced. This is an easy consequence of the fact that the sum of the residues of a meromorphic 1-form on a compact Riemann surface is zero. From a geometrical point of view, the flux of a halfcatenoid is the value of its normal vector at infinity times for the length of its neck and the flux of a plane is zero. Kusner was the first author who proposed the above problem in its right terms.

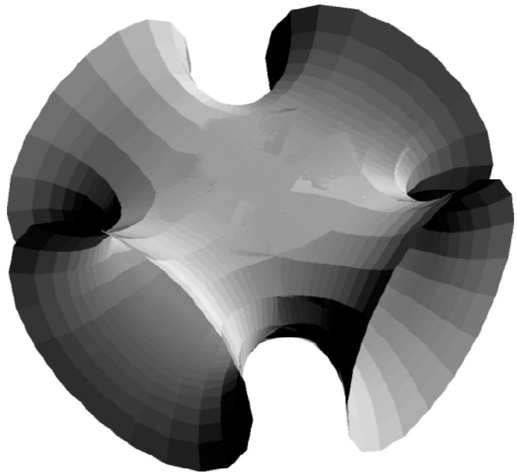

Figure 15. Jorge and Meeks' four-oid.

In the genus zero case, Kato, Umehara and Yamada [**41**], [**42**], [**43**] reduce the above Plateau problem at infinity, using Weierstrass representation, to a system of algebraic equations. So, they prove that, for generic data, this problem admits a solution.

Let $X : M \to \mathbb{R}^3$ be an r-oid. Then M is conformally diffeomorphic to $\overline{M}-\{P_1, \dots, P_r\}$, where $\overline{M}$ is compact. We will say that M is *Alexandrov embedded* if $\overline{M}$ bounds a compact 3-manifold $\overline{\Omega}$ and the immersion X extends to a proper local diffeomorphism $f : \overline{\Omega} - \{P_1, \dots, P_r\} \to \mathbb{R}^3$. In the line of Kato-Umehara-Yamada theorems, Cosin and Ros [**12**] have obtained the existence of a genus zero r-oid, $r \geq 3$, satisfying:

1. The normal vectors at the ends lie in the plane $x_3 = 0$.
2. The surface is Alexandrov-embedded and symmetric with respect to the plane $x_3 = 0$.
3. The polygon whose edges are the ordered flux vectors of M bounds an immersed disc in the plane.

Minimal surfaces with embedded planar ends. Let $X : M \to \mathbb{R}^3$ a complete minimal immersion with finite total curvature. Then (reinterpreting Theorems 5.1 and 5.2) there is associated to X a $C^{1,\alpha}$ immersion of a compact Riemann $\overline{M}$ into $\mathbb{S}^3$,

$$\overline{X} : \overline{M} \to \mathbb{S}^3 \equiv \mathbb{R}^3 \cup \{\infty\},$$

such that $M = \overline{M} - \{P_1, \ldots , P_n\}$ and $\overline{X}|_M = X$. The immersion $\overline{X}$ is possibly branched at the ends $P_1, \ldots , P_n$. However, if the ends are embedded and asymptotic to planes, Bryant [**4**] observed that $\overline{X}$ is regular at the ends. Furthermore, this author also noted that these surfaces give extrema for the *Willmore functional*: $W = \int H^2 \, dA$.

Concerning minimal surfaces with embedded flat ends, we know the following results:

- There exist examples of genus zero and 4, 6 and n ends, $n \geq 8$. In case of 4 and 6 ends, the classification is known [**4**], [**47**], [**74**].
- There are no examples of genus zero and 3, 5, and 7 ends [**5**].
- The moduli space of genus zero examples with $2p$ ends, $2 \leq p \leq 7$, has dimension $4(p-1)$ [**5**].
- There exist rectangular tori with four ends [**14**].
- There is a real two-dimensional family of four-ended immersed examples on each conformal torus [**46**].
- There are no three-ended tori [**46**].

Concerning to the last point, Kusner conjectured that:

Conjecture 1 (Kusner). *There are no complete, orientable, minimal surfaces with finite total curvature and three embedded planar ends.*

In genus two case, Pirola [**81**] have obtained a partial answer to this conjecture, by proving that there are no three-ended untwisted genus two surfaces.

5.3. Nonorientable examples.

Let $X' : M' \to \mathbb{R}^3$ be a complete nonorientable minimal immersion with finite total curvature. We call (M, I, g, η) the Weierstrass data of X' (for more details see Subsection 1.1).

Under these assumptions, using Huber's and Osserman's theorems (see Theorems 5.1 and 5.2), we obtain that M is conformally diffeomorphic to a compact Riemann surface $\overline{M}$ punctured in a finite number of points $\{P_1, \ldots, P_r\}$ and (g, η) extends meromorphically to $\overline{M}$. Furthermore, I extends meromorphically to $\overline{M}$ and we have:

$$r = 2s, \quad \{P_1, \ldots, P_r\} = \{Q_1, \ldots, Q_s, I(Q_1), \ldots, I(Q_s)\}.$$

Therefore, g has a well defined degree and $\mathcal{C}(M) = -4\pi \deg(g)$. Let $G : M' \to \mathbb{RP}^2$ be the generalized Gauss map. As we mentioned in Subsection 1.1, the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{g} & \overline{\mathbb{C}} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_0} \\ M' & \xrightarrow{G} & \mathbb{RP}^2 \end{array}$$

where $p_0 : \overline{\mathbb{C}} \to \mathbb{RP}^2 \equiv (\overline{\mathbb{C}}/\langle I_0 \rangle)$ is the natural projection, and $I_0(z) = -1/\overline{z}$.

As $\deg(p) = \deg(p_0) = 2$, then $\deg(G)$ is also well-defined and $\deg(g) = \deg(G)$. In particular $\mathcal{C}(M') = -2\pi \deg(G)$.

Note that
$$\overline{M}' = \frac{\overline{M}}{\langle I \rangle}$$
is a compact nonorientable conformal surface, and

$$M' = \overline{M}' - \{p(Q_1), \ldots, p(Q_s)\}.$$

On the other hand, Jorge-Meeks formula can be reformulated as follows:

$$\deg(g) = -\chi(\overline{M}') + \sum_{i=1}^{s} (\nu_i + 1) \tag{25}$$

where
$$\nu_i = \text{Maximum}\{\text{ord}(\Phi_j, Q_i), \quad j = 1, 2, 3\} - 1$$
and $\text{ord}(\Phi_j, Q_i)$ is the order of the pole of Φ_j at Q_i.

In the nonorientable case, we have stronger restrictions on the topology of M' (or M). Meeks showed that:

Theorem 5.10 (Meeks [62]). *Let $\overline{M}'$ be a compact nonorientable conformal surface, and $M' = \overline{M}' - \{P_1, \ldots, P_s\}$. If $X' : M' \to \mathbb{R}^3$ is a complete minimal immersion with finite total curvature, then the Euler characteristic $\chi(\overline{M}')$ of $\overline{M}'$ and $\mathcal{C}(M')/2\pi$ are congruent modulo 2.*

This theorem is consequence of the following topological lemma.

Lemma 5.11 (Meeks [62]). *Let M_1 and M_2 be two compact surfaces without boundary, and consider $p : M_1 \to M_2$ a branched covering map. Then:*

1. *$\chi(M_2)$ odd implies that $\chi(M_1)$ and $\deg(p)$ are both either even or odd.*
2. *$\chi(M_2)$ even yields that $\chi(M_1)$ is even too.*

The proof of this result can be found in the above mentioned Meek's article, and we omit it.

Proof of Theorem 5.10: Consider $G : \overline{M}' \to \mathbb{RP}^2$ the generalized Gauss map of X'. Then G is a branched covering map. Taking into account that $\chi(\mathbb{RP}^2) = 1$, we apply Lemma 5.11 and obtain $\deg(G) = -\mathcal{C}(M')/(2\pi) \equiv \chi(\overline{M}')(\mathrm{mod}\, 2)$. ■

As a consequence of the monotonicity formula (Theorem 5.5), Kusner proved the following theorem:

Theorem 5.12 (Kusner [47]). *Let $X' : M' \to \mathbb{R}^3$ be a connected complete nonorientable minimal immersion with finite total curvature. Following (25), define*

$$n(M') = \sum_{i=1}^{n} \nu_i.$$

Then, for any $p \in \mathbb{R}^3$, the cardinal number of ${X'}^{-1}(p)$ is at most $n(M') - 1$.

No properly embedded surface in $\mathbb{R}^3$ is nonorientable. Hence, and as a consequence of Theorem 5.12, we have

Corollary 5.13 (Kusner [47]). *There are no complete nonorientable minimal surfaces with finite total curvature and two embedded ends.*

5.3.1. Nonorientable minimal surfaces of least total curvature.

Let $X' : M' \to \mathbb{R}^3$ be a complete nonorientable minimal surface with finite total curvature. As in the orientable case, we say that X' has critical total curvature iff $|\mathcal{C}(M')| \leq |\mathcal{C}(M'')|$, where $X'' : M'' \to \mathbb{R}^3$ is any complete nonorientable minimal surface with the same genus as M'. Looking at the formula of Jorge and Meeks (25), this means that the degree of the generalized Gauss map is the least possible among the surfaces with the same genus.

We know that $M' = \overline{M}' - \{P_1, \ldots, P_n\}$, and from Theorem 5.3 we have $\nu_i \geq 1$, $i = 1, \ldots, n$. Taking into account Remark 1 and the formula of Jorge and Meeks (25), it is not hard to deduce that $|\mathcal{C}(M')|$ is critical if and only if $\mathcal{C}(M') = -2\pi(\text{genus}(M') + 2)$ (i.e., $\text{degree}(G) = \text{genus}(M') + 2$), and so either $n = 1$ and $\nu_1 = 3$ or $n = 2$ and $\nu_1 = \nu_2 = 1$. The second case cannot occur (see Corollary 5.13), and so only the first one holds. Thus, surfaces of this kind have only one end of weight 3.

Meeks' minimal Möbius strip. We consider $M = \mathbb{C} - \{0\}$ and $I(z) = -1/\overline{z}$. Define

$$g(z) = z^2 \frac{z+1}{z-1},$$

$$\eta = i\frac{(z-1)^2}{z^4}\, dz.$$

So, the Weierstrass 1-forms are

$$\Phi_1 = \frac{i}{2}\left[\frac{(z-1)^2}{z^4} - (z+1)^2\right] dz,$$

$$\Phi_2 = -\frac{1}{2}\left[\frac{(z-1)^2}{z^4} + (z+1)^2\right] dz,$$

$$\Phi_3 = i\frac{z^2-1}{z^2}\, dz$$

which obviously satisfy (3) and (8). Furthermore, it is clear that

$$\text{Residue}(\Phi, 0) = \text{Residue}(\Phi, \infty) = 0.$$

Hence, Φ has no real periods, and so the minimal immersion $X : M \to \mathbb{R}^3$, $X = \text{Real}(\int \Phi)$ is well defined.

As $\deg(g) = 3$, then $\mathcal{C}(M) = -12\pi$.

Taking $M' = M/\langle I \rangle$, X induces a complete nonorientable minimal immersion with finite total curvature, $X' : M' \to \mathbb{R}^3$, satisfying $\mathcal{C}(M') = -6\pi$. Observe that M' is homeomorphic to $\mathbb{RP}^2 - \{p(0)\}$, which has the topological type of a Möbius strip. The surface $X'(M')$ has two symmetries. The nontrivial one is induced by $T(z) = \overline{z}$, and corresponds to a reflection about the x_2-axis, which is contained in the surface.

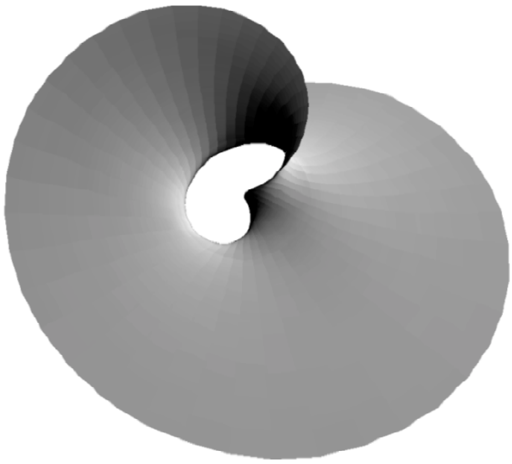

Figure 16. Meeks' minimal Möbius strip.

A minimal Klein bottle with one end. This section is devoted to construct the complete minimal Klein Bottle, which was discovered in [**51**]. This surface has four symmetries and only one end.

Let $\overline{M}_r$ be the conformal torus:

$$\overline{M}_r = \left\{ (z,u) \in (\mathbb{C} \cup \{\infty\})^2 / z^2 = \frac{u(u-r)}{ru+1} \right\}$$

and label $0 = (0,0)$, $\infty = (\infty, \infty)$, $r = (r, 0)$, $-1/r = (-1/r, \infty) \in \overline{M}_r$. Let I denote the antiholomorphic involution without fixed points defined as follows:

$$I : \overline{M}_r \longrightarrow \overline{M}_r$$
$$I(z,u) = \left(\frac{1}{\overline{z}}, \frac{-1}{\overline{u}} \right).$$

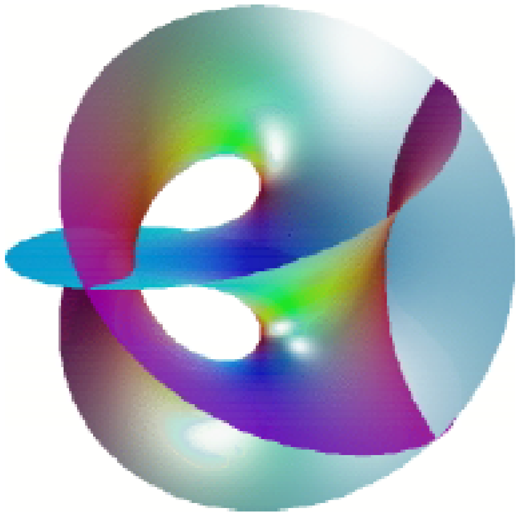

Figure 17. The minimal Klein bottle of total curvature -8π.

Define $M_r = \overline{M}_r - \{0, \infty\}$, $r \in \mathbb{R}$. M_r is a Riemann surface of genus 1, and I leaves M_r invariant. We define also the conformal mappings:

$$J, S : \overline{M}_r \longrightarrow \overline{M}_r$$

$$J(z,u) = (-z,u), \quad S(z,u) = (\overline{z}, \overline{u}).$$

Note that S has order 2 and J order k. The group generated by J and S is isomorphic to the dihedral group $\mathcal{D}(2)$ with 4 elements, and leaves M_r invariant. Moreover, this group fixes both $r, -1/r \in M_r$ and $J \circ I = I \circ J$, $S \circ I = I \circ S$. So, it can be induced, in a natural way, on the nonorientable conformal surfaces $M_r' = M_r/\langle I \rangle$. We want to define a proper conformal minimal immersion of M_r' into $\mathbb{R}^3$, for a suitable r.

First, define the following meromorphic Weierstrass data:

$$g = z\frac{u-1}{u+1} \quad \eta = i\frac{(u+1)^2}{u^2 z}\,du \tag{26}$$

on M_r.

Then $\Phi_1 = \frac{1}{2}\eta(1-g^2)$, $\Phi_2 = \frac{i}{2}\eta(1+g^2)$, $\Phi_3 = \eta g$ satisfy (3) on M_r, and (8). So, as we have said at the end of Subsection 1.1, if $X : M_r \to \mathbb{R}^3$, $X = \mathrm{Real}\int(\Phi_1, \Phi_2, \Phi_3)$ is well defined, then it induces a minimal immersion $X' : M_r' \to \mathbb{R}^3$ satisfying: $X = p \circ X'$, where $p : M_r \to M_r'$ is the natural projection.

Theorem 5.14. *There exists $r_0 \in \mathbb{R} - \{0, -1\}$ such that X is well defined.*

Proof: X is well defined if and only if Φ_j, $j = 1, 2, 3$ have no real periods, that is, $\text{Real} \int_\gamma \Phi_j = 0$, for every closed curve, γ in M_r. It is easy to check that $\text{Residue}(\Phi_j, 0) = \text{Residue}(\Phi_j, \infty) = 0$, $j = 1, 2, 3$. So, it suffices to prove $\text{Real} \int_\gamma \Phi_j = 0$ for any closed curve γ lying in $\overline{M}_r$ (not containing the ends).

On the other hand, if γ is a closed curve in $\overline{M}_r$,

$$\int_\gamma \Phi_j = \int_{I^*(\gamma)} I^*(\Phi_j) = \int_{I^*(\gamma)} \overline{\Phi}_j,$$

and so:

$$\text{Real} \left(\int_\gamma \Phi_j \right) = \frac{1}{2} \int_{\gamma + I^*(\gamma)} \Phi_j.$$

Therefore, what remains is to show that on a homology basis Γ of $\overline{M}_r$:

$$\int_{\gamma + I^*(\gamma)} \Phi_j = 0, \quad \gamma \in \Gamma.$$

A suitable homology basis of $\overline{M}_r$ may be constructed as follows. Let $c_1(t)$, $c_2(t)$ be two oriented differentiable curves in the u-plane illustrated in the Figure 18 in the case $r > 0$. We suppose $c_1(0) = (\varepsilon_1, 0)$, $\varepsilon_1 > \text{Maximum}\{0, -1/r\}$, and $c_2(0) = (\varepsilon_2, 0)$, $\varepsilon_2 > \text{Maximum}\{0, r\}$.

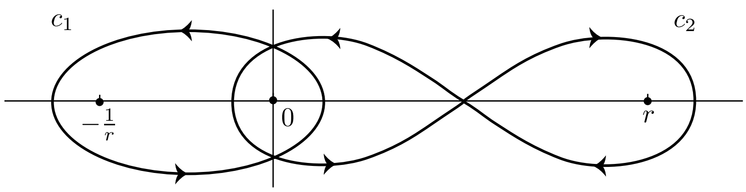

Figure 18. The curves c_1 and c_2.

The winding number of c_1 around $0, -1/r$ is 1, and 0 around r. The winding number of c_2 around 0 is 1, around r is -1 and around $-1/r$ is 0.

