A PROOF OF THE WEAK (1,1) INEQUALITY FOR SINGULAR INTEGRALS WITH NON DOUBLING MEASURES BASED ON A CALDERÓN-ZYGMUND DECOMPOSITION

Xavier Tolsa

Abstract

Given a doubling measure μ on $\mathbb{R}^d$, it is a classical result of harmonic analysis that Calderón-Zygmund operators which are bounded in $L^2(\mu)$ are also of weak type $(1,1)$. Recently it has been shown that the same result holds if one substitutes the doubling condition on μ by a mild growth condition on μ. In this paper another proof of this result is given. The proof is very close in spirit to the classical argument for doubling measures and it is based on a new Calderón-Zygmund decomposition adapted to the non doubling situation.

1. Introduction

Let μ be a positive Radon measure on $\mathbb{R}^d$ satisfying the growth condition

$$(1.1) \qquad \mu(B(x,r)) \leq C_0\, r^n \quad \text{for all } x \in \mathbb{R}^d,\, r > 0,$$

where n is some fixed number with $0 < n \leq d$. We do *not* assume that μ is doubling [μ is said to be doubling if there exists some constant C such that $\mu(B(x,2r)) \leq C\,\mu(B(x,r))$ for all $x \in \text{supp}(\mu)$, $r > 0$]. Let us remark that the doubling condition on the underlying measure μ on $\mathbb{R}^d$ is an essential assumption in most results of classical Calderón-Zygmund theory. However, recently it has been shown that a big part of the classical theory remains valid if the doubling assumption on μ is

substituted by the size condition (1.1) (see for example the references cited at the end of the paper).

In this note we will prove that Calderón-Zygmund operators (CZO's) which are bounded in $L^2(\mu)$ are also of weak type $(1,1)$, as in the usual doubling situation. This result has already been proved in [**To1**] in the particular case of the Cauchy integral operator, and by Nazarov, Treil and Volberg [**NTV2**] in the general case. The proof that we will show here is different from the one of [**NTV2**] (and also from the one of [**To1**], of course) and it is closer in spirit to the classical proof of the corresponding result for doubling measures. The basic tool for the proof is a decomposition of Calderón-Zygmund type for functions in $L^1(\mu)$ obtained in [**To4**].

Our purpose in writing this paper is not only to obtain another proof in the non doubling situation of the basic result that CZO's bounded in $L^2(\mu)$ are of weak type $(1,1)$, but to show that the Calderón-Zygmund decompositon of [**To4**] is a good substitute of its classical doubling version.

Let us introduce some notation and definitions. A kernel $k(\cdot,\cdot)$ from $L^1_{\mathrm{loc}}((\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(x,y) : x = y\})$ is called a Calderón-Zygmund kernel if

1. $|k(x,y)| \leq \dfrac{C}{|x-y|^n}$,

2. there exists $0 < \delta \leq 1$ such that

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}}$$

if $|x-x'| \leq |x-y|/2$.

Throughout all the paper we will assume that μ is a Radon measure on $\mathbb{R}^d$ satisfying (1.1). The CZO associated to the kernel $k(\cdot,\cdot)$ and the measure μ is defined (at least, formally) as

$$Tf(x) = \int k(x,y)\,f(y)\,d\mu(y).$$

The above integral may be not convergent for many functions f because $k(x,y)$ may have a singularity for $x = y$. For this reason, one introduces the truncated operators T_ε, $\varepsilon > 0$:

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x,y)\,f(y)\,d\mu(y),$$

and then one says that T is bounded in $L^p(\mu)$ if the operators T_ε are bounded in $L^p(\mu)$ uniformly on $\varepsilon > 0$. It is said that T is bounded from

$L^1(\mu)$ into $L^{1,\infty}(\mu)$ (or of weak type $(1,1)$) if

$$\mu\{x : |T_\varepsilon f(x)| > \lambda\} \leq C\frac{\|f\|_{L^1(\mu)}}{\lambda}$$

for all $f \in L^1(\mu)$, uniformly on $\varepsilon > 0$. Also, T is bounded from $M(\mathbb{C})$ (the space of complex Radon measures) into $L^{1,\infty}(\mu)$ if

$$\mu\{x : |T_\varepsilon \nu(x)| > \lambda\} \leq C\frac{\|\nu\|}{\lambda}$$

for all $\nu \in M(\mathbb{C})$, uniformly on $\varepsilon > 0$. In the last inequality, $T_\varepsilon \nu(x)$ stands for $\int_{|x-y|>\varepsilon} k(x,y)\,d\nu(y)$ and $\|\nu\| \equiv |\nu|(\mathbb{R}^d)$.

The result that we will prove in this note is the following.

Theorem 1.1. *Let μ be a Radon measure on $\mathbb{R}^d$ satisfying the growth condition (1.1). If T is a Calderón-Zygmund operator which is bounded in $L^2(\mu)$, then it is also bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$. In particular, it is of weak type $(1,1)$.*

2. The proof

First we will introduce some additional notation and terminology. As usual, the letter C will denote a constant which may change its value from one occurrence to another. Constants with subscripts, such as C_0, do not change in different occurrences.