Let $\gamma_i(t) = (z(c_i(t)), c_i(t))$, $i = 1, 2$ be the unique lift of c_i to $\overline{M}_r$, $i = 1, 2$, satisfying $\arg(z(c_1(0))) = \frac{\pi i}{2}$, $\arg(z(c_2(0))) = 0$, respectively. The set

$$\{J^h \circ \gamma_i,\, S^j \circ \gamma_i,\, h \in \{0,1\},\, j \in \{0,1\},\, i \in \{1,2\}\}$$

contains a homology basis of $\overline{M}_r$. As $J \circ I = I \circ J$, $S \circ I = I \circ S$, then

$$\int_{J^*(\gamma_i)+I^*(J^*(\gamma_i))} \Phi_j = \int_{\gamma_i+I^*(\gamma_i)} (J)^*(\Phi_j)$$

and analogously:

$$\int_{S(\gamma_i)+I^*(S(\gamma_i))} \Phi_j = \int_{\gamma_i+I^*(\gamma_i)} (S \circ I)^*(\Phi_j)$$

for $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$. Then $J^*({}^t\Phi) = A \cdot {}^t\Phi$, $(S \circ I)^*({}^t\Phi) = B \cdot^t \Phi$, where $A, B \in \mathcal{O}(3, \mathbb{R})$ are the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, $\int_{\gamma+I^*(\gamma)} \Phi_j = 0$, for every $\gamma \in \Gamma$ if and only if

$$\int_{\gamma_i+I^*(\gamma_i)} \Phi_j = 0, \quad j \in \{1, 2, 3\},\, i \in \{1, 2\}.$$

Since $I^*(\gamma_1) = \gamma_1$ and $I^*(\gamma_2) = \gamma_1 - \gamma_2 + (J)^*(\gamma_1)$, the map X has no real periods if and only if $\int_{\gamma_1} \Phi_j = 0$, $j = 1, 2, 3$. But Φ_3 is exact, Φ_1, Φ_2 have no residues and hence above equations are equivalent to

$$\int_{\gamma_1} \eta g^2 = 0. \tag{27}$$

Remember that $\eta g^2 = i\frac{(u-1)^2(u-r)}{u(ru+1)}\frac{du}{z}$. Then take $f = \frac{(-\frac{2(2r+1)}{r}u+2)(u-r)}{z}$, and observe that:

$$-i\eta g^2 + df = 2(a_0 + a_1 u)\frac{du}{z}, \quad \text{where}$$

$$a_0 = 2r - 1, \quad a_1 = \frac{-1 - 3r}{r}.$$

Integrating by parts, $\int_{\gamma_1} \eta g^2 = i \int_{\gamma_1} 2(a_0 + a_1 u)\frac{du}{z}$, and so equation (27) is equivalent to:

$$\int_0^{-1/r} (a_0 + a_1 u)\frac{du}{z} = 0.$$

Up to the change $u = -t/r$, above integral vanishes if and only if:

$$\int_0^1 (b_0 + b_1 t)\frac{dt}{w_r(t)} = 0, \quad \text{where :}$$

$$b_0 = r^2 a_0;\ b_1 = -r a_1,\ w_r(t) = \sqrt{\frac{t(t+r^2)}{1-t}} > 0,\ t \in]0,1[.$$

Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(r) = \int_0^1 (b_0 + b_1 t)\frac{dt}{w_r}. \tag{28}$$

It is clear that $r < -1/2$ implies $b_0,\ b_1 < 0$, and $r > -1/3$ yields $b_0,\ b_1 > 0$. So f vanishes at least once on $]-1/2, -1/3[$, and never vanishes on $\mathbb{R} -]-1/2, -1/3[$. In fact, $f < 0$ on $]-\infty, -1/2[$, $f > 0$ on $]-1/3, +\infty[$. ■

For arbitrary genus greater than 2, it is still open the following conjecture:

Conjecture 2. *There are complete, nonorientable, minimal surfaces of genus g and least total curvature, for any $g > 2$.*

5.3.2. Highly symmetric nonorientable examples.

If the group of symmetries is large enough, elementary topological arguments determine, up to conformal transformations, the underlying complex structure of such a surface. Then, it is not hard to describe the Weierstrass data arising out of these kinds of examples and obtain uniqueness theorems.

Basically, two ways exist to construct new examples of highly symmetric minimal surfaces:

- In the first one the genus of $\overline{M}$ is fixed and the number of ends increases. Among these surfaces we emphasize a family of immersed projective planes with p ($p \geq 3$, p odd) embedded flat ends and total curvature $-2\pi(2p-1)$, by Kusner [**47**].

To be more precise, given p odd, $p \geq 3$, define

$$M_p = \overline{\mathbb{C}} - \left\{ z \in \mathbb{C} / z^{2p} + 2\frac{\sqrt{2p-1}}{p-1} z^p - 1 = 0 \right\}, \quad I(z) = -\frac{1}{\overline{z}},$$

$$g_p = \frac{z^{p-1}(z^p - \sqrt{2p-1})}{\sqrt{2p-1}z^p + 1},$$

$$\eta_p = \frac{i(\sqrt{2p-1}z^p + 1)^2}{(z^{2p} + 2\frac{\sqrt{2p-1}}{p-1} z^p - 1)^2}.$$

It is straightforward to check that (Φ_1, Φ_2, Φ_3) has no residues at the ends, and satisfies (8). So, the minimal minimal immersion $X_p = \text{Real}(\int \Phi)$ is well defined, and induces a minimal immersion X_p' of the nonorientable surface $M_p' = M_p / \langle I \rangle$ in $\mathbb{R}^3$. The surface $X_p'(M_p')$ contains p straight lines which lie in a plane and meet at equal angles. The dihedral group of order $2p$ acts on $X_p'(M_p')$ by reflections about these lines.

- In the second one, the number of ends are fixed and the genus of $\overline{M}$ increases. Inside these kinds of minimal surfaces we emphasize a family of complete nonorientable highly symmetric minimal surfaces with arbitrary topology and one end, constructed by the authors of this survey in [**54**], [**55**]. For each topology the authors constructed the most symmetric example. Furthermore, if the Euler characteristic of the closed associated surface is even, the examples minimize the energy (or the degree of the Gauss map) among the surfaces with their symmetry.

  The Weierstrass data are:

$$\overline{M}_{k\,m\,r} = \left\{ (z, w) \in \mathbb{C}^2 : z^k = \frac{w(w^m - r)}{r w^m + 1} \right\}$$

$$M_{k\,m\,r} = \overline{M}_{k\,m\,r} - \{(0,0), (\infty, \infty)\}$$

$$I_1 : \overline{M}_{k\,m\,r} \longrightarrow \overline{M}_{k\,m\,r}$$

$$I_1(z, w) = \left(\frac{1}{\overline{z}}, -\frac{1}{\overline{w}} \right)$$

$$g = z^{k-1} \frac{w^m - 1}{w^m + 1} \quad g\,\omega = i \frac{w^{2m} - 1}{w^{m+1}}\, dw$$

where $k \geq 2$, $m \geq 1$ and m is odd, r is suitable and $r \in \mathbb{R} - \{0, -1\}$.

When $k \geq 2$, k even, we also have:

$$\overline{M}_{k\,m\,r} = \left\{ (z,w) \in \mathbb{C}^2 : z^k = \frac{w(w^m - r)}{r w^m + 1} \right\}$$

$$M_{k\,m\,r} = \overline{M}_{k\,m\,r} - \{(0,0),(\infty,\infty)\}$$

$$I_2 : \overline{M}_{k\,m\,r} \longrightarrow \overline{M}_{k\,m\,r}$$

$$I_2(z,w) = \left(-\frac{1}{z}, -\frac{1}{w} \right)$$

$$g = z^{k-1} \frac{w^m - 1}{w(w^m+1)} \quad g\,\omega = i \frac{w^{2m} - 1}{w^{m+1}}\, dw$$

where as above $m \geq 1$, m odd, r is suitable and $r \in \mathbb{R} - \{0, -1\}$.

For each k, m the surfaces $M_{k\,m\,r}$ intersect the $x_1\ x_2$-plane in km straight lines which meet at equal angles at the origin, and the dihedral group $\mathcal{D}(k\,m)$ acts on $M_{k\,m\,r}$ by reflections about these lines.

If $k = 2$ and $m = 1$, we obtain the once punctured Klein bottle of total curvature -8π described above.

5.4. Uniqueness results for minimal surfaces of least total curvature.

Following the formula of Jorge and Meeks, there are essentially three numbers which determine the geometry of a complete orientable minimal surface with finite total curvature: the genus of the surface, the number of ends and the degree of the Gauss map. A natural way to obtain classification results is to fix some of these variables and study the arising moduli space of minimal surfaces.

The most classical results of classification are due to Osserman and Schoen.

Theorem 5.15 (Osserman [73]). *A complete minimal surface in $\mathbb{R}^3$ with finite total curvature -4π is the catenoid or Enneper's surface.*

Proof: Let $X : M \to \mathbb{R}^3$ be a complete orientable minimal surface with total curvature -4π. From Huber and Osserman theorems (Theorems 5.1 and 5.2), $M = \overline{M} - \{P_1, \dots, P_r\}$ and the Weierstrass data (g, η) extend meromorphically to $\overline{M}$. Since $\mathcal{C}(M) = -4\pi \deg(g)$, then g has degree 1, and so, it is a biholomorphism. In particular, $\overline{M} = \overline{\mathbb{C}}$. From the formula of Jorge and Meeks (see Theorem 5.3), we infer that either $r = 2$ and $\nu_1 = \nu_2 = 1$ or $r = 1$ and $\nu_1 = 3$.

- In the first case, we can put, up to a Möbius transformation, $M = \mathbb{C} - \{0\}$, $g = z$ and $\eta g = B\,dz/z$. Since ηg has no real periods, $B \in \mathbb{R}$, which corresponds to the Weierstrass representation of a catenoid.
- In the second case, we can put $M = \mathbb{C}$ and $g = z$. So, $\eta g = Bz\,dz$, which corresponds to Enneper's surface.

This concludes the proof. ■

Theorem 5.16 (Schoen [87]). *The only complete minimal surface in $\mathbb{R}^3$ with finite total curvature and two embedded ends is the catenoid.*

The proof of this theorem consists of an elegant use of Alexandrov's reflection principle. We refer to [**87**].

If we also deal with nonorientable surfaces, we have:

Theorem 5.17 (Meeks [62]). *The only complete minimal surfaces in $\mathbb{R}^3$ with total curvature greater than -8π are: the plane, the catenoid, Enneper's surface and Meeks' minimal Möbius strip.*

Proof: By Theorem 5.15, Enneper's surface and the catenoid are the only orientable surfaces with total curvature -4π.

Claim 1. *There are no complete nonorientable minimal surfaces with total curvature -2π.*

Suppose that $X' : M' \to \mathbb{R}^3$ is complete nonorientable minimal surfaces with total curvature -2π. Using Jorge-Meeks formula (25) one has:

$$1 = -\chi(\overline{M}') + \sum_{i=1}^{r} (\overline{\nu}_i + 1),$$

which implies $\chi(\overline{M}') = r = 1$ and $\overline{\nu}_1 = 1$. So, Remark 1 leads to a contradiction.

Claim 2. *There are no complete nonorientable minimal surfaces with total curvature -4π.*

We proceed once again by contradiction. If $X' : M' \to \mathbb{R}^3$ is a complete nonorientable minimal surfaces with total curvature -4π, then Jorge-Meeks formula (25) says:

$$2 = -\chi(\overline{M}') + \sum_{i=1}^{r} (\overline{\nu}_i + 1).$$

As $\mathcal{C}(M')/(2\pi) \equiv -\chi(\overline{M}')(\operatorname{mod} 2)$ (Theorem 5.10), then we deduce $\chi(\overline{M}') = 0$, $r = 1$ and $\overline{\nu}_1 = 1$, and so Remark 1 once again leads to a contradiction.

Claim 3. *The only complete minimal surface in $\mathbb{R}^3$ with total curvature -6π is Meeks' minimal Möbius strip.*

Let $X' : M' \to \mathbb{R}^3$ be a complete minimal surface in $\mathbb{R}^3$ with total curvature -6π. As $\mathcal{C}(M')$ is not a multiple of -4π, then M' is nonorientable. As in the above two claims, we use formula (25), Remark 1 and Theorem 5.10 to obtain that $\chi(\overline{M}') = 1$ and the number of ends r is either one or two.

If $r = 2$, then (25) leads to $\overline{\nu}_1 = \overline{\nu}_2 = 1$. This kind of surface does not exist by Theorem 5.12.

Hence $r = 1$, and from (25), $\overline{\nu}_1 = 3$. Label (M, g, η, I) the Weierstrass representation of X'. Then, up to Möbius transformations, $M = \mathbb{C} - \{0\}$ and $I(z) = -1/\overline{z}$. After a rigid motion in $\mathbb{R}^3$, we can assume that g has a zero at 0 and a pole at ∞. Since $\mathcal{C}(M) = -6\pi$, then $\deg(g) = 3$.

We will distinguish three cases:

Case 1: The multiplicity of g at the ends is 3.

So, $g = cz^3$ and $\eta = B\frac{dz}{z^4}$, $c, B \in \mathbb{C} - \{0\}$. Since $I^*(\eta g) = \overline{\eta g}$, then $cB \in i\mathbb{R}$. Thus, $\eta\, g$ has real periods, which is impossible.

Case 2: The multiplicity of g at the ends is 2.

In this case $g = cz^2(z - b)/(z - a)$, where $a, b, c \in \mathbb{C} - \{0\}$. After a rotation of the coordinates of M, we may assume $a \in \mathbb{R}^+$, and up to a rotation in $\mathbb{R}^3$, $c \in \mathbb{R}$. Since $g \circ I = -1/\overline{g}$, $b = -1/a$ and $c = a$. On the other hand, taking (25) and (8) into account, it is clear that $\eta = \frac{i(z-a)^2}{z^4}\, dz$. Since Φ has no real periods, it is not hard to check that $a = 1$. This corresponds to Meeks' example.

Case 3: The multiplicity of g at the ends is 1.

In this case, and as in the preceding one,

$$g = cz\frac{(z+1/a)(a+1/\overline{b})}{(z-a)(z-b)}, \quad \eta = B\frac{(z-a)^2(z-b)^2}{z^4}\, dz,$$

where $a, c \in \mathbb{R}^+$, $B \in i\mathbb{R}$. Since Φ has no real periods, then η, ηg and ηg^2 are exact. Calculating the residues, one gets Residue$(\eta, 0) = -2B(a+b)$, and so $a = -b$. Thus, Residue$(\eta g, 0) = -cB(a^2 + 1/a^2) \neq 0$, which is absurd. ■

Concerning to complete nonorientable minimal surfaces with total curvature -8π, one has the following result:

Theorem 5.18 ([51]). *The only complete nonorientable minimal surface with total curvature -8π is, up to scaling and rigid motions, the one-ended Klein bottle described in paragraph 5.3.1.*

We omit the proof.

Next, we deal with the classification of complete orientable minimal surfaces with total curvature -8π. From the formula of Jorge and Meeks, there are three topological possibilities:

- The surface has genus zero, and the number of ends n is $1, 2$ or 3. The sum of the weights of the ends is $6 - n$. In this case, the classification is merely an algebraic exercise. We refer to [**50**].
- The genus of the surface and the number of ends are 1. In this case, the weight of the end is 3. See Theorem 5.19 below.
- The genus of the surface is 1 and the number of ends are 2. In this case, both ends are embedded. From Theorem 5.16, there are no such surfaces.
- The genus of the surface is 2 and the number of ends is 1. In this case the end is embedded, which contradicts Remark 1.

Hence, we are going to restrict our interest to the genus one case, and prove the following theorem:

Theorem 5.19 ([2], [50]). *The only orientable complete minimal surface of genus one with total curvature -8π is the Chen-Gackstatter example.*

Proof: We sketch the proof of this theorem given in [**50**].

During the proof, we will use some basic results about compact Riemann surfaces. We refer to [**20**] for a good setting.

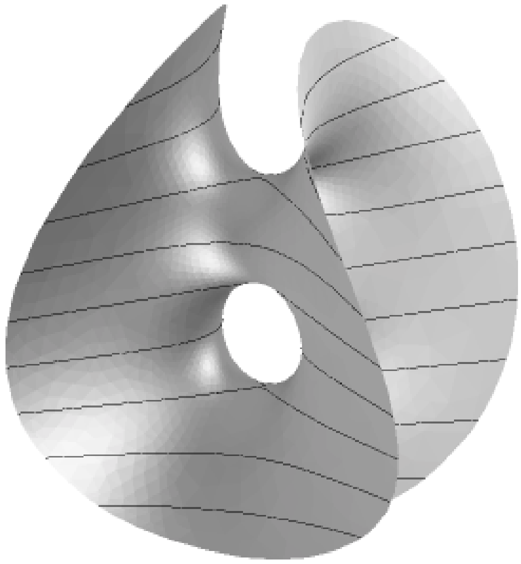

Figure 19. Chen and Gackstatter's surface of genus one.

Let $X : M \to \mathbb{R}^3$ be a conformal complete minimal immersion of a genus one surface with total curvature -8π. Following Theorem 5.2, M is conformally equivalent to a compact genus one Riemann surface punctured in a finite set of points (the ends of the surface). Moreover, if (g, η) is the Weierstrass representation of M, then the Gauss map g is a meromorphic function of degree two on M, and the 1-forms Φ_j, $j = 1, 2, 3$, defined as in (4), are meromorphic on $\overline{M}$. As we have mentioned above, the formula of Jorge and Meeks (Theorem 5.3) implies that M has only one end, and so:

$$M = \overline{M} - \{P\}.$$

After a suitable rigid motion, we assume that the normal vector at the unique end is $(0, 0, 1)$, i.e., $g(P) = \infty$.

Recall that meromorphic 1-forms and functions on a torus have the same number of zeroes and poles. Accordingly to the formula of Jorge and Meeks, the weight of the end is 3, and so the meromorphic 1-form η has either a double pole at P (if P is a regular point of g) or is holomorphic on $\overline{M}$ (if P is a ramification point of g). Furthermore, from (3) and in the first case, η has a double zero at the other pole of g and no more zeroes.

We will distinguish two cases:

1. P is a ramification point of g, and then the divisors $[g]$ and $[\eta]$ satisfy (see [**20**]): $[g] = \frac{YZ}{P^2}$, $[\eta] = 1$, where $Y, Z \in M$.
2. P is not a ramification point of g: $[g] = \frac{YZ}{PQ}$, $[\eta] = \frac{Q^2}{P^2}$.

Suppose firstly that P is a ramification point of g. Since g has degree two, the Riemann-Hurwitz formula (see [**20**]) implies that g has four ramification points: $P_1, P_2, P_3, P_4 = P$. Label $z_i = g(P_i)$, $i = 1, 2, 3, 4$. As the normal vector of X at the end is $(0, 0, 1)$, then $z_4 = \infty$. Classical theory of compact Riemann surfaces (see [**20**]) yields that $\overline{M}$ is conformally equivalent to the algebraic curve:

$$\{(z, w) \in \overline{\mathbb{C}}^2 : w^2 = (z - z_1)(z - z_2)(z - z_3)\},$$

with its natural complex structure, and up to this identification, $P = (\infty, \infty)$ and $g = z$. As we mentioned, η is holomorphic on $\overline{M}$, and thus

$$\eta = A\frac{dz}{w},$$

where $A \in \mathbb{C}^*$. On the other hand, the transformation $T : (z, w) \mapsto (z, -w)$ satisfies $T^*(\phi_j) = -\phi_j$. Therefore, viewed on $X(M)$ and up to a translation, it is the restriction of the symmetry with respect to the origin. The points P_i, $i = 1, 2, 3$ are points of M fixed by T, and so $X(P_i) = (0, 0, 0)$, $i = 1, 2, 3$, and so the origin is a triple point of the surface $x(M)$. However, the total weight of the immersion is 3, which contradicts Theorem 5.6. This proves that the first case is impossible.

Consider the second one, and assume that P is a regular point of g. As we have said above, we write Q as the other pole of g. In this case, η has a double pole at P and a double zero at Q, i.e., $[\eta] = \frac{Q^2}{P^2}$. Label θ_0 as a holomorphic nonzero 1-form on $\overline{M}$ and define $z = \eta/\theta_0$. It follows that z is a degree two meromorphic function on $\overline{M}$ satisfying:

$$[z] = \frac{Q^2}{P^2}.$$

Label Q_1 and Q_2 as the two other ramification points of z, and put $z_1 = z(Q_1)$, $z_2 = z(Q_2)$. Up to an affine Möbius transformation $z \to a_1 z + a_2$, we can suppose that $z_1 = 1$, $z_2 = -1$. Hence, writing $a = z(Q)$, and up to biholomorphisms,

$$\overline{M} = \{(z, w) \in \overline{\mathbb{C}} : w^2 = (z - a)(z^2 - 1)\},$$
$$P = (\infty, \infty) \quad M = \overline{M} - \{P\}.$$

Since $[g] = \frac{YZ}{PQ}$ and $[\eta] = \frac{Q^2}{P^2}$, Riemann-Roch Theorem implies easily that

$$g = A\frac{w}{z-a} + B, \quad \eta = C\frac{z-a}{w}\,dz,$$

where $A, B, C \in \mathbb{C}$, $A, C \neq 0$. Note that $a^2 \neq 1$: otherwise, $\overline{M}$ would be the Riemann sphere, which is absurd.

Claim 1. *The constant B is equal to zero.*

Let $\{\gamma_1, \gamma_2\}$ be a canonical homology basis of $\mathcal{H}_1(\overline{M}, \mathbb{Z})$. We can choose γ_1 and γ_2 as the closed curves given by the lifts to $\overline{M}$ of the slits $[-1, a]$ and $[a, 1]$ in the z-plane. Define $\tau_1 = \frac{z-a}{w}\,dz$, $\tau_2 = \frac{w}{z-a}\,dz$, and write:

$$f_i = \frac{1}{2}\int_{\gamma_i} \tau_1, \quad g_i = \frac{1}{2}\int_{\gamma_i} \tau_2,$$

where $i = 1, 2$.

Up to a suitable choice of the orientation of γ_i, and using standard bilinear relations (see [**20**]), we deduce

$$f_1 g_2 - f_2 g_1 = \frac{4\pi i}{3}(1 - a^2). \tag{29}$$

On the other hand, the 1-forms Φ_j have no real periods, and so:

$$BCf_i \in i\mathbb{R}, \quad Cf_i = \overline{CB}^2\,\overline{f}_i + \overline{CA}^2\,\overline{g}_i,$$

where $i = 1, 2$. If $B \neq 0$, the last equation gives

$$(1 + |B|^2) f_i = \frac{-\overline{B}A^2}{B} g_i,$$

which contradicts (29) and the fact $a^2 \neq 1$. This proves the claim.

Therefore, $g = A\frac{w}{z-a}\,dz$ and $\eta = C\frac{z-a}{w}\,dz$. Then, $\Phi_3 = AC\,dz$ is exact. Since Φ_1 and Φ_2 have no real periods along γ_j, $j = 1, 2$, then

$$f_1\overline{g}_2 - f_2\overline{g}_1 = 0. \tag{30}$$

To solve this equation, we have to carry out a careful analysis. Introduce the following notation.

Consider the algebraic curve $\{(a, y) \in \overline{\mathbb{C}}^2 : y^2 = a^2 - 1\}$, and denote by Ω the region in $a^{-1}(\overline{\mathbb{C}} - ([\infty, -1] \cup [1, \infty]))$ which contains the point $(0, i)$. Identify the points in Ω with the complex numbers in $a(\Omega)$. Let $\overline{\Omega}$ be the closure of Ω on the Riemann surface of the polynomial $y^2 = a^2 - 1$. The boundary of Ω contains two copies of the real segments $[-\infty, -1[\cup]1, +\infty]$, but $a^{-1}(\{1, -1\})$ contains only two points. We shall denote ∞_i, $i = 1, 2$, as the two points of $\partial(\Omega)$ lying in $a^{-1}(\infty)$.

Straightforward arguments imply that the functions f_i, g_i, $i = 1, 2$, are holomorphic on Ω and have continuous extensions to $\partial(\Omega)$ (taking possibly infinite values on the set $\{\infty_1, \infty_2\}$). Furthermore,

- For $i = 1, 2$,

$$f_i'(a) = \frac{a}{2(a^2-1)} f_i(a) + \frac{3}{4(1-a^2)} g_i(a)$$

$$g_i'(a) = \frac{1}{1-a^2} f_i(a) - \frac{3a}{2(1-a^2)} g_i(a).$$

- $f_1(\overline{a}) = \overline{f_1(a)}$, $g_1(\overline{a}) = \overline{g_1(a)}$, $f_2(a) = if_1(-a)$ and $g_2(a) = -ig_1(-a)$.

These analytical properties imply the following claims. The proof can be found in [**50**].

Claim 2. *The function $\frac{f_1 g_2}{f_2 g_1}$ is well defined, holomorphic and never vanishes on Ω. Moreover, it has a continuous extension to $\overline{\Omega}$, satisfying:*

$$\left|\frac{f_1 g_2}{f_2 g_1}\right|(a) \neq 0, 1, \quad \forall\, a \in \partial\Omega - \{\infty_1, \infty_2\},$$

and

$$\frac{f_1 g_2}{f_2 g_1}(\infty_i) = 1.$$

Claim 3. *If $a \in i\mathbb{R}$, then $\left|\frac{f_1 g_2}{f_2 g_1}\right|(a) = 1$. Moreover, the only solution of (30) on the imaginary axis is $a = 0$.*

Now we can conclude the proof of the theorem. Consider the function:

$$h \stackrel{\text{def}}{=} \log\left(\left|\frac{f_1 g_2}{f_2 g_1}\right|\right) : \overline{\Omega} \longrightarrow \mathbb{R}.$$

From Claim 2, h is well defined and continuous on $\overline{\Omega}$, and harmonic on Ω.

The symmetries of the functions f_i, g_i, imply that the nodal set of h; $N = \{a \in \overline{\Omega} : h(a) = 0\}$, is invariant under the transformations $a \to \overline{a}$ and $a \to -a$. Moreover, it is easy to deduce that N contains the set $i\mathbb{R} \cup \{\infty_1, \infty_2\}$ (this is part of Claim 3). On the other hand, Claim 2 gives that $N \cap (\partial(\Omega) - \{\infty_1, \infty_2\}) = \emptyset$.

Since h is nonconstant, the maximum principle for harmonic functions implies that it is not possible to have compact domains in $\overline{\Omega}$ bounded by curves in N. Thus, taking into account the above arguments, it is not hard to infer that $N = i\mathbb{R} \cup \{\infty_1, \infty_2\}$.

Since any solution of (30) lies in $N - \{\infty_1, \infty_2\}$, then $a \in i\mathbb{R}$. Therefore, Claim 3 yields $a = 0$, which corresponds to the Chen-Gackstatter genus one example. ■

The Cheng-Gackstatter genus two surface also admits the following uniqueness theorem:

Theorem 5.20 (López, Martín, Rodríguez [57]). *The Chen-Gackstatter genus two surface is the only complete minimal immersion in $\mathbb{R}^3$ of genus two, total curvature -12π and eight symmetries.*

A natural conjecture asserts that this theorem is true without any symmetry assumption. In fact, Hoffman and Meeks proposed the following:

Conjecture 3. *The moduli space of complete, orientable, minimal surfaces with genus k, $k \in \mathbb{N}$, and total curvature $-4\pi(k+1)$ is discrete (probably a unique point).*

In the nonorientable case, the corresponding conjecture asserts:

Conjecture 4. *The moduli space of complete, nonorientable, minimal surfaces with genus m, $m \in \mathbb{N}$, and total curvature $-2\pi(m+2)$ is also discrete.*

6. Properly embedded minimal surfaces

Embedded minimal surfaces are more natural; they correspond to our primitive notion of a surface, the boundary of a solid region. At the beginning of the eighties, this theory gathered new speed. This is particularly thanks to C. Costa [**15**], D. Hoffman and W. H. Meeks [**30**], [**32**] who disproved a longstanding conjecture which said that the only complete embedded minimal surfaces in $\mathbb{R}^3$ of finite topological type are the plane, the catenoid and the helicoid. This conjecture turned out to be false as there is a family of complete embedded minimal surfaces defined on a genus $k-1$ ($k>1$) compact Riemann surface with three points removed.

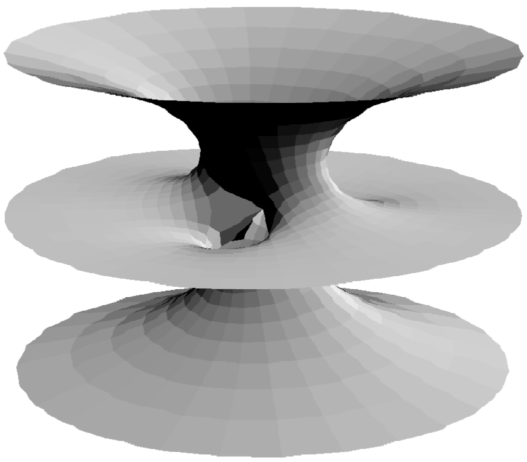

Figure 20. Costa's surface.

A natural question is to decide under what conditions finite total curvature is equivalent to finite topology. As the helicoid shows, this result is false for properly embedded minimal surfaces in $\mathbb{R}^3$ with only one end. Furthermore, Hoffman, Karcher and Wei [**28**], [**29**] have recently discovered an one-ended, genus one, properly embedded minimal surface with infinite total curvature (see also Bobenko's paper [**3**]).

Inspired in previous results by Meeks and Rosenberg [**67**], Collin proved the following theorem:

Theorem 6.1 (Collin [9]). *Let A be a properly embedded minimal annulus whose boundary is a Jordan curve. Suppose that A is contained in a half-space of $\mathbb{R}^3$ and the boundary of A lies in the boundary of the half-space. Then A has finite total curvature, and so, it is asymptotic to either a plane or a half-catenoid.*

This theorem proves a classical conjecture by Nitsche. Using the main result in [**67**], any annular end of a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end lies in a half-space. Therefore, Theorem 6.1 implies that this end has finite total curvature. As a corollary,

Corollary 6.2 (Collin [9]). *A properly embedded minimal surface in $\mathbb{R}^3$ with finite topology and more than one end has finite total curvature.*

Note that from Theorem 5.1 the converse is also true.

Embedded minimal surfaces have some special properties. If $X : M \to \mathbb{R}^3$ is an embedding then $\mathbb{R}^3 - X(M)$ consists of two connected components. Outside of a sufficiently large compact set, the ends of M are ordered from top to bottom. Therefore, up to a rotation, the normal limit vector at the ends are $(0, 0, \pm 1)$ and they alternate from one end to the next. Since the ends are embedded, they are either planar ends or catenoid ends (see Definition 2). In particular the logarithmic growth rates are also ordered: $r_1 \leq r_2 \leq \cdots \leq r_s$, $r_1 r_s < 0$. Furthermore, from (52), $\sum_{i=1}^{s} r_i = 0$. For details see [**36**], [**73**].

6.1. Examples with finite topology and more than one end.

Corollary 6.2 and Theorem 5.16 imply that the only properly embedded minimal surface in $\mathbb{R}^3$ with two ends is the catenoid. As we have mentioned above, properly embedded minimal surfaces with three or more ends have finite total curvature. Hoffman and Meeks in [**33**] (see [**27**] for a complete discussion) constructed a one-parameter family $\mathcal{F}_k$ of complete embedded minimal surfaces of genus $k - 1$ ($k > 1$) with three ends and $2k$ symmetries. These surfaces are deformations of the examples of Hoffman and Meeks in [**32**] (Costa's example for $k = 2$). A complete list of figures of these surfaces can be found in [**27**].

Today we have more families of examples for which only computational evidences of embeddedness are known. We emphasize the family of four ended examples with high topology, by Wohlgemuth [**95**], and the Weber-Wolf family [**93**].

It is also remarkable Kapouleas' work [**38**]. His method of construction amounts to desingularizing the circles of intersection of a collection of coaxial catenoids and planes. The desingularization process uses Scherk's singly periodic surfaces for an approximate construction which is subsequently corrected by singular perturbation methods. So, this author shows complete embedded minimal surfaces with arbitrarily many (at least three) ends. The examples are highly symmetric, and the genus takes arbitrarily high values.

6.1.1. Properly embedded minimal surfaces with three ends: Costa-Hoffman-Meeks and Hoffman-Meeks families.

In this subsection we deal with the period problem associated to the Hoffman-Meeks family of embedded minimal surfaces with three ends. This matter has been studied in depth by Hoffman and Karcher in [**27**]. Furthermore, in that work these authors have been able to give a uniqueness theorem in terms of the symmetry (see Theorem 6.9).

The exposition of the period problem given here is different from the one in [**27**].

Let $\overline{M}_{k\,a}$, $k \in \mathbb{N}$, $k \geq 2$, $a \in \mathbb{R} - \{0, -1\}$, be the compact Riemann surface

$$\overline{M}_{k\,a} = \left\{ (z, w) \in (\mathbb{C} \cup \{\infty\})^2 : w^k = \frac{(z+1)(z-a)}{z} \right\}.$$

Let $P_0 = (0, \infty)$, $P_1 = (-1, 0)$, $P_2 = (\infty, \infty)$ and $P_3 = (a, 0)$, and define

$$M_{k\,a} = \overline{M}_{k\,a} - \{P_1, P_2, P_3\}.$$

Consider the conformal mappings of $\overline{M}_{k\,a}$

$$J(z, w) = (z, \theta w), \quad \theta = e^{\frac{2\pi i}{k}}$$

$$S(z, w) = (\overline{z}, \overline{w}).$$

The group generated by J and S is the dihedral group $\mathcal{D}(k)$ with $2k$ elements, it leaves $M_{k\,a}$ invariant and fixes P_i, $i = 0, 1, 2, 3$.

It will be useful to construct a homology basis of $\overline{M}_{k\,a}$. We distinguish two cases:

- Suppose $a > 0$. Let $\beta_i(t)$, $i = 1, 2$, be the oriented simple closed curves in the z-plane illustrated in Figure 21. We assume that $\beta_1(0) \in \mathbb{R}$, $\beta_1(0) > a$, $\beta_2(0) \in \mathbb{R}$, $0 < \beta_2(0) < a$. Let $b_i(t)$ be the unique lift of $\beta_i(t)$ to $\overline{M}_{k\,a}$, $i = 1, 2$, satisfying $w(b_1(0)) \in \mathbb{R}_+$ and $\mathrm{Arg}(w(b_2(0))) = \frac{\pi}{k}$.
- Suppose $a < 0$. Write $a_0 = \mathrm{Minimum}\{a, -1\}$, $a_1 = \mathrm{Maximum}\{a, -1\}$ and let $\beta_i(t)$, $i = 1, 2$, the oriented closed curves in the z-plane illustrated in the Figure 22. We assume that $\beta_1(0) \in \mathbb{R}$, $\beta_1(0) > 0$, $\beta_2(0) \in \mathbb{R}$, $a_1 < \beta_2(0) < 0$. Let $b_i(t)$ be the unique lift of $\beta_i(t)$ to $\overline{M}_{k\,a}$, $i = 1, 2$, satisfying $w(b_1(0)) \in \mathbb{R}_+$, $\mathrm{Arg}(w(b_2(0))) = \frac{\pi}{k}$.

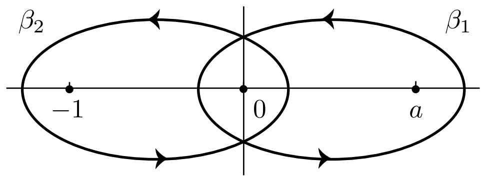

Figure 21. β_1 and β_2 for $a > 0$.

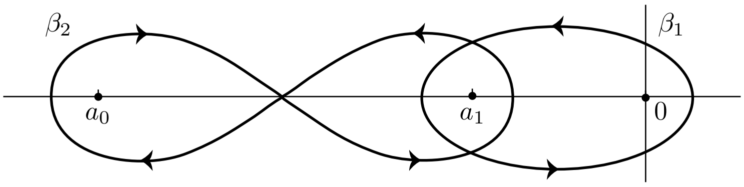

Figure 22. β_1 and β_2 for $a < 0$.

In the following, we identify d and its homology class $[d]$, for any closed curve d in $\overline{M}_{k\,a}$. The desired homology basis of $\overline{M}_{k\,a}$ is:

$$\mathcal{B} = \left\{(J^h)_*(b_i) : h \in \{0, \ldots, k-2\},\, i \in \{1, 2\}\right\}.$$

We leave the topological details to the reader.