By a cube $Q \subset R^d$ we mean a closed cube with sides parallel to the axes. We denote its side length by $\ell(Q)$ and its center by x_Q. Given $\alpha > 1$ and $\beta > \alpha^n$, we say that Q is (α, β)-doubling if $\mu(\alpha Q) \leq \beta\,\mu(Q)$, where αQ is the cube concentric with Q with side length $\alpha\,\ell(Q)$. For definiteness, if α and β are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube.

Before proving Theorem 1.1 we state some remarks about the existence of doubling cubes.

Remark 2.1. Because μ satisfies the growth condition (1.1), there are a lot of "big" doubling cubes. To be precise, given any point $x \in \mathrm{supp}(\mu)$ and $c > 0$, there exists some (α, β)-doubling cube Q centered at x with $l(Q) \geq c$. This follows easily from (1.1) and the fact that $\beta > \alpha^n$. Indeed, if there are no doubling cubes centered at x with $l(Q) \geq c$, then $\mu(\alpha^m Q) > \beta^m \mu(Q)$ for each m, and letting $m \to \infty$ one sees that (1.1) cannot hold.

Remark 2.2. There are a lot of "small" doubling cubes too: if $\beta > \alpha^d$, then for μ-a.e. $x \in \mathbb{R}^d$ there exists a sequence of (α, β)-doubling

cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$. This is a property that any Radon measure on $\mathbb{R}^d$ satisfies (the growth condition (1.1) is not necessary in this argument). The proof is an easy exercise on geometric measure theory that is left for the reader.

Observe that, by the Lebesgue differentiation theorem, for μ-almost all $x \in \mathbb{R}^d$ one can find a sequence of $(2, 2^{d+1})$-doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ such that

$$\lim_{k \to \infty} \frac{1}{\mu(Q_k)} \int_{Q_k} f \, d\mu = f(x).$$

As a consequence, for any fixed $\lambda > 0$, for μ-almost all $x \in \mathbb{R}^d$ such that $|f(x)| > \lambda$, there exists a sequence of cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ such that

$$\limsup_{k \to \infty} \frac{1}{\mu(2Q_k)} \int_{Q_k} |f| \, d\mu > \frac{\lambda}{2^{d+1}}.$$

In the following lemma we will prove an easy but essential estimate which will be used below. This result has already appeared in previous works ([**DM**], [**NTV2**]) and it plays a basic role in [**To2**] and [**To4**] too.

Lemma 2.3. *If $Q \subset R$ are concentric cubes such that there are no (α, β)-doubling cubes (with $\beta > \alpha^n$) of the form $\alpha^k Q$, $k \geq 0$, with $Q \subset \alpha^k Q \subset R$, then,*

$$\int_{R \setminus Q} \frac{1}{|x - x_Q|^n} \, d\mu(x) \leq C_1,$$

where C_1 depends only on α, β, n, d and C_0.

Proof: Let N be the least integer such that $R \subset \alpha^N Q$. For $0 \leq k \leq N$ we have $\mu(\alpha^k Q) \leq \mu(\alpha^N Q)/\beta^{N-k}$. Then,

$$\begin{aligned}
\int_{R \setminus Q} \frac{1}{|x - x_Q|^n} \, d\mu(x) &\leq \sum_{k=1}^{N} \int_{\alpha^k Q \setminus \alpha^{k-1} Q} \frac{1}{|x - x_Q|^n} \, d\mu(x) \\
&\leq C \sum_{k=1}^{N} \frac{\mu(\alpha^k Q)}{\ell(\alpha^k Q)^n} \\
&\leq C \sum_{k=1}^{N} \frac{\beta^{k-N} \mu(\alpha^N Q)}{\alpha^{(k-N)n} \ell(\alpha^N Q)^n} \\
&\leq C \frac{\mu(\alpha^N Q)}{\ell(\alpha^N Q)^n} \sum_{j=0}^{\infty} \left(\frac{\alpha^n}{\beta} \right)^j \leq C. \qquad \square
\end{aligned}$$

The Calderón-Zygmund decomposition mentioned above has been obtained in Lemma 7.3 of [**To4**] and in that paper it has been used to show that if a linear operator is bounded from a suitable space of type H^1 into $L^1(\mu)$ and from $L^\infty(\mu)$ into a space of type BMO, then it is bounded in $L^p(\mu)$. We will use a slight variant of this decompositon to prove Theorem 1.1. Let us state the result that we need in detail.