Consider the following meromorphic data $M_{k\,a}$:

$$g_m = A\frac{zw}{mz+1} \qquad \eta_m g_m = B\frac{mz+1}{(z+1)(z-a)}\,dz$$

where $m \in \mathbb{R}$, $A \in \mathbb{R} - \{0\}$, $B \in \mathbb{C}$, $|B| = 1$. Define as in (4)

$$2\Phi_1^m = (1 - g_m^2)\eta_m,$$
$$2\Phi_2^m = i(1 + g_m^2)\eta_m,$$
$$\Phi_3^m = g_m \eta_m.$$

Let

$$X_m(P) = \text{Real}\int_{P_0}^{P} \left(\Phi_1^m, \Phi_2^m, \Phi_3^m\right), \quad P \in M_{k\,a}.$$

In general X_m is a multivalued conformal minimal immersion.

It is interesting to translate the period problem associated to X_m to a simpler language. First observe that Φ_3^m has no real periods if and only if $r_i(a,m) = \text{Residue}(\Phi_3^m, P_i) \in \mathbb{R}$, $i = 1,2,3$. An easy computation gives

$$r_1(a,m) = Bk\frac{m-1}{1+a},$$
$$r_2(a,m) = -Bkm, \tag{31}$$
$$r_3(a,m) = Bk\frac{ma+1}{1+a}$$

and therefore $\text{Real}(\int \Phi_3^m)$ is well defined if and only if $B \in \mathbb{R}$, that is, $B \in \{-1,1\}$.

In what follows and up to rigid motions we will assume that $B = 1$.

It is clear that the residues of Φ_1^m and Φ_2^m vanish.

If we put $\Phi^m = \begin{pmatrix} \Phi_1^m \\ \Phi_2^m \\ \Phi_3^m \end{pmatrix}$, then $J^*(\Phi^m) = \mathcal{R} \cdot \Phi^m$ where $\mathcal{R} \in \mathcal{O}(3)$ is the matrix

$$\mathcal{R} = \begin{pmatrix} \cos(\frac{2\pi}{k}) & -\sin(\frac{2\pi}{k}) & 0 \\ \sin(\frac{2\pi}{k}) & \cos(\frac{2\pi}{k}) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking into account that $\mathcal{B}$ is a homology basis of $\overline{M}_{k\,a}$ and the last equality, we deduce that X_m has no real periods if and only if

$$\text{Real}\left(\int_{b_i} \Phi_j^m\right) = 0, \quad i,j = 1,2. \tag{32}$$

Let τ_1, τ_2, τ_3 be the following 1-forms on $\overline{M}_{k\,a}$

$$\tau_1 = \frac{dz}{w},$$
$$\tau_2 = \frac{zdz}{w},$$
$$\tau_3 = \frac{dz}{w^{k-1}}$$

and observe that

$$\eta_m g_m^2 = A\tau_3$$

$$\eta_m = \frac{1}{A}\left(q_0\tau_1 + q_1\tau_2 - d\Big(\frac{c_0 + c_1 z + c_2 z^2}{w}\Big)\right)$$

where

$$q_0 = \frac{(k+1)(1-a)}{2(2k-1)} q_1 + \frac{(k-1)m}{a} \quad , q_1 = \frac{2(2k-1)(m-1)(1+am)}{a(1+a)^2}$$

$$c_0 = \frac{k}{a},$$
$$c_1 = \frac{k(-1+a+2m+2a^2m+am^2-a^2m^2)}{a(1+a)^2},$$
$$c_2 = \frac{2k(m-1)(1+am)}{a(1+a)^2}.$$

Define on $\mathbb{R} - \{0, -1\}$ the following functions

$$f_1(a) = \frac{1}{\theta}\int_{b_1} \tau_1, \quad g_1(a) = \frac{1}{\theta}\int_{b_1} \tau_2, \qquad h_1(a) = \frac{1}{\theta}\int_{b_1} \tau_3$$
$$f_2(a) = \frac{1}{\overline{\xi}}\int_{b_2} \tau_1, \quad g_2(a) = -\frac{1}{\overline{\xi}}\int_{b_2} \tau_2, \quad h_2(a) = \frac{1}{\xi}\int_{b_2} \tau_3$$

where $\theta = e^{\frac{\pi i}{k}} - e^{-\frac{\pi i}{k}}$ and $\xi = e^{\frac{2\pi i}{k}} - 1$.

Deforming the curves b_i, $i = 1, 2$ on the real axis, an analytic continuation argument gives:

- for $a > 0$

$$f_1(a) = \int_0^a \frac{d\,z}{|w|}, \quad g_1(a) = \int_0^a \frac{z d\,z}{|w|}, \quad h_1(a) = \int_0^a \frac{d\,z}{|w|^{k-1}}$$
$$f_2(a) = \int_{-1}^0 \frac{d\,z}{|w|}, \quad g_2(a) = \int_{-1}^0 \frac{z d\,z}{|w|}, \quad h_2(a) = \int_{-1}^0 \frac{d\,z}{|w|^{k-1}}$$

- for $a < 0$

$$f_1(a) = -\int_{a_1}^0 \frac{d\,z}{|w|}, \quad g_1(a) = -\int_{a_1}^0 \frac{z d\,z}{|w|}, \quad h_1(a) = -\int_{a_1}^0 \frac{d\,z}{|w|^{k-1}}$$
$$f_2(a) = -\int_{a_0}^{a_1} \frac{d\,z}{|w|}, \quad g_2(a) = \int_{a_0}^{a_1} \frac{z d\,z}{|w|}, \quad h_2(a) = -\int_{a_0}^{a_1} \frac{d\,z}{|w|^{k-1}}.$$

Notice that $f_i(a)$, $g_i(a)$, $h_i(a) \in \mathbb{R}_+ \ \forall\, a > 0$, $i \in \{1, 2\}$, and $f_i(a)$, $h_i(a) \in \mathbb{R}_- \ \forall\, a < 0$, $i \in \{1, 2\}$, $g_1(a) \in \mathbb{R}_+$, $g_2(a) \in \mathbb{R}_-$, $\forall\, a < 0$.

From above definitions (32) becomes

$$(-1)^{i+1}q_0 f_i + q_1 g_i = -A^2 h_i \quad i = 1, 2. \tag{33}$$

Call $\Omega = \{(a,m) \in \mathbb{R}^2 : a \neq 0, -1\}$, $\Omega^+ = \{(a,m) \in \Omega : a > 0\}$ and $\Omega^- = \{(a,m) \in \Omega : a < 0\}$.

Definition 3. Define by $\mathcal{C}$ the set of points of Ω for which there exists $A \in \mathbb{R} - \{0\}$ such that (33) holds (i.e. X_m has no real periods). Label $\mathcal{C}^+ = \mathcal{C} \cap \Omega^+$ and $\mathcal{C}^- = \mathcal{C} \cap \Omega^-$.

Remark 3. For any point $(a,m) \in \mathcal{C}$, the conformal transformations J and S yield on X_m a rotation around the x_3-axis by angle $\frac{2\pi}{k}$ and a symmetry with respect to the plane $x_2 = 0$, respectively.

Let $\pi_1(a,m) = a$, $\pi_2(a,m) = m$ the two natural projections from the (a,m)-plane into the a and m-axis, respectively.

Theorem 6.3 (Hoffman, Karcher [27]). *The set $\mathcal{C}^+$ is a regular curve in Ω^+ and*

$$\pi_{1|\mathcal{C}^+} : \mathcal{C}^+ \longrightarrow]0, +\infty[$$

is a diffeomorphism.

Proof: Let $\Lambda = \{(a,m) \in \Omega^+ : (m-1)(ma+1) < 0\}$.

Define $\varphi : \Omega^+ \longrightarrow \mathbb{R}$ by

$$\varphi = h_2(q_0 f_1 + q_1 g_1) + h_1(q_0 f_2 - q_1 g_2).$$

First, we observe that $\mathcal{C}^+ = \{(a,m) \in \Lambda : \varphi(a,m) = 0\}$. If $\varphi(a,m) = 0$ and $(a,m) \in \Lambda$, then there exists $\lambda \in \mathbb{R}$ such that $\left((-1)^{i+1}q_0 f_i + q_1 g_i\right)(a,m) = \lambda h_i(a,m)$ $i = 1, 2$. Hence $(q_1(g_1 f_2 + g_2 f_1))(a,m) = \lambda(h_1 f_2 + h_2 f_1)(a,m)$, and then $\lambda q_1(a,m) > 0$. As $q_1 < 0$ on Λ we deduce that $\lambda < 0$ and taking $A = \sqrt{-\lambda} \in \mathbb{R}$, (33) holds. Conversely, if $(a,m) \in \mathcal{C}^+$ then (33) implies: $\varphi(a,m) = 0$ and $(q_1(g_1 f_2 + g_2 f_1))(a,m) = -A^2(h_1 f_2 + h_2 f_1)(a,m)$. In particular $q_1(a,m) < 0$ and thus $(a,m) \in \Lambda$.

For each $a \in]0, +\infty[$, we label $\ell_a = (\pi_1)^{-1}(a)$.

We want to show that ℓ_a meets $\mathcal{C}^+$ in a single point. It is clear that ℓ_a intersects the boundary of Λ in two points: $(a, -\frac{1}{a})$ and $(a, 1)$. On the other hand we have

$$\varphi(a,1) = \frac{k-1}{a}(h_2(a) f_1(a) + h_1(a) f_2(a)) > 0$$

$$\varphi\left(a, -\frac{1}{a}\right) = -\frac{k-1}{a^2}(h_2(a) f_1(a) + h_1(a) f_2(a)) < 0.$$

By an intermediate value argument the function φ vanishes at a point of $\ell_a \cap \Lambda$. Furthermore, since $\varphi_{|\ell_a}$ is a polynomial function of m of degree less than or equal to two it has only one root $m(a) \in [-\frac{1}{a}, 1]$ (counting multiplicities).

Hence $\mathcal{C}^+ = \{(a, m(a)) : a \in]0, \infty[\}$ is a graph on the positive a-axis and it is not hard to check that the function $a \longmapsto m(a)$ is continuous on $]0, +\infty[$. We are going to prove that in fact it is differentiable. To see this we note firstly that $\mathcal{C}^+$ does not meet the zero set $\mathcal{S}$ of $q_0 f_1 + q_1 g_1$: if $(a, m) \in \mathcal{C}^+$ and $(q_0 f_1 + q_1 g_1)(a, m) = 0$ then (33) gives $h_1(a) = 0$, a contradiction. Therefore defining $\rho : \Omega^+ - \mathcal{S} \longrightarrow \mathbb{R}$ by

$$\rho = \frac{q_0 f_2 - q_1 g_2}{q_0 f_1 + q_1 g_1} + \frac{h_2}{h_1}$$

we have that $\mathcal{C}^+ = \{(a, m) \in \Lambda - \mathcal{S} : \rho(a, m) = 0\}$. On the other hand on $\Lambda - \mathcal{S}$

$$\frac{\partial \rho}{\partial m} = -\frac{2(k-1)(2k-1)(1+am^2)(f_2 g_1 + f_1 g_2)}{(a(a+1)(q_0 f_1 + q_1 g_1))^2} < 0.$$

Hence applying the implicit function theorem the function $a \longmapsto m(a)$ is differentiable and so $\mathcal{C}^+$ is a regular curve in Ω^+ which projects homeomorphically on the positive a-axis. ∎

Define $\Sigma = \{(a, m) \in \Omega^+ : ((2 + a)m - 1)((2a + 1)m + 1) < 0\}$.

Remark 4. The involutive automorphism $\mathcal{I}$ on Ω defined by $\mathcal{I}(a,m) = (\frac{1}{a}, -am)$ leaves Λ, Σ, $\mathcal{C}^-$ and $\mathcal{C}^+$ invariant. Furthermore if $(a, m) \in \mathcal{C}$ then the surfaces associated to (a, m) and $(1/a, -am)$ are, up to change of variables, scaling and rigid motions, the same.

An important fact is that, for any $(a, m(a)) \in \mathcal{C}^+$,

$$r_1(a, m(a)) < r_2(a, m(a)) < r_3(a, m(a)). \tag{34}$$

These inequalities easily follow from the theorem

Theorem 6.4 (Hoffman, Karcher [27]). *The set $\mathcal{C}^+$ is contained in Σ.*

Proof: To prove that $\mathcal{C}^+ \subset \Sigma$ we need to work harder than in Theorem 6.3.

The boundary of Σ has two connected components $\mu_1 = \{(a, \frac{1}{2+a}) : a > 0\}$ and $\mu_2 = \{(a, -\frac{1}{2a+1}) : a > 0\}$.

We want to prove that $\varphi_{|\mu_1} > 0$ and $\varphi_{|\mu_2} < 0$, where φ was defined in the proof of Theorem 6.3. Notice that from (ii) in Lemma 6.5 above inequalities are equivalent to $\psi_{|\mu_1} > 0$ and $\psi_{|\mu_2} < 0$, where $\psi = \varphi f_1/h_1$.

Firstly, we notice that $\mu_1 \cap \mathcal{S} = \emptyset$, where $\mathcal{S}$ is the zero set of $q_0 f_1 + q_1 f_1$, as in Theorem 6.3. For, we define

$$h(a) = (q_0 f_1 + q_1 f_1)\left(a, \frac{1}{2+a}\right) = \frac{(3ak + a - 4)f_1(a) + 4(1-2k)g_1(a)}{a(2+a)^2}.$$

Using (i) in Lemma 6.5 and substituting it is easy to see that

$$(35)\ h(a) = 0 \Longrightarrow h'(a) = \frac{(1-k)(16 + 8a + a^2 + 8ak + 3a^2k)f_1(a)}{4a^2(1+a)(2+a)^2k} < 0.$$

On the other hand, for $a > 0$ it is clear that

$$f_1(a) = \int_0^a \frac{dz}{|w|} = a\int_0^1 \sqrt[k]{\frac{t}{(1+at)(1-t)}}\, dt$$

$$g_1(a) = \int_0^a \frac{z\, dz}{|w|} = a^2\int_0^1 t\sqrt[k]{\frac{t}{(1+at)(1-t)}}\, dt.$$

Hence one obtains

$$(36)\quad \lim_{a\to 0^+} \frac{f_1(a)}{a} = \frac{\pi}{k\sin(\pi/k)}, \qquad \lim_{a\to+\infty} f_1(a)a^{1/k-1} = \frac{k}{k-1}$$

$$(37)\quad \lim_{a\to 0^+} \frac{g_1(a)}{a^2} = \frac{(k+1)\pi}{2k^2\sin(\pi/k)}, \quad \lim_{a\to+\infty} g_1(a)a^{1/k-2} = \frac{k^2}{(k-1)(2k-1)}.$$

From (36) and (37) one has $\lim_{a\to 0^+} h(a) = -\dfrac{\pi}{k\sin(\pi/k)}$. If h vanishes in $]0,+\infty[$ and a_0 is the lowest root of h then the above limit says us that $h'(a_0) \geq 0$, which is contrary to (35). Thus $h(a) < 0\ \forall\, a \in]0,+\infty[$ and so $\mu_1 \cap \mathcal{S} = \emptyset$.

Now we observe that $\rho_{|\mu_1}$ is negative, where ρ was defined in the proof of Theorem 6.3. From (ii) in Lemma 6.5 it is obvious that

$$\rho = \frac{f_2}{f_1} + \frac{q_0 f_2 - q_1 g_2}{q_0 f_1 + q_1 g_1}$$

and so

$$\rho\left(a, \frac{1}{2+a}\right) = \frac{f_2(a)}{f_1(a)} + \frac{(4 - a - 3ak)f_2(a) + 4(1-2k)g_2(a)}{(4 - a - 3ak)f_1(a) + 4(2k-1)g_1(a)}.$$

In what follows we write $u(a) = \rho\left(a, \frac{1}{2+a}\right)$. From (i) in Lemma 6.5 once again, we obtain

$$u'(a) = \frac{(2k-1)(f_1 g_2 + f_2 g_1)}{k a^3 (2+a)^4 (a+1) h^2} \left((k-1)((4+a)^2 + ak(8+3a)) - \left(\frac{a(2+a)^2\, h}{f_1}\right)^2 \right).$$

Note that $u'(a) > 0$. To see this we define $v :]0, +\infty[\longrightarrow \mathbb{R}$ by

$$\begin{aligned} v(a) &= \sqrt{(k-1)((4+a)^2 + ak(8+3a))} + a(2+a)^2 \frac{h(a)}{f_1(a)} \\ &= \sqrt{(k-1)((4+a)^2 + ak(8+3a))} + 3ak + a - 4 + 4(1-2k)\frac{g_1(a)}{f_1(a)}. \end{aligned}$$

It is clear that $u'(a)/v(a) > 0$. Taking into account the assertion (i) of Lemma 6.5

$$\frac{d}{da}\left(\frac{g_1(a)}{f_1(a)}\right) = \frac{k+1}{k(1+a)} + \frac{2a-1}{a(1+a)}\left(\frac{g_1(a)}{f_1(a)}\right) - \frac{2k-1}{a(1+a)k}\left(\frac{g_1(a)}{f_1(a)}\right)^2. \tag{38}$$

Using this equation and substituting, we deduce that $v(a) = 0$ implies

$$v'(a) = \frac{(k-1)[(a-4)(a+4)^2 + 4(8-6a+a^3)k + 3a(8+4a+a^2)k^2]}{2a(1+a)k\sqrt{(k-1)((4+a)^2 + ak(8+3a))}} > 0. \tag{39}$$

On the other hand, from (36) and (37) we obtain that $\lim_{a \to 0^+} v(a) = 4(\sqrt{k-1} - 1) > 0$. A similar discussion for the function h gives $v(a) > 0$ $\forall\, a \in \mathbb{R}_+$, and so $u'(a) > 0$, $\forall\, a \in \mathbb{R}_+$.

From (36), (37) and the following formulae:

$$f_1\left(\frac{1}{a}\right) = a^{\frac{1-k}{k}} f_2(a), \quad f_2\left(\frac{1}{a}\right) = a^{\frac{1-k}{k}} f_1(a),$$

$$g_1\left(\frac{1}{a}\right) = a^{\frac{1-2k}{k}} g_2(a), \quad g_2\left(\frac{1}{a}\right) = a^{\frac{1-2k}{k}} g_1(a)$$

we obtain $\lim_{a\to+\infty} a\cdot u(a) = -\dfrac{2(k+1)\pi}{k^2\sin(\pi/k)}$. Hence $u(a)$ is increasing and so $u(a)<0$ for all $a>0$, i.e., $\rho_{|\mu_1}<0$.

As $h(a)<0\ \forall\, a>0$ we obtain that $\psi_{|\mu_1}>0$.

To estimate the sign of $\psi_{|\mu_2}$ observe that the above formulae lead to:

$$(q_0f_1+q_1g_1)\circ\mathcal{I} = -a^{\frac{2k+1}{k}}\cdot(q_0f_2-q_1g_2),$$
$$(q_0f_2-q_1g_2)\circ\mathcal{I} = -a^{\frac{2k+1}{k}}\cdot(q_0f_1+q_1g_1)$$

and thus one gets

$$\psi\circ\mathcal{I} = -a^{\frac{2+k}{k}}\psi.$$

Therefore, using that $\mathcal{I}(\mu_1)=\mu_2$ we deduce $\psi_{|\mu_2}<0$. An intermediate value argument yields $\mathcal{C}^+\subset\Sigma$. ■

Remark 5. The minimal surfaces $X_{m(a)}$, $(a,m(a))\in\mathcal{C}^+$, form the curve of embedded examples with three ends shown by Hoffman and Meeks in [**33**]. The surface X_0 associated to the point $(1,0)$ is just Hoffman-Meeks genus $k-1$ example in [**32**], and $X_{m(a)}$ provides a smooth deformation of this surface. Furthermore since (34) holds at the point $(1,0)$ and the logarithmic growth rates are continuous functions of a, Theorem 6.4 implies that the inequalities (34) hold at any point of $\mathcal{C}^+$. This fact yields a good control of the logarithmic growth rates of the ends along the deformation and together with the embeddedness of the Hoffman-Meeks surface, they have a strong influence on the proof of the embeddedness of this family of surfaces. For details see [**32**], [**27**] and [**33**].

If $k=2$ and $a=1$ ($m(1)=0$) we obtain Costa's example. For $k=2$ and $a\in]0,1[$ we get the Hoffman-Meeks deformation of Costa surface, [**15**].

6.1.2. Some analytical nonexistence and uniqueness theorems.

In Paragraph 6.1.3 we will obtain a classification theorem for the family $\mathcal{F}_k$ in terms of their symmetries, which generalizes that by Hoffman and Karcher (Theorem 6.9). To do this, we will need some analytical results of nonexistence and uniqueness.

Along this subsection we assume $k>2$. The case $k=2$ has been treated extensively by Costa in [**13**]. We follow the notation established in Paragraph 6.1.1. We start with the following technical lemma:

Lemma 6.5. *The functions f_i, g_i, h_i, $i = 1, 2$, satisfy:*

(i) *For $a \in \mathbb{R} - \{0, -1\}$ and $i \in \{1, 2\}$*

$$f_i'(a) = \frac{k-a}{ka(a+1)} f_i(a) + (-1)^{i+1} \frac{2k-1}{ka(a+1)} g_i(a)$$

$$g_i'(a) = (-1)^{i+1} \frac{k+1}{k(a+1)} f_i(a) + \frac{2k-1}{k(a+1)} g_i(a).$$

(ii) *For $a > 0$*

$$(f_1 h_2 - f_2 h_1)(a) = 0.$$

Proof: Taking $w = w(a)$ into account, one formally gets:

$$\frac{\partial \tau_1}{\partial a} = \frac{k-a}{ka(a+1)} \tau_1 + \frac{2k-1}{ka(a+1)} \tau_2 + d\, v$$

$$\frac{\partial \tau_2}{\partial a} = \frac{k+1}{k(a+1)} \tau_1 + \frac{2k-1}{k(a+1)} \tau_2 + d\,(a\, v)$$

where $v = -\frac{z(z+1)}{a(a+1)w}$. Hence, using the definitions of f_i, g_i, $i = 1, 2$, (ii) holds.

To obtain (iii) we need to compute the intersection matrix of the homology basis $\mathcal{B}$. Given $c_1, c_2 \in \mathcal{H}_1(\overline{M}_{k\,a}, \mathbb{Z})$ we label $c_1 \cdot c_2$ as the intersection number of c_1 and c_2 (see [**20**]). If we write $d_i = (J^{i-1})_*(b_1)$, $e_i = (J^{i-1})_*(b_2)$, $i = 1, \ldots, k-1$, it is not hard to check that for $j, h \in \{1, \ldots, k-1\}$:

$$d_j \cdot d_h = 0, \quad e_j \cdot e_h = 0$$

$$d_j \cdot e_h = \begin{cases} 0 & j - h < 0 \\ 0 & j - h > 1 \\ -1 & j = h \\ 1 & j = h + 1 \end{cases}$$

$$e_h \cdot d_j = -d_j \cdot e_h.$$

Hence arranging the basis $\mathcal{B}$ as follows

$$(d_1, \ldots, d_{k-1}, e_1, \ldots, e_{k-1})$$

the intersection matrix $D_{\mathcal{B}}$ is given by

$$D_{\mathcal{B}} = \left(\begin{array}{c|c} 0 & G \\ \hline -{}^tG & 0 \end{array} \right)$$

where $G = (g_{j,h})_{1\leq i,\, j\leq k-1}$ is the matrix defined by:

$$g_{j,h} = \begin{cases} 0 & j-h<0 \\ 0 & j-h>1 \\ -1 & j=h \\ 1 & j=h+1. \end{cases}$$

Classical bilinear relations of Riemann applied to the 1-forms τ_1 and τ_3 say:

$$-\vec{v}_1 \cdot D_{\mathcal{B}}^{-1} \cdot^t \vec{v}_3 = 2\pi i \,\mathrm{Residue}(f\tau_3, P_2)$$

where

$$\vec{v}_1 = \left(\int_{d_1}\tau_1, \dots, \int_{d_{k-1}}\tau_1, \int_{e_1}\tau_1, \dots, \int_{e_{k-1}}\tau_1\right)$$

$$\vec{v}_3 = \left(\int_{d_1}\tau_3, \dots, \int_{d_{k-1}}\tau_3, \int_{e_1}\tau_3, \dots, \int_{e_{k-1}}\tau_3\right)$$

$$D_{\mathcal{B}}^{-1} = \left(\begin{array}{c|c} 0 & -{}^tG^{-1} \\ \hline G^{-1} & 0 \end{array}\right)$$

and $\tau_1 = df$ locally around P_2. It is easy to see that $k>2$ implies $\mathrm{Residue}(f\tau_3, P_2) = 0$. Taking that $J^*(\tau_1) = e^{-\frac{2\pi i}{k}}\tau_1$, $J^*(\tau_3) = e^{\frac{2\pi i}{k}}\tau_3$ and the definitions of f_i, h_i, $i=1,2$, d_j, e_j, $j=1,\dots,k-1$ into account, (iii) holds. ■

The Hoffman-Meeks surface is the only element in $\mathcal{C}^+$ with a flat end. This is a consequence of the following

Theorem 6.6. *The function $m(a)$ vanishes only at the point $a=1$, i.e. $\mathcal{C}^+ \cap \pi_2^{-1}(0) = \{(1,0)\}$.*

Proof: Since $\mathcal{I}(\mathcal{C}^+) = \mathcal{C}^+$, it suffices to prove that $m(a) \neq 0\ \forall\, a \in]0,1[$. Define $f:]0,1[\to \mathbb{R}$ by $f(a) = \rho(a,0)$ where ρ was defined in the proof of the Theorem 6.3. From (ii) in Lemma 6.5 we obtain that

$$f(a) = \frac{(1-a)(k+1)f_2 + 2(1-2k)g_2}{(1-a)(k+1)f_1 + 2(2k-1)g_1} + \frac{f_2}{f_1}.$$

Using the assertion (i) of Lemma 6.5, one has

$$f'(a) = \frac{(2k-1)(f_1g_2+f_2g_1)[(1+a)^2(k^2-1)f_1^2 - ((1-a)(k+1)f_1 + 2(2k-1)g_1)^2]}{a(1+a)kf_1^2((1-a)(k+1)f_1 + 2(2k-1)g_1)^2}.$$

If we define $y(a) = (1+a)\sqrt{k^2-1} - (1-a)(k+1) - 2(2k-1)\frac{g_1}{f_1}$ it is clear that $f'(a)/y(a) > 0$. Using (38) and substituting one obtains that

$$y(a) = 0 \Rightarrow y'(a) = \frac{(a-1)\sqrt{k^2-1}}{ak} < 0. \tag{40}$$

From (36) and (37) one has $\lim_{a\to 0} y(a) = \sqrt{k^2-1} - k - 1 < 0$. If y vanishes in $]0,1[$ and a_0 is the lowest root of y in this interval, then the above limit says us that $y'(a) \geq 0$, which is contrary to (40). Thus $y(a) < 0$ $\forall\, a \in]0,1[$ and hence $f'(a) < 0$ in $]0,1[$. As $m(1) = 0$ then $f(1) = 0$ and so $f(a) > 0$ $\forall\, a \in]0,1[$. ■

Theorem 6.7. *The set of points in $\mathcal{C}^-$ providing embedded minimal surfaces is void.*

Proof: Take (a,m) a point of $\mathcal{C}^-$. From Remark 4 and without loss of generality we can assume that $a < -1$.

The equation (33) implies

$$q_1(a,m)(g_1(a)f_2(a) + f_1(a)g_2(a)) = -A^2(h_1(a)f_2(a) + f_1(a)h_2(a)).$$

As $A \in \mathbb{R}$ then $q_1(a,m)(g_1(a)f_2(a) + f_1(a)g_2(a)) < 0$.

We want to see that $g_1(a)f_2(a) + f_1(a)g_2(a)$ is positive. Applying the assertion (i) of Lemma 6.5, it is clear that

$$\frac{d}{da}\left(f_2 g_1 + f_1 g_2\right) = \frac{2ka + k - 2a}{a(1+a)k}\left(f_2 g_1 + f_1 g_2\right).$$

Integrating the above ordinary differential equation, we obtain that

$$f_2 g_1 + f_1 g_2 = K \cdot |a||a+1|^{\frac{k-2}{k}}, \quad \forall\, a \in \mathbb{R} - \{0,-1\}$$

where K is constant on each connected component of $\mathbb{R} - \{0,-1\}$.

We are going to find the value of K in the interval $]-\infty,-1[$. Observe that

$$K = \lim_{a\to-\infty} \frac{f_2 g_1 + f_1 g_2}{|a||a+1|^{\frac{k-2}{k}}}.$$

From the definitions of f_i, g_i, $i = 1, 2$, for $a < -1$ we have:

$$f_1(a) = -\int_{-1}^{0} \frac{dz}{|w|} = -\int_0^1 \sqrt[k]{\frac{t}{(1-t)(-a-t)}}\, dt$$

$$g_1(a) = -\int_{-1}^{0} \frac{z\,dz}{|w|} = \int_0^1 t\sqrt[k]{\frac{t}{(1-t)(-a-t)}}\, dt$$

$$f_2(a) = -\int_{a}^{-1} \frac{dz}{|w|} = -|a+1|^{\frac{k-2}{k}} \int_0^1 \sqrt[k]{\frac{(1+a)t-a}{t(1-t)}}\, dt$$

$$g_2(a) = \int_{a}^{-1} \frac{z\,dz}{|w|} = -|a+1|^{\frac{k-2}{k}} \int_0^1 ((1+a)t-a)\sqrt[k]{\frac{(1+a)t-a}{t(1-t)}}\, dt.$$

Using these expressions

$$\lim_{a\to-\infty} |a|^{\frac{1}{k}} f_1(a) = -\frac{\pi}{k\sin(\pi/k)}, \quad \lim_{a\to-\infty} |a|^{\frac{1}{k}} g_1(a) = \frac{\pi(k+1)}{2k^2\sin(\pi/k)}$$

$$\lim_{a\to-\infty} |a|^{\frac{1-k}{k}} f_2(a) = -\frac{k}{k-1}, \qquad \lim_{a\to-\infty} |a|^{\frac{1-2k}{k}} g_2(a) = -\frac{k^2}{(k-1)(2k-1)}$$

and so $K = \dfrac{\pi k}{\sin(\pi/k)(k-1)(2k-1)}$. In particular, $f_2 g_1 + f_1 g_2 > 0$ in $]-\infty, -1[$.

Then we deduce that $q_1(a, m) < 0$. Suppose X_m is an embedding. As $M_{k\,a}$ has three ends and $g(P_1) = g(P_3) = 0$ then P_1 and P_3 are the top and the bottom ends (or viceversa) and P_2 is the middle end of $M_{k\,a}$. In particular their logarithmic growth rates satisfy $r_1(a, m) \cdot r_3(a, m) < 0$ (see the beginning of Subsection 6.1). This implies that

$$q_1(a, m) = \frac{2(2k-1)r_1(a, m)r_3(a, m)}{ak^2} > 0$$

which is a contradiction. ■

To finish this section we will prove an analytic uniqueness theorem for the Hoffman-Meeks surface X_0.

For $r \in]-2, 2[$ and $k > 2$, define

$$\overline{M}_{k\,r} = \left\{ (u, w) \in (\mathbb{C} \cup \{\infty\})^2 : w^k = \frac{u^2 + ru + 1}{u} \right\}.$$

Let now $P_0 = (0, \infty)$, $P_1 = (d(r), 0)$, $P_2 = (\infty, \infty)$ and $P_3 = (\overline{d(r)}, 0)$, where $d(r) = \frac{-r+i\sqrt{4-r^2}}{2}$. Put $M_{k\,r} = \overline{M}_{k\,r} - \{P_1, P_2, P_3\}$.

Consider on $M_{k\,r}$ the following meromorphic data:

$$g = Auw \quad \eta g = B\frac{d\,u}{u^2 + ru + 1}$$

where $A \in \mathbb{R} - \{0\}$ and $B \in \mathbb{C}$, $|B| = 1$. Define Φ_j, $j = 1, 2, 3$, as in (4) and let

$$X(P) = \text{Real}\int_{P_0}^{P} (\Phi_1, \Phi_2, \Phi_3), \quad P \in M_{k\,r}.$$

Theorem 6.8. *The minimal immersion X has no real periods if and only if $r = 0$. This case leads to Hoffman-Meeks genus $k-1$ example.*

Proof: Up to the change $u \longmapsto -u$, we will assume $r > 0$.

Suppose X has no real periods. In particular, Residue$(\Phi_3, P_i) \in \mathbb{R}$, $i = 1, 2, 3$ and so $B \in i\mathbb{R}$, that is, $B \in \{-i, i\}$. In what follows and without loss of generality we will suppose $B = i$.

Let γ_1 and γ_2 be the oriented closed curves in the u-plane illustrated in Figure 23 below. We suppose $\gamma_i(0)$, $i = 1, 2$, are the points indicated in Figure 3.

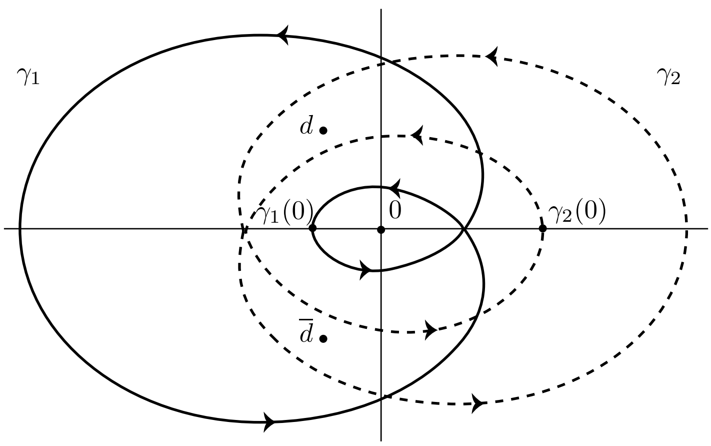

Figure 23. γ_1 and γ_2.

Let c_i, $i = 1, 2$, be the unique lifts of γ_i, $i = 1, 2$, to $\overline{M}_{k\,r}$ satisfying: $\text{Arg}(w(c_1(0))) = -\frac{\pi}{k}$, $\text{Arg}(w(c_2(0))) = \frac{2\pi}{k}$. Let τ_1, τ_2 denote the following 1-forms

$$\tau_1 = \frac{d\,u}{(u^2 + ru + 1)uw}, \quad \tau_2 = \frac{(1-k)u(2 + ru)w}{(u^2 + ru + 1)^2}\,d\,u.$$

Observe that $\eta = \frac{i}{A}\tau_1$ and $\eta g^2 = iA\tau_2 + d\,f$, where:

$$f = k\frac{u^2 w}{u^2 + ru + 1}.$$

Define

$$f_1(r) = \frac{1}{\overline{\theta}}\int_{c_1}\tau_1, \qquad f_2(r) = -\frac{1}{\overline{\xi}}\int_{c_2}\tau_1,$$

$$g_1(r) = -\frac{1}{\theta}\int_{c_1}\tau_2, \quad g_2(r) = \frac{1}{\xi}\int_{c_2}\tau_2$$

where $\theta = e^{\frac{-2\pi i}{k}} - 1$, $\xi = e^{\frac{3\pi i}{k}} - e^{\frac{\pi i}{k}}$. Deforming c_1 and c_2 on the real axis, an analytic continuation argument gives:

$$\begin{aligned} f_1(r) &= \int_0^{+\infty}\frac{d\,u}{(u^2+ru+1)u|w|}, \\ f_2(r) &= -\int_{-\infty}^{0}\frac{d\,u}{(u^2+ru+1)u|w|} \end{aligned} \tag{41}$$

$$\begin{aligned} g_1(r) &= \int_0^{+\infty}\frac{(k-1)u(2+ru)|w|}{(u^2+ru+1)^2}\,d\,u, \\ g_2(r) &= \int_{-\infty}^{0}\frac{(1-k)u(2+ru)|w|}{(u^2+ru+1)^2}\,d\,u. \end{aligned} \tag{42}$$

As Φ_1 and Φ_2 have no real periods we deduce

$$\int_{c_i}\tau_1 = -A^2\overline{\int_{c_i}\tau_2}, \quad i = 1, 2$$

and thus $(g_1 f_2 - g_2 f_1)(r) = 0$.

Assertion: The functions f_i, g_i satisfy:

$$f_i''(r) = \frac{(2+3k)r}{k(4-r^2)}f_i'(r) + \frac{1+2k}{k^2(4-r^2)}f_i(r) \tag{43}$$

$$g_i''(r) = \frac{(-2+3k)r}{k(4-r^2)}g_i'(r) + \frac{1-2k}{k^2(4-r^2)}g_i(r). \tag{44}$$

To see this, note that $w = w(r)$ and formally

$$\frac{\partial^2\tau_1}{\partial r^2} = \frac{(2+3k)r}{k(4-r^2)}\frac{\partial\tau_1}{\partial r} + \frac{1+2k}{k^2(4-r^2)}\tau_1 + d\,j_1$$

$$\frac{\partial^2(\eta g^2)}{\partial r^2} = \frac{(-2+3k)r}{k(4-r^2)}\frac{\partial(\eta g^2)}{\partial r} + \frac{1-2k}{k^2(4-r^2)}\eta g^2 + d\,(iAj_2)$$

where j_1 and j_2 are the meromorphic functions

$$j_1 = \frac{u^2 - rku - 2k - 1}{k(4-r^2)(u^2+ru+1)^2 w}, \quad j_2 = \frac{u^2((2k-1)u^2 + rku + 1)w}{k(4-r^2)(u^2+ru+1)^2}.$$

Now the assertion follows immediately.

As $f_1(r), g_1(r) > 0 \; \forall\, r \in [0,2[$, we can define the function $\rho : [0,2[\to \mathbb{R}$

$$\rho = \frac{f_2}{f_1} - \frac{g_2}{g_1}.$$

We will observe that ρ is a increasing function. We obtain this by proving that $\left(\frac{f_2}{f_1}\right)' > 0$ and $\left(\frac{g_2}{g_1}\right)' < 0$, that is, $f_2' f_1 - f_2 f_1' > 0$, $g_2' g_1 - g_2 g_1' < 0$.

From the above Assertion we have

$$(f_2' f_1 - f_2 f_1')' = \frac{(2+3k)r}{k(4-r^2)}(f_2' f_1 - f_2 f_1'),$$

$$(g_2' g_1 - g_2 g_1')' = \frac{(-2+3k)r}{k(4-r^2)}(g_2' g_1 - g_2 g_1')$$

and then

$$f_2' f_1 - f_2 f_1' = C_1 (4-r^2)^{-\frac{3k+2}{2k}}, \quad g_2' g_1 - g_2 g_1' = C_2 (4-r^2)^{\frac{2-3k}{2k}}.$$

To finish the proof it remains only to check that $C_1 > 0$ and $C_2 < 0$.

It is clear that $f_i(0) > 0$, $g_i(0) > 0$, $i = 1, 2$. On the other hand

$$\frac{\partial \tau_1}{\partial r} = -\frac{k+1}{k(u^2+ru+1)^2 w}\, d\,u, \qquad \frac{\partial \eta g^2}{\partial r} = iA \frac{(1-k)u^2 w}{k(u^2+ru+1)^2}\, d\,u$$

and so

$$f_1'(0) = -\frac{k+1}{k}\int_0^{+\infty} \frac{d\,u}{(u^2+1)^2|w|} < 0$$

$$f_2'(0) = \frac{k+1}{k}\int_{-\infty}^{0} \frac{d\,u}{(u^2+1)^2|w|} > 0$$

$$g_1'(0) = \frac{k-1}{k}\int_0^{+\infty} \frac{u^2|w|}{u^2+1}\, du > 0$$

$$g_2'(0) = -\frac{k-1}{k}\int_{-\infty}^{0} \frac{u^2|w|}{u^2+1}\, du < 0.$$

This implies that $C_1 = 4^{\frac{3k+2}{2k}}(f_2' f_1 - f_2 f_1')(0) > 0$ and $C_2 = 4^{\frac{3k-2}{2k}}(g_2' g_1 - g_2 g_1')(0) < 0$.

Since $\rho(0) = 0$ and ρ is increasing, $\rho(r) > 0 \; \forall\, r > 0$ and then we could solve the period problem only for $r = 0$, which corresponds to Hoffman-Meeks example. ■

6.1.3. Uniqueness results for embedded examples with three ends.

The aim of this subsection is to characterize the minimal embeddings $X_{m(a)} : M_{k\,a} \to \mathbb{R}^3$, $k > 2$, $a > 0$, described in Paragraph 6.1.1. They are the unique surfaces of maximal symmetry among complete minimal embeddings of genus $k-1$ with three ends and finite total curvature. For $a \neq 1$ the symmetry group $\mathrm{Sym}(M_{k\,a})$ is isomorphic to the dihedral group with $2k$ elements $\mathcal{D}(k)$ and is generated by a rotation about the x_3-axis by angle $\frac{2\pi}{k}$ and a symmetry with respect to the (x_1, x_3)-plane.

Hoffman and Karcher have previously obtained the following uniqueness theorem:

Theorem 6.9 (Hoffman, Karcher [27]). *Let $X : M \to \mathbb{R}^3$ be a complete embedded minimal surface of finite total curvature and three catenoid ends. Suppose that M has genus $k-1$ and k vertical planes of symmetry intersecting in a common vertical line. Then, up to scaling and rigid motion, $X = X_{m(a)}$, for a suitable $a \in]0,1[$.*

For $a = 1$, the immersion $X_{m(1)} = X_0$ is the Hoffman-Meeks surface of genus $k-1$. The group $\mathrm{Sym}(M_{k\,1})$ is isomorphic to $\mathcal{D}(2k)$ and is generated by a rotation about the x_3-axis by angle $\frac{\pi}{k}$ followed by a symmetry with respect to the (x_1, x_2)-plane and a symmetry with respect to the (x_1, x_3)-plane. The corresponding uniqueness theorem for this surface was firstly obtained by Hoffman and Meeks.

Theorem 6.10 (Hoffman, Meeks [32]). *Suppose $X : M \to \mathbb{R}^3$ is a complete embedded minimal surface with finite total curvature, genus $k-1$, $k > 1$, and three ends. If the symmetry group of $X(M)$ has at least $4k$ elements, then, up to homothety and rigid motion, $X = X_0$.*

In the case of genus one, a more general theorem was proved by Costa:

Theorem 6.11 (Costa [13]). *The only complete embedded minimal surfaces of genus one and three ends are the surfaces $\{X_{m(a)} : M_{2\,a} \to \mathbb{R}^3,\ a \in]0,1]\}$.*

The following notes are due to the authors of this survey and D. Rodríguez. We generalize Theorems 6.9 and 6.10 (see Corollary 6.13 and Theorem 6.14). The fundamental ideas here are inspired in the uniqueness part of the paper [**32**].

Let $X : M \to \mathbb{R}^3$ be a complete embedded minimal surface of genus $k-1$, $k > 2$, with three ends and finite total curvature. We assume that $\mathrm{Sym}(M)$ has at least $2k$ elements. From Theorem 5.1 M is conformally equivalent to $\overline{M} - \{P_1, P_2, P_3\}$, where $\overline{M}$ is a compact Riemann surface of genus $k-1$. The points removed correspond to the ends and from Theorem 5.2 the Weierstrass data (g, η) of X extends meromorphically to $\overline{M}$. Since the ends of M are parallel, we can suppose, up to one rotation if necessary, $g(P_i) \in \{0, \infty\}$, $i = 1, 2, 3$.

A symmetry of $X(M)$ induces in a natural way a conformal automorphism of M which extends to $\overline{M}$ leaving the set $\{P_1, P_2, P_3\}$ invariant. Since the subgroup of holomorphic transformations has index either one or two in $\mathrm{Sym}(M)$, then Hurwitz's Theorem (see [**20**]) implies that $\mathrm{Sym}(M)$ is finite. Thus after a suitable choice of the origin, $\mathrm{Sym}(M)$ is a finite group $\mathcal{G}$ of orthogonal linear transformations. Furthermore, since the normal vectors at the ends are vertical $\mathcal{G}$ leaves the x_3-axis invariant. Basic topics on embedded minimal surfaces indicate that, up to re-indexing the ends and without loss of generality, $g(P_1) = g(P_3) = 0$, $g(P_2) = \infty$, and that P_1 is the highest end, P_2 is the middle end and P_3 is the lowest end (see Subsection 6.1).

In what follows we do not distinguish between T and $T_{|X(M)}$, the latter being viewed as conformal transformation of M, $\forall T \in \mathcal{G}$.

Observe that any symmetry in $\mathcal{G}$ leaves the set $\{P_1, P_3\}$ invariant and so it fixes the point P_2. Let $\mathcal{H}$ be the subgroup of holomorphic transformations in $\mathcal{G}$. If we take D a conformal disk centered at P_2 invariant by $\mathcal{G}$ then $\{T_{|D} : T \in \mathcal{G}\}$ is a finite group of conformal automorphisms of the disk fixing the origin. Hence this group is either cyclic (i.e. $\mathcal{G} = \mathcal{H}$) or $|\mathcal{H}| = |\mathcal{G}|/2$ and is isomorphic to $\mathcal{D}(|\mathcal{H}|)$. Since $\mathcal{H}$ has at least k elements and $k > 2$, $\mathcal{H}$ is generated by J, where J is either a rotation around the x_3-axis or a rotation around the x_3-axis followed by a symmetry with respect to the (x_1, x_2)-plane. Moreover, if $\mathcal{G}$ is a dihedral group there exists an antiholomorphic transformation $S \in \mathcal{G}$ satisfying $J \circ S \circ J = S$. The isometry S corresponds to either a symmetry with respect to a plane containing the x_3-axis or a reflection around a straight line orthogonal to the x_3-axis and meeting this axis at the origin.

In the remaining part of this section, we assume that J is the generator of $\mathcal{H}$ corresponding either to a rotation around the x_3-axis by an angle of $\frac{2\pi}{\mathrm{ord}(J)}$ or a rotation around the x_3-axis by an angle of $\frac{2\pi}{\mathrm{ord}(J)}$ followed by a symmetry with respect to the (x_1, x_2)-plane, where $\mathrm{ord}(J)$ is the order of J and $\mathrm{ord}(J) = |\mathcal{H}|$. In the second case the number $\mathrm{ord}(J)$ is even. Label J_0 as the rotation around the x_3-axis with the lowest positive angle in $\mathcal{H}$. Note that either $J_0 = J$ or $J_0 = J^2$.

We have fixed the following notation: for $Q \in \overline{M}$ we denote

$$I(Q) = \{T \in \mathcal{H} : T(Q) = Q\}$$

as the isotropy group of Q in $\mathcal{H}$ and label $\mu(Q) = |I(Q)|$ as the cardinal of $I(Q)$. We also denote $\mathrm{orb}(Q) = \left\{Q, J(Q), \ldots, J^{|\mathcal{H}|-1}(Q)\right\}$ as the orbit of Q associated to $\mathcal{H}$. Notice that $\mathrm{orb}(Q)$ has $\frac{|\mathcal{H}|}{\mu(Q)}$ elements.

Label $\mathcal{H}_0 = \langle J_0 \rangle$ and define $I_0(Q)$, $\mu_0(Q)$ and $\mathrm{orb}_0(Q)$, for each $Q \in \overline{M}$, as above.

Theorem 6.12 (López, Martín, Rodríguez). *If* $\mathrm{ord}(J_0) \geq k$ *then there exists* $a \in \mathbb{R}_+$ *such that, up to conformal transformations and rigid motions in* $\mathbb{R}^3$*,* $M = M_{k\,a}$ *and* $X = X_{m(a)}$.

Proof: Since J_0 is a rotation around the x_3-axis then $J_0(P_i) = P_i$, $i = 1, 2, 3$, and so the formula of Riemann-Hurwitz gives:

$$4 - 2k = |\mathcal{H}_0|\chi\left(\overline{M}/\mathcal{H}_0\right) - \left(3|\mathcal{H}_0| - 3 + \sum_{Q \in M} (\mu_0(Q) - 1)\right).$$

Hence

$$1 - 2k + 3|\mathcal{H}_0| + \sum_{Q \in M} (\mu_0(Q) - 1) = |\mathcal{H}_0|\chi\left(\overline{M}/\mathcal{H}_0\right).$$

As $|\mathcal{H}_0| \geq k$, the left hand side of this formula is positive and then $\chi\left(\overline{M}/\mathcal{H}_0\right) > 0$, which implies that $\overline{M}/\mathcal{H}_0$ is a sphere and $\chi\left(\overline{M}/\mathcal{H}_0\right) = 2$. Therefore

$$1 - 2k + |\mathcal{H}_0| + \sum_{Q \in M} (\mu_0(Q) - 1) = 0. \tag{45}$$

Let $\mathrm{orb}_0(Q_1), \ldots, \mathrm{orb}_0(Q_s)$ be the different nontrivial orbits of $\mathcal{H}_0$ on M (i.e., $\mu_0(Q_i) > 1$, $i = 1, \ldots, s$ and if $Q \in M - \cup_{i=1}^s \mathrm{orb}_0(Q_i)$ then $\mu_0(Q) = 1$). We label $m_i = \frac{|\mathcal{H}_0|}{\mu_0(Q_i)}$, $i = 1, \ldots, s$. Since ${J_0}^{m_i}$ is a rotation around the x_3-axis that fixes ${J_0}^l(Q_i)$, $l = 0, \ldots, m_i - 1$, then these points are mapped by X into the x_3 axis. Furthermore, as J_0 is a rotation then $X(Q_i) = X({J_0}^l(Q_i))$, $l = 0, \ldots, m_i - 1$. Our embeddedness assumption implies $m_i = 1$, $i = 1, \ldots, s$ and then (45) becomes

$$1 - 2k + |\mathcal{H}_0| + s(|\mathcal{H}_0| - 1) = 0.$$

Since $|\mathcal{H}_0| \geq k$ the last equation implies that either $s = 0$ and $|\mathcal{H}_0| = 2k-1$ or $s = 1$ and $|\mathcal{H}_0| = k$.

If $s = 0$ then $X(M)$ does not meet the x_3-axis and so the number of points in M with the same vertical normal vector is a multiple of $|\mathcal{H}_0| = 2k-1$. (The rotation J_0 maps a point with vertical normal vector into another point with the same normal vector.) Since the ends P_1 and P_3 have the same normal vector, we can deduce that the degree of the Gauss map g is $2 + n(2k-1)$, $n > 0$. This number is greater than $k+1$ which contradicts the formula of Jorge-Meeks (Theorem 5.3).

Therefore $s = 1$ and $|\mathcal{H}_0| = k$. Let P_0 be the unique fixed point of J_0 in M. Since P_0 is a point of M fixed by a conformal automorphism of order k which corresponds to a rotation around the x_3-axis by an angle of $\frac{2\pi}{k}$, then the normal vector at this point is vertical and the multiplicity of the Gauss map g at P_0 is $nk-1$, $n > 0$. Furthermore any other point $Q \in M - \{P_0\}$ with vertical normal vector does not lie on the x_3-axis and so ${J_0}^l(Q)$, $l = 0, \ldots, k-1$, are k different points with the same vertical normal vector. Taking into account that $g(P_1) = g(P_3) = 0$, $g(P_2) = \infty$ and $\deg(g) = k+1$ we deduce that $n = 1$, $g(P_0) = 0$ and either $g^{-1}(\infty) = \{P_2\}$ or $g^{-1}(\infty) = \{P_2, Q_0, \ldots, {J_0}^{k-1}(Q_0)\}$, where $Q_0 \in \overline{M} - \{P_0, P_1, P_3\}$ is not a branch point of g. In other words the divisor associated to g is

$$[g] = \frac{P_1 \cdot P_3 \cdot P_0^{k-1}}{P_2 \cdot \prod_{i=0}^{k-1} J_0^i(Q_0)}. \tag{46}$$

The mapping $u : \overline{M} \longrightarrow \overline{M}/\mathcal{H}_0$ is a k-fold cyclic branched covering of $\overline{M}/\mathcal{H}_0 = \mathbb{C} \cup \{\infty\}$. Without loss of generality we may choose $u(P_0) = 0$, $u(P_1) = -1$, $u(P_2) = \infty$. We also write $u(P_3) = a \in \mathbb{C}$ and $u(Q_0) = -1/m$, $m \in \mathbb{C}$ (of course, $m = 0$ means $u(Q_0) = \infty$). If we define $N = M - P_0$ then $u_{|N} : N \to \mathbb{C} - \{0, -1, a\}$ is a k-fold unbranched cyclic covering. Moreover, the conformal structure of N determines the conformal structure of $\overline{M}$. We may determine $u_{|N}$ as follows. Remember that J_0 is the generator of $\mathcal{H}_0$ corresponding to a counterclockwise rotation around the x_3-axis by an angle of $\frac{2\pi}{k}$. Let α_i, $i = 1, 2, 3$ be a counterclockwise circuit around 0, a and -1 respectively, and $\widetilde{\alpha_i}$ its lift to N. The end points of $\widetilde{\alpha_i}$ will differ by a deck transformation of the form $J_0^{k_i}$, $0 \leq k_i \leq k-1$, $i = 1, 2, 3$. The choice of J_0 and the fact that we have oriented $\overline{M}$ with downward-pointing normal vectors at P_0, P_1 and P_3 implies that J_0 has rotation number $2\pi/k$ at P_1 and P_3 and rotation number $-2\pi/k$ at P_0 and P_2. Hence $k_2 \equiv k_3 \equiv 1 \operatorname{mod}(k)$ and $k_1 \equiv -1 \operatorname{mod}(k)$. The numbers k_1, k_2 and k_3 determine the induced map from $\Pi_1(\mathbb{C} - \{0, -1, a\})$ onto $\mathbb{Z}_k$ whose kernel corresponds to $u_*(\Pi_1(N)) \subset \Pi_1(\mathbb{C}-\{0,-1,a\})$. Any k-fold cyclic covering of $\mathbb{C}-\{0,-1,a\}$ is equivalent to $u_{|N}$ if the associated representation has the same kernel. In particular the cyclic covering defined by the z-projection of

$$\left\{(z,w) \in (\mathbb{C}-\{0,-1,a\}) \times (\mathbb{C}-\{0\}) : w^k = \frac{(z+1)(z-a)}{z}\right\}$$

is equivalent to $u_{|N}$. The extension of this covering to the Riemann surface

$$\overline{M}_{k\,a} = \left\{(z,w) \in (\mathbb{C}\cup\{\infty\})^2 : w^k = \frac{(z+1)(z-a)}{z}\right\}$$

is conformally equivalent to u. In particular $M = M_{k\,a}$, $u = z$ and $J_0(z,w) = (z, e^{\frac{2\pi i}{k}} w)$.

Furthermore from (46) and taking into account that the ends are embedded (i.e. $\nu_i = 1$, $i = 1,2,3$, see Theorem 5.3) we have up to rigid motions and scaling:

$$g = A\frac{zw}{mz+1}, \quad \eta g = B\frac{mz+1}{(z+1)(z-a)}\,dz \tag{47}$$

where $A \in \mathbb{R}$ and $B \in \mathbb{C}$, $|B| = 1$.

We distinguish two cases: $J_0 = J^2$ and $J_0 = J$.

In the first case J is a rotation followed by a symmetry and then $J(P_1) = P_3$, $J(P_3) = P_1$, $J(P_2) = P_2$ and since J_0 has only a fixed point P_0 in M, then $J(P_0) = P_0$. The conformal transformation J can be induced on $\overline{M}/\mathcal{H}_0 = \mathbb{C}\cup\{\infty\}$ giving a nontrivial involutive holomorphic automorphism of $\mathbb{C}\cup\{\infty\}$ fixing 0 and ∞. This automorphism must be $z \mapsto -z$ and then $a = 1$. Furthermore, since J maps points with vertical normal vector into other points with vertical normal vector we deduce that $J^{2l+1}(Q_0)$ $l = 0, \dots, k-1$ are points which appear in the divisor associated to g. Moreover it is clear that $J(Q_0) \neq P_0, P_1, P_3$ and so $Q_0 = P_2$, i.e. $m = 0$. Hence the Weierstrass data in (47) correspond to the Hoffman-Meeks surface.

Suppose now $J_0 = J$, i.e., J is a rotation around the x_3-axis. In this case $\mathcal{H}_0 = \mathcal{H}$ and $|\mathcal{G}| = 2k$. Let S be a symmetry in $\mathcal{G}-\mathcal{H}$. From the definition of $\mathcal{H}$, S is an antiholomorphic transformation on $\overline{M}$ fixing P_2 which corresponds to either a symmetry with respect to a plane containing the x_3-axis or a reflection about a straight line orthogonal to the x_3-axis and meeting it at the origin. Since $J \circ S \circ J = S$ and P_0 is the unique point fixed by $J = J_0$ on M, S fixes P_0 too. Inducing S on $\overline{M}/\mathcal{H}_0$ we obtain an antiholomorphic involution in $\mathbb{C}\cup\{\infty\}$ which fixes 0 and ∞. This implies that $z \circ S = \theta\overline{z}$, where $|\theta| = 1$.

If S is a symmetry with respect to a plane containing the x_3-axis then S fixes P_1 and P_3, and so the function $z \mapsto \theta\overline{z}$ fixes -1 and a, which implies $\theta = 1$ and $a \in \mathbb{R} - \{0, -1\}$. In particular and without loss of generality $S(z, w) = (\overline{z}, \overline{w})$. Furthermore, since S maps points with vertical normal vector into other points with vertical normal vector then it leaves invariant $\operatorname{orb}(Q_0)$ and so $m \in \mathbb{R}$. Thus Theorem 6.3 and Theorem 6.7 leads to $X = X_{m(a)}$.

If S is a reflection about a straight line orthogonal to the x_3-axis then $S(P_1) = P_3$ and P_2 is a flat end. This implies that $m = 0$, $\theta = -a$ and without loss of generality $S(z, w) = (-a\overline{z}, \sqrt{[k]{-a}}\overline{w})$. Making $u = z/\sqrt{-a}$, we obtain $M = M_{k:\ r}$ where $r = 2\operatorname{Real}(\sqrt{-a})$ and the Weierstrass data (47) are, up to natural transformations, those studied in Theorem 6.8. This theorem leads to Hoffman-Meeks surface which has $4k$ symmetries, which is a contradiction. ■

If P_1, P_2 and P_3 are catenoid ends then $\operatorname{Sym}(M)$ does not contain any rotation followed by a symmetry, i.e. $J_0 = J$. Hence we get the following

Corollary 6.13 (López, Martín, Rodríguez). *If $X : M \to \mathbb{R}^3$ has three catenoid ends and $\operatorname{Sym}(M)$ contains $2k$ elements or more then, up to natural transformations, $X = X_{m(a)}$, $a \in \mathbb{R}_+ - \{1\}$.*

In the end, we are going to obtain a new characterization of the Hoffman-Meeks surface which improves on the one by Hoffman and Meeks in [**30**].

Theorem 6.14 (López, Martín, Rodríguez). *If M has $2k+3$ symmetries or more then X is the Hoffman-Meeks surface X_0.*

Proof: First we observe that $J_0 = J^2$. If $J_0 = J$ then $\operatorname{Sym}(M)$ contains a rotation about the x_3-axis of order greater than $k+1$ and then Theorem 6.12 leads to $X = X_{m(a)}$ for a suitable $a > 0$. No such surfaces contain a rotation of this order, which is a contradiction.

Hence $J_0 = J^2$ and $\mathcal{H}$ is generated by a rotation followed by a symmetry. We deduce that P_2 is a flat end, P_1 and P_3 are catenoid ends, $J(P_2) = P_2$, $J(P_1) = P_3$ and $J(P_3) = P_1$. From the Riemann-Hurwitz formula we obtain

$$4 - 2k = |\mathcal{H}|\chi\left(\overline{M}/\mathcal{H}\right) - \left(2|\mathcal{H}| - 3 + \sum_{Q \in M} (\mu(Q) - 1)\right).$$

Since $|\mathcal{H}| \geq k$ we deduce $\chi\left(\overline{M}/\mathcal{H}\right) > 0$ and so $\overline{M}/\mathcal{H}$ is a sphere and $\chi\left(\overline{M}/\mathcal{H}\right) = 2$. Substituting in the above formula $\chi\left(\overline{M}/\mathcal{H}\right)$ for 2, we get

$$\sum_{Q \in M} (\mu(Q) - 1) = 2k - 1. \tag{48}$$

Let $\mathrm{orb}(Q_1), \ldots, \mathrm{orb}(Q_s)$ be the different nontrivial orbits of $\mathcal{H}$ on M (i.e., $\mu(Q_i) > 1$, $i = 1, \ldots, s$ and if $Q \in M - \cup_{i=1}^{s} \mathrm{orb}(Q_i)$ then $\mu(Q) = 1$). If we label $m_i = \frac{|\mathcal{H}|}{\mu(Q_i)}$, $i = 1, \ldots, s$, then (48) gives

$$\sum_{i=1}^{s} (|\mathcal{H}| - m_i) = 2k - 1. \tag{49}$$

Since $|\mathcal{H}|$ is even then at least one of the numbers m_i is odd. On the other hand, if m_i is odd then J^{m_i} is a rotation around the x_3-axis followed by a symmetry with respect to the (x_1, x_2)-plane and it fixes only the origin of $\mathbb{R}^3$. As X is an embedding, there is at most one point of M mapped by X into the origin, and so there exists a unique odd number m_i and $m_i = 1$. Up to re-indexing we can assume that $m_1 = 1$.

Therefore m_i is even, $i \geq 2$, and J^{m_i} is a rotation, $i \geq 2$. If $m_i > 2$, $i \in \{2, \ldots, s\}$, then Q_i and $J^2(Q_i)$ are two different points lying in $\mathrm{orb}(Q_i)$. Since J^{m_i} is a rotation around the x_3-axis then $X(Q_i)$ and $X(J^2(Q_i))$ lie in the x_3-axis. Moreover J^2 is a rotation around the x_3-axis too, and thus $X(Q_i) = X(J^2(Q_i))$, which contradicts the fact that X is an embedding. Hence $m_i = 2 \; \forall \, i \geq 2$ and (49) becomes

$$|\mathcal{H}| + (s-1)(|\mathcal{H}| - 2) = 2k.$$

Since $|\mathcal{H}| \geq k + 2$ we get $s = 1$ and $|\mathcal{H}| = 2k$.

In particular $\mathrm{ord}(J_0) = k$. Using Theorem 6.12 we obtain $X = X_{m(a)}$, $a \in \mathbb{R}_+$. Taking into account that the unique surface of the family, which has more than $2k$ symmetries, is the Hoffman-Meeks example, then the theorem holds. ■

A standing conjecture asserts that:

Conjecture 5. *The moduli space of properly embedded minimal surfaces in $\mathbb{R}^3$ with finite topology and three ends consists of the family $\{X_{m(a)}(M_{k\,a}),\ a \in]0,1],\ k \in \mathbb{N}\}$.*

As we have mentioned before, Costa's theorem (Theorem 6.11) proves that the conjecture is true for $k = 1$.

This conjecture suggest a more general question: the study of the moduli space of embedded complete minimal surfaces in $\mathbb{R}^3$, up to homotheties, with finite total curvature and fixed topology. In this sense, they are remarkable the works by Ros [**84**] and by Pérez and Ros [**77**], [**78**]. So, in the first paper, it is proved that if the genus is one and the number of ends is five or more then the moduli space above is *compact* in strong sense. In the other two papers, and under suitable conditions of nondegeneration, it is proved that this moduli space is a real analytical sub-manifold of a finite dimensional Euclidean complex space.

6.2. Properly embedded minimal surfaces of genus zero.

It gradually became a question of increasing interest to classify the properly embedded minimal planar domains.

In case of finite topology, Corollary 6.2 and Theorem 6.15 below show that the only such surface with two or more ends is the catenoid. For simply connected properly embedded minimal surfaces, it is still open the following conjecture:

Conjecture 6 (Osserman, Meeks). *The only properly embedded, simply connected, nonflat, minimal surface in $\mathbb{R}^3$ is the helicoid.*

Partial answers to this conjecture can be found in [**66**], [**97**], [**98**], [**83**].

A more general conjecture asserts:

Conjecture 7 (Meeks). *The only properly embedded genus zero minimal surfaces are the plane, the catenoid, the helicoid and Riemann's minimal examples.*

Riemann's minimal examples will be described in Paragraph 6.2.2. As we will comment in Theorem 6.19, Meeks, Pérez and Ros have proved that Conjecture 7 is true if, in addition, we suppose that the surface has infinitely many symmetries.

6.2.1. Properly embedded minimal surfaces with vertical flux.

Let $X : M \to \mathbb{R}^3$ be a minimal immersion. Given γ a closed curve in M, we define the *flux* of X along γ as

$$\operatorname{Flux}(\gamma) = \operatorname{Im}\left(\int_\gamma \Phi\right),$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ are the Weierstrass data of X. From the definition, it is clear that $\operatorname{Flux}(\gamma)$ only depends on the homology class of γ in $H_1(M, \mathbb{Z})$.

On the other hand, if $n(s)$ represents the conormal vector of X along the curve γ, then it is straightforward to check that

$$\mathrm{Flux}(\gamma) = \int_\gamma n(s)\, ds.$$

Furthermore, the following assertions are equivalent:

1. $\mathrm{Flux}(\gamma) = 0$, $\forall\, \gamma \in H_1(M, \mathbb{Z})$.
2. Φ_1, Φ_2 and Φ_3 are exact.
3. η, $g\,\eta$ and $g^2\,\eta$ are exact.
4. The conjugate immersion $X^* = \mathrm{Im}\left(\int \Phi\right)$ is well defined.

For any $\lambda > 0$, we consider on M the Weierstrass data $g_\lambda \stackrel{\text{def}}{=} \lambda\, g$ and $\eta_\lambda \stackrel{\text{def}}{=} \frac{1}{\lambda}\,\eta$. They define in general a multivalued unbranched minimal immersion $X_\lambda : M \to \mathbb{R}^3$, given by

$$X_\lambda(P) = \mathrm{Real} \int_{P_0}^P \frac{1}{\lambda} \left(\frac{1}{2}(1 - \lambda^2 g^2), \frac{i}{2}(1 + \lambda^2 g^2), \lambda g \right) \eta. \tag{50}$$

If all the fluxes of X are *vertical* vectors, we easily conclude that Φ_1 and Φ_2 are exact, and we can check that this fact occurs if and only if X_λ is well defined, $\forall\, \lambda > 0$. If the deformation $\{X_\lambda\}_{\lambda > 0}$ exists on M, it is obviuous that the third coordinate function of X_λ is independent on λ.

Suppose A is an annulus, A homeomorphic to $\overline{\mathbb{D}}^*$, and consider $X : A \to \mathbb{R}^3$ a minimal immersion. We define the flux of the minimal end $X(A)$ as $\mathrm{Flux}(\gamma)$, where γ is any curve in A generating $H_1(A, \mathbb{Z})$. If $X : M \to \mathbb{R}^3$ is a complete minimal immersion with finite topology, and we label E_i, $i = 1, \dots, r$, as the topological ends of M, then

$$\sum_{i=1}^r f_i = 0, \tag{51}$$

where f_i is the flux of the end E_i (i.e., the flux of the minimal end $X(D_i)$, where D_i is a neighborhood of E_i homeomorphic to $\overline{\mathbb{D}}^*$,) $i = 1, \dots, r$. This result is an easy consequence of Stokes' Theorem.

In particular, if X has finite total curvature, the sum of the fluxes of the ends is zero, and in this sense, the surface is balanced. If P is an embedded end of M with limit normal vector v_P, then the flux of P is

$$\mathrm{Residue}(\Phi^t v_P) a_P,$$

where a_P is the logarithmic growth of P. So, if all the ends $P_1, \dots, P_r$ are embedded and parallel, one has

$$\sum_{i=1}^{r} a_{P_i} = 0. \tag{52}$$

Other balancing formulae for minimal surfaces can be found in [**76**].

Remark 6. Any annular embedded end with finite total curvature and vertical limit normal vector at the end has vertical flux.

Thus, any properly embedded minimal surface with finite total curvature, genus zero, and vertical normal vectors at the ends, has vertical flux. So, the above deformation exists for these kinds of surfaces.

Theorem 6.15 (López, Ros [59]). *Let $X : M \to \mathbb{R}^3$ be a complete, genus zero, embedded minimal surface with finite total curvature. Then $X(M)$ is the catenoid or the plane.*

Proof: As we have mentioned above, any embedded genus zero minimal surface has vertical flux. In fact, we are going to prove that:

The only complete, properly embedded, minimal surfaces with finite total curvature and vertical flux are the plane and the catenoid.

The presentation here of this result follows that of Pérez and Ros [**79**].

Suppose that $X(M)$ is not a plane. Then, by Theorems 5.1, 5.2 and 5.3, M is conformally equivalent to $\overline{M} - \{P_1, \dots, P_r\}$, where $\overline{M}$ is a compact Riemann surface, the Weierstrass data (g, η) extends meromorphically to $\overline{M}$, and the ends are asymptotic to planes or half-catenoids.

As $X(M)$ has vertical flux, then the deformation $X_\lambda : M \to \mathbb{R}^3$, $\lambda > 0$, given in (50), is well defined. It is clear that X_λ is complete, has finite total curvature and embedded ends.

Claim 1. *Let $X : M \to \mathbb{R}^3$ be a complete nonflat minimal immersion with vertical flux. Consider $P \in M$ satisfying $g(P) \in \{0, \infty\}$, then for all conformal disc centered at P, $D(P)$, there is $\lambda > 0$ such that $X_\lambda|_{D(P)}$ is not an embedding.*

Firstly, observe that the set $g_\lambda^{-1}(\{0, \infty\})$ does not depend on λ. Suppose that $X(M)$ is not a plane, and consider $P \in M$ a point with vertical normal vector. Up to a rigid motion, we can assume that $g(P) = 0$. Take $(D(\epsilon), z)$ a local coordinate centered at P, in such a way that:

$$g(z) = z^k, \quad \eta = (a + z\,h(z))\,dz,$$

where $k \in \mathbb{N}$, $k > 0$, $a \in \mathbb{C} - \{0\}$, h is a holomorphic function on $D(\epsilon) = \{z \in \mathbb{C}/|z| < \epsilon\}$. If we consider $(D(\lambda^{1/k}\epsilon), \zeta = \lambda^{1/k}z)$, then the Weierstrass data of X_λ can be locally expressed as:

$$g_\lambda(\zeta) = \zeta^k, \quad \eta_\lambda = \frac{1}{\lambda^{1+1/k}} \left(a + \frac{\zeta}{\lambda^{1/k}} h\left(\frac{\zeta}{\lambda^{1/k}}\right)\right) d\zeta.$$

Thus, the homothetic shrinking $\widetilde{X}_\lambda = \lambda^{1+1/k} X_\lambda$ is another complete minimal immersion. As $\lambda \to \infty$, $\widetilde{X}_\lambda$ converges uniformly over compact subsets of $\mathbb{C}$ to a minimal immersion $\widetilde{X}_\infty$ whose Weierstrass representation is:

$$g_\infty(\zeta) = \zeta^k, \quad \eta_\infty = a\, d\zeta.$$

This minimal immersion is complete, but not embedded. Therefore, if λ is large enough X_λ is not an embedding. This proves the claim.

In order to prove the next claim, we need some previous results.

Consider $X : D(\epsilon)^* \to \mathbb{R}^3$ be an embedded end with well defined limit normal vector (i.e. g is well defined at 0, and so the total curvature is finite), and assume that $X(D(\epsilon)^*)$ is not a planar domain. Up to a rigid motion, we suppose that $g(0) = 0$. By Theorem 5.4, the Weierstrass data of the end are:

$$g(z) = z^k, \quad \eta = \left(\frac{a}{z^2} + h(z)\right) dz, \quad a \in \mathbb{C} - \{0\}, \tag{53}$$

where h is holomorphic and k is a positive integer. If $k = 1$, then a must be real, and so we have a catenoid end with logarithmic growth a. If $k > 1$ the end is flat. The flux of X either vanishes (flat end) or is vertical (catenoid end). Hence, X_λ is well defined on $D(\epsilon)^*$. Furthermore, if X is a catenoid end (resp. flat end) then X_λ is a catenoid end (resp. flat end).

We obtain from (53) that

$$X_\lambda(z) = \phi_\lambda(z) + F(z, \lambda), \tag{54}$$

where $\phi_\lambda : D(\epsilon)^* \to \mathbb{R}^3$ denotes either a parametrization of the end of the (x_1, x_2)-plane or a parametrization of an end of the vertical catenoid symmetric respect to the origin with logarithmic growth a, and F is a finite-valued smooth function on $D(\epsilon)\times]0, +\infty[$.

Claim 2. *Let $X : D(\epsilon)^* \to \mathbb{R}^3$ be a planar end with finite total curvature. If $X(D(\epsilon)^*)$ is not a planar domain, then there is $\lambda > 0$ such that X_λ is not an embedding.*

As in Claim 1, we consider the conformal coordinate $(D(\lambda^{1/k}\epsilon)^*, \zeta = \lambda^{1/k}z)$. Then, taking (53) into account, the Weierstrass data of X_λ are:

$$g_\lambda(\zeta) = \zeta^k, \quad \eta_\lambda = \frac{1}{\lambda^{1+1/k}} \left(\frac{a}{\zeta^2} + \frac{1}{\lambda^{2/k}} h\left(\frac{\zeta}{\lambda^{1/k}}\right)\right) d\zeta.$$

Thus, the homothetical shrinking $\widetilde{X}_\lambda = \lambda^{1+1/k} X_\lambda$ is another complete minimal immersion. As $\lambda \to \infty$, $\widetilde{X}_\lambda$ converges uniformly over compact subsets of $\mathbb{C} - \{0\}$ to a minimal immersion $\widetilde{X}_\infty : \mathbb{C} - \{0\} \to \mathbb{R}^3$ whose Weierstrass representation is:

$$g_\infty(\zeta) = \zeta^k, \quad \eta_\infty = \frac{a}{\zeta^2}\, d\zeta.$$

As $k \geq 2$, this surface has an embedded end at 0 and a nonembedded one at ∞. So, for λ large enough, X_λ is not an embedding.

Claim 3. *X_λ is an embedding,* $\forall\, \lambda > 0$.

Let $B = \{\lambda > 0 / X_\lambda \text{ is an embedding}\}$. Observe that $1 \in B$, and so $B \neq \emptyset$. If $\lambda_0 \in B$, then, using Theorem 1.6, one has that the distance between two ends of X_{λ_0} is positive. So, from (54), this distance is either infinite, for all $\lambda > 0$, or a continuous function on λ. Then, there exists $\epsilon, R > 0$ such that $X_\lambda(M) \cap (\mathbb{R}^3 - B(0,R))$ is embedded, $\forall\, \lambda \in]\lambda_0 - \epsilon, \lambda_0 + \epsilon[$. If X_λ were not injective for some $\lambda \in]\lambda_0 - \epsilon, \lambda_0 + \epsilon[$, then self-intersections of $X_\lambda(M)$ would be in $\overline{B}(0,R)$, and so we would arrive to a contradiction, by using the classical maximum principle (Theorem 1.5). Hence, $]\lambda_0 - \epsilon, \lambda_0 + \epsilon[\subset B$ which implies that B is open.

Now take $\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence in B converging to $\lambda_0 > 0$. Assume that X_{λ_0} is not injective. Then, there are two points $x, y \in M$ satisfying $X_{\lambda_0}(x) = X_{\lambda_0}(y)$. The convergence of $\{X_{\lambda_n}\}_{n\in\mathbb{N}}$ to X_{λ_0} uniformly over compact subsets of M and Theorem 1.5 insure that there exist neighborhoods $N(x)$, $N(y)$, of x and y, respectively, such that $X_{\lambda_0}(N(x)) = X_{\lambda_0}(N(y))$. So, the image set $X_{\lambda_0}(M)$ is an embedded minimal surface with finite total curvature and

$$X_{\lambda_0} : M \longrightarrow X_{\lambda_0}(M)$$

is a finitely sheeted covering map. By Theorem 1.6, there exists an embedded tubular neighborhood of $X_{\lambda_0}(M)$ in $\mathbb{R}^3$. Label $\pi : U \to X_{\lambda_0}(M)$, and $d : U \to \mathbb{R}$ the orthogonal projection and the oriented distance, respectively. From (54), it is clear that $X_{\lambda_n}(M) \subset U$, for $n \in \mathbb{N}$ large enough. Hence,

$$\pi \circ X_{\lambda_n} : M \longrightarrow X_{\lambda_0}(M)$$

is a proper local diffeomorphism, i.e., a finitely sheeted covering map. As X_{λ_n} is an embedding, then $d \circ X_{\lambda_n}$ is a continuous map which separates the points in the fiber $(\pi \circ X_{\lambda_n})^{-1}(\{P\})$, $\forall\, P \in X_{\lambda_0}(M)$. So, $\pi \circ X_{\lambda_n}$ has only one sheet. As $\{\pi \circ X_{\lambda_n}\}_{n\in\mathbb{N}}$ converges to $\pi \circ X_{\lambda_0} = X_{\lambda_0}$ uniformly over compact subsets of M, we deduce that X_{λ_0} is injective, which is contrary to our assumption. This contradiction proves that B is closed.

Thus, an elementary connectedness argument gives that $B =]0, +\infty[$, which concludes the claim.

Claim 4. *The surface $X(M)$ is a catenoid.*

Taking into account Claims 1, 2 and 3, we deduce that X has neither planar ends nor points with vertical normal vector. Hence, the third coordinate function X_3 is proper and has no critical points. So, M is an annulus and Theorems 5.3 and 5.15 imply that $X(M)$ is a catenoid. ■

6.2.2. Properly embedded minimal cylinders with planar ends.

The first examples of periodic minimal surfaces with more than one end were discovered by B. Riemann [**82**] in 1867. Riemann constructed a one-parametric family of properly embedded minimal surfaces, $\{R_\lambda : \lambda > 0\}$, which are invariant under a translation, T_λ. He proved also that every minimal surface expressible as a union of circles in parallel planes is either a subset of some R_λ or a subset of the catenoid. Furthermore, Enneper obtained the same conclusion without assuming that the planes of the folitation are parallel.

Let us introduce the Riemann minimal examples. Consider, for each $\lambda > 0$, the compact genus one Riemann surface

$$\overline{M}_\lambda = \{(z, w) \in \overline{\mathbb{C}}^2 : w^2 = z(z - \lambda)(\lambda z + 1)\},$$

with its natural complex structure. Define

$$M_\lambda = \overline{M}_\lambda - \{(0,0), (\infty, \infty)\}$$
$$g = z, \quad \eta g = B\, dz/w,$$

where $B \in \mathbb{R}$. Let $[\alpha]$ be the homology class of the closed curve in $\overline{M}_\lambda$ obtained by lifting the slit $[0, \lambda]$. We suppose that α is an element of $[\alpha]$ lying in M_λ.

Let $p : N_\lambda \to \overline{M}_\lambda$ be the conformal covering determined by: $p_*(\mathcal{H}_1(N_\lambda, \mathbb{Z})) = \{m[\alpha] : m \in \mathbb{Z}\}$. Up to conformal transformations, $N_\lambda = \mathbb{C}^*$. We label $\widetilde{M_\lambda} = p^{-1}(M_\lambda)$, and observe that $\widetilde{M_\lambda}$ is conformally equivalent to $\mathbb{C}^*$ punctured in a sequence of points diverging to 0 and ∞. We write $\widetilde{g} = g \circ p$, and $\widetilde{\eta} = p^*(\eta)$. We also label $\widetilde{\alpha}$ as a lift of the curve α in $\widetilde{M_\lambda}$.

Consider the minimal immersion $Y_\lambda : \widetilde{M}_\lambda \to \mathbb{R}^3$ given by:

$$Y_\lambda(P) = \text{Real} \int_{P_0}^{P} (\widetilde{\Phi}_1, \widetilde{\Phi}_2, \widetilde{\Phi}_3),$$

where $\widetilde{\Phi}_j$, $j = 1, 2, 3$, were defined in (4), and $P_0 \in \widetilde{M}_\lambda$. It is clear that the 1-forms η, ηg^2 and ηg have no residues at the ends of M_λ, and so the same holds for their pull-backs at the ends of $\widetilde{M}_\lambda$. Furthermore, $\int_\alpha \eta g = \int_{\widetilde{\alpha}} \widetilde{\eta}\widetilde{g} \in i\mathbb{R}$, and so the map Y is well defined (i.e., it has no real periods) if and only if

$$\int_{\widetilde{\alpha}} \widetilde{\eta} = \overline{\int_{\widetilde{\alpha}} \widetilde{\eta}\widetilde{g}^2},$$

which is equivalent to

$$\int_\alpha \eta = \overline{\int_\alpha \eta g^2}. \tag{55}$$

The transformation $I(z, w) = (-1/\overline{z}, \overline{w}/\overline{z}^2)$ satisfies $I_*\alpha = -\alpha$ and $I^*\eta = -\overline{\eta g^2}$. Therefore, (55) trivially holds.

The arising family of surfaces $R = \{R_\lambda = Y_\lambda(\widetilde{M}_\lambda)\}$ are the so called Riemann minimal examples. These surfaces are invariant under the translation T_λ defined by the vector

$$\vec{v}_\lambda = \text{Real}\left(\int_{[\beta]} (\Phi_1, \Phi_2, \Phi_3)\right),$$

where $[\beta]$ is the homology class of a lift to $\overline{M}_\lambda$ of the slit $[-1/\lambda, 0]$.

The ends of R_λ are embedded and planar, and the quotient $R_\lambda/\langle T_\lambda\rangle$ is a genus one, embedded minimal surface in $\mathbb{R}^3/\langle T_\lambda\rangle$ with two ends and total curvature -8π. F. J. López, M. Ritoré and F. Wei [**58**] have characterized Riemann's examples as the only embedded minimal tori with two planar ends in $\mathbb{R}^3/T_\lambda$. From this point of view, and very recently, Meeks, Pérez and Ros [**64**] have obtained the best possible theorem, generalizing the last result for a finite arbitrary number of ends. See Theorem 6.17 below.

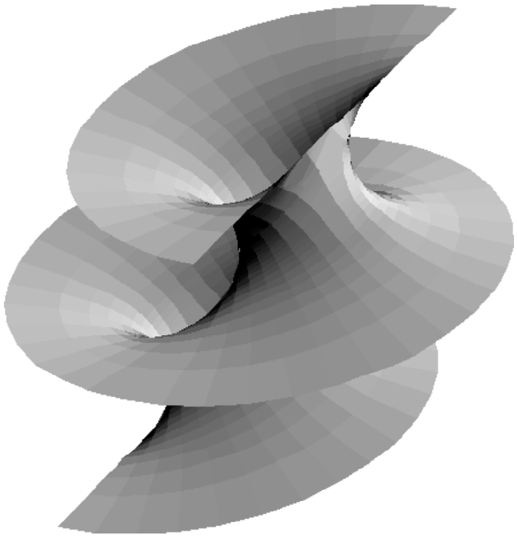

Figure 24. A fundamental piece of R_1.

We first include here the following classical theorem.

Theorem 6.16 (Riemann [82]). *A nonflat minimal surface foliated by pieces of circles or lines in parallel planes is, up to scaling and rigid motions, a piece of either some R_λ, $\lambda > 0$, the catenoid or the helicoid.*

Proof: Let $X : M \to \mathbb{R}^3$ be a surface satisfying the hypotheses in the theorem. Take (D, z) a conformal disk in M such that $\eta g = dz$. The level curve $x_3 = c$, $c \in \mathbb{R}$, corresponds in D to the curve $z_c(y) = c + iy$. Therefore, it is straightforward to check that

$$k_c(y) = \left[\frac{|g|}{1+|g|^2}\,\mathrm{Real}\left(\frac{g'}{g}\right)\right]|_{z=z_c(y)},$$

where k_c is the planar curvature of the curve $X \circ z_c$. In what follows, we write $k(c, y) = k_c(y)$, and observe that from our assumptions, this function just depends on c. In particular,

$$\frac{\partial k}{\partial y} = 0.$$

By a straightforward computation, this equality is equivalent to

$$\mathrm{Im}\left(\frac{3}{2}\left(\frac{g'(z)}{g(z)}\right)^2 - \frac{g''(z)}{g(z)} - \frac{1}{1+|g(z)|^2}\left(\frac{g'(z)}{g(z)}\right)^2\right) = 0. \tag{56}$$

If we define

$$f_1(z) = \frac{1}{2}\left(\frac{g'(z)}{g(z)}\right)^2 - \frac{g''(z)}{g(z)}, \quad f_2(z) = g(z)\left(\frac{g''(z)}{g(z)} - \frac{3}{2}\left(\frac{g'(z)}{g(z)}\right)^2\right),$$

then (56) becomes

$$\mathrm{Im}(f_1) = \mathrm{Im}(\overline{g} f_2). \tag{57}$$

This implies that $\mathrm{Im}(\overline{g} f_2)$ is harmonic. Since g and f_2 are meromorphic, it is not hard to deduce that

$$f_2 = r_1 g + a,$$

where $r_1 \in \mathbb{R}$ and $a \in \mathbb{C}$. By using (57), one has

$$f_1 + \overline{a} g - r_2 = 0,$$

where $r_2 \in \mathbb{R}$.

These two equations imply that $r_1 = r_2 = r \in \mathbb{R}$ and

$$g'(z)^2 = g(z)\left(\overline{a} g(z)^2 - 2rg(z) - a\right). \tag{58}$$

If $a = 0$, then $g'(z) = \pm i\sqrt{2r} g(z)$, and so, $g(z) = Ae^{\pm i\sqrt{2r}z}$, $A \in \mathbb{C}^*$. If $r < 0$, we get a piece of a catenoid. When $r > 0$, we obtain a piece of the helicoid. Remember that Catalan in [**7**] obtained a previous characterization of the helicoid as the only ruled minimal surface in $\mathbb{R}^3$.

Assume that $a \neq 0$. Up to a rotation about the x_3-axis (i.e., the change $g \longmapsto e^{it} g$, where $a = |a| e^{it}$), we can suppose that $a \in]0, +\infty[$. Moreover, up to the change $z = \mu^{-1}\zeta$, where $\mu^4 - a^2 + 2r\mu^2 = 0$, (58) becomes

$$g'(\zeta)^2 = g(\zeta)(g(\zeta) - \lambda)(\lambda g(\zeta) + 1),$$

where $\lambda = a/\mu^2$. Up to scaling and rigid motions, this surface corresponds to a piece of R_λ. ■

The Meeks-Pérez-Ros uniqueness theorem asserts:

Theorem 6.17 (Meeks, Pérez, Ros [64]). *Let M be a properly embedded minimal surface in $\mathbb{R}^3/T$ of genus one and a finite number of planar ends, T being a nontrivial translation. Then, M is a quotient of a Riemann example.*

Sketch of the Proof: Let $\mathcal{S}$ denote the space of properly embedded minimal (oriented) tori in a quotient of $\mathbb{R}^3$ by a translation T, which depends on the surface, with $2n$ horizontal planar (ordered) ends. In $\mathcal{S}$ we consider the uniform topology on compact subsets of $\mathbb{R}^3$. By the maximum principle at infinity (Theorem 1.6), the ends are separated by a positive distance. Furthermore, embeddedness insures that, up to a rotation, the normal limit vector at the ends are $(0,0,\pm1)$ and they alternate from one end to the next. Note that if we rotate around the x_3-axis a $M \in \mathcal{S}$, we get a different element of $\mathcal{S}$. The allowed orders for the ends with normal vector $(0,0,-1)$, $P_1,\dots,P_n$, and for the ends with normal vector $(0,0,1)$, $Q_1,\dots,Q_n$, will be those in which the list $(P_1,Q_1,P_2,Q_2,\dots,P_n,Q_n)$ corresponds to consecutive ends in the quotient space. We will indentify in $\mathcal{S}$ two surfaces which differ by a translation that preserves both orientation and the order of the above list of ends.

A surface $M \in \mathcal{S}$ cuts transversally any horizontal plane nonasymptotic to its ends in a compact Jordan curve γ. We will orient γ in such a way that $\mathrm{Flux}(\gamma)$ has positive third coordinate. As the flux vanishes around the ends (they are planar ends), it follows that $\mathrm{Flux}(\gamma)$ does not depend on the height of the plane. Results in [**79**] say us that $\mathrm{Flux}(\gamma) = (\mathrm{Flux}(\gamma)_1, \mathrm{Flux}(\gamma)_2, \mathrm{Flux}(\gamma)_3)$ is not vertical. We will rescale our surfaces so that $\mathrm{Flux}(\gamma)_3 = 1$. We define the *Flux map*:

$$F : \mathcal{S} \longrightarrow \mathbb{R}^2 - \{0\}$$
$$F(M) = (\mathrm{Flux}(\gamma)_1, \mathrm{Flux}(\gamma)_2)\,.$$

Denote by $\mathcal{R}$ the subset of $\mathcal{S}$ consisting of the Riemann examples and their rotations around the x_3-axis. The set $\mathcal{R}$ is open and closed in $\mathcal{S}$. Indeed, it is proved in [**75**] that any small deformation of a Riemann example must be another Riemann example. This gives the openness of $\mathcal{R}$. Remember that Riemann examples are characterized by the fact of being foliated by circles and lines in horizontal planes (Theorem 6.16). So, if a sequence of Riemann examples converges to a surface $M \in \mathcal{S}$, then horizontal sections on M will de circles or lines and, thus, $M \in \mathcal{R}$.

The theorem is a consequence of the following three facts:

1. The map F is proper.
2. The map F is open.
3. There exists $\epsilon > 0$ such that if $M \in \mathcal{S}$ satisfies $|F(M)| < \epsilon$, then $M \in \mathcal{R}$.

The proof of these assertions can be found in [**64**].

Indeed, suppose $\mathcal{S}' \overset{\text{def}}{=} \mathcal{S} - \mathcal{R} \neq \emptyset$. As $\mathcal{S}'$ is open and closed in $\mathcal{S}$ and F is proper and open, it follows that its restriction to $\mathcal{S}'$ is also an open and proper map. So, $F(\mathcal{S}') = \mathbb{R}^2 - \{0\}$. This contradicts assertion 3, and completes the proof. ■

This theorem has the following consequences:

Theorem 6.18 (Meeks, Pérez, Ros [64]). *If M is a properly embedded periodic minimal planar domain in $\mathbb{R}^3$ with two limit ends, then M is one of the Riemann examples.*

Proof: Since every periodic minimal surface with more than one end has a top and bottom limit end, it follows that the middle ends are simple ends, which in the case of finite genus means annular ends (see [**21**]). The structure theorem in [**6**] implies the existence of a nontrivial screw motion or a translation Λ which preserves the surface and such that the quotient surface $M/\langle\Lambda\rangle$ has genus one and finitely many planar parallel ends. In particular, $M/\langle\Lambda\rangle$ has finite total curvature. By a result of Pérez and Ros [**79**] we have that Λ must be a translation. From Theorem 6.17 we conclude the proof. ■

Theorem 6.19 (Meeks, Pérez, Ros [64]). *Let M be a properly embedded minimal surface in $\mathbb{R}^3$ with genus zero. If the symmetry group of M is infinite, then M is one the following surfaces: a plane, a catenoid, a helicoid or a Riemann example.*

Proof: Recently, Kusner, Meeks and Rosenberg have proved that an embedded genus zero minimal surface that is not the plane or the helicoid must have exactly two limit ends. Then, the theorem is a consequence of Theorem 6.18. ■

This theorem has the following corollary:

Corollary 6.20 (Meeks, Pérez, Ros [64]). *A properly embedded minimal surface of $\mathbb{R}^3$ with genus zero has infinite symmetry group if and only if it is foliated by circles and/or lines in parallel planes.*

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