Lemma 2.4 (Calderón-Zygmund decomposition). *Assume that μ satisfies (1.1). For any $f \in L^1(\mu)$ and any $\lambda > 0$ (with $\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$ if $\|\mu\| < \infty$) we have:*

(a) *There exists a family of almost disjoint cubes $\{Q_i\}_i$ (that is, $\sum_i \chi_{Q_i} \le C$) such that*

$$\frac{1}{\mu(2Q_i)} \int_{Q_i} |f| \, d\mu > \frac{\lambda}{2^{d+1}}, \tag{2.1}$$

$$\frac{1}{\mu(2\eta Q_i)} \int_{\eta Q_i} |f| \, d\mu \le \frac{\lambda}{2^{d+1}} \quad \text{for } \eta > 2, \tag{2.2}$$

$$|f| \le \lambda \quad a.e. \ (\mu) \ on \ \mathbb{R}^d \setminus \bigcup_i Q_i. \tag{2.3}$$

(b) *For each i, let R_i be a $(6, 6^{n+1})$-doubling cube concentric with Q_i, with $l(R_i) > 4l(Q_i)$ and denote $w_i = \frac{\chi_{Q_i}}{\sum_k \chi_{Q_k}}$. Then, there exists a family of functions φ_i with $\mathrm{supp}(\varphi_i) \subset R_i$ and with constant sign satisfying*

$$\int \varphi_i \, d\mu = \int_{Q_i} f \, w_i \, d\mu, \tag{2.4}$$

$$\sum_i |\varphi_i| \le B \, \lambda \tag{2.5}$$

(where B is some constant), and

$$\|\varphi_i\|_{L^\infty(\mu)} \mu(R_i) \le C \int_{Q_i} |f| \, d\mu. \tag{2.6}$$

Let us remark that other related decompositons with non doubling measures have been obtained in [**NTV2**] and [**MMNO**]. However, these results are not suitable for our purposes.

Although the proof of the lemma can be found in [**To4**], for the reader's convenience we have included it in the last section of the present paper.

Proof of Theorem 1.1: We will show that T is of weak type $(1,1)$. By similar arguments, one gets that T is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$. In this case, one has to use a version of the Calderón-Zygmund decomposition in the lemma above suitable for complex measures (see the end of the proof for more details).

Let $f \in L^1(\mu)$ and $\lambda > 0$. It is straightforward to check that we may assume $\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$. Let $\{Q_i\}_i$ be the almost disjoint family of cubes of Lemma 2.4. Let R_i be the smallest $(6, 6^{n+1})$-doubling cube of the form $6^k Q_i$, $k \geq 1$. Then we can write $f = g + b$, with

$$g = f \chi_{\mathbb{R}^d \setminus \bigcup_i Q_i} + \sum_i \varphi_i$$

and

$$b = \sum_i b_i := \sum_i (w_i f - \varphi_i),$$

where the functions φ_i satisfy (2.4), (2.5), (2.6) and $w_i = \frac{\chi_{Q_i}}{\sum_k \chi_{Q_k}}$.

By (2.1) we have

$$\mu\left(\bigcup_i 2Q_i\right) \leq \frac{C}{\lambda}\sum_i \int_{Q_i}|f|\,d\mu \leq \frac{C}{\lambda}\int |f|\,d\mu.$$

So we have to show that

$$(2.7) \qquad \mu\left\{x \in \mathbb{R}^d \setminus \bigcup_i 2Q_i : |T_\varepsilon f(x)| > \lambda\right\} \leq \frac{C}{\lambda}\int |f|\,d\mu.$$

Since $\int b_i\,d\mu = 0$, $\mathrm{supp}(b_i) \subset R_i$ and $\|b_i\|_{L^1(\mu)} \leq C\int_{Q_i}|f|\,d\mu$, using condition 2 in the definition of a Calderón-Zygmund kernel (which implies Hörmander's condition), we get

$$\int_{\mathbb{R}^d \setminus 2R_i}|T_\varepsilon b_i|\,d\mu \leq C\int |b_i|\,d\mu \leq C\int_{Q_i}|f|\,d\mu.$$

Let us see that

$$(2.8) \qquad \int_{2R_i \setminus 2Q_i}|T_\varepsilon b_i|\,d\mu \leq C\int_{Q_i}|f|\,d\mu$$

too. On the one hand, by (2.6) and using the $L^2(\mu)$ boundedness of T and that R_i is $(6, 6^{n+1})$-doubling we get

$$\int_{2R_i} |T_\varepsilon \varphi_i| \, d\mu \leq \left(\int_{2R_i} |T_\varepsilon \varphi_i|^2 \, d\mu \right)^{1/2} \mu(2R_i)^{1/2}$$

$$\leq C \left(\int |\varphi_i|^2 \, d\mu \right)^{1/2} \mu(R_i)^{1/2}$$

$$\leq C \int_{Q_i} |f| \, d\mu.$$

On the other hand, since $\operatorname{supp}(w_i f) \subset Q_i$, if $x \in 2R_i \setminus 2Q_i$, then $|T_\varepsilon(w_i f)(x)| \leq C \int_{Q_i} |f| \, d\mu / |x - x_{Q_i}|^n$, and so

$$\int_{2R_i \setminus 2Q_i} |T_\varepsilon(w_i f)| \, d\mu \leq C \int_{2R_i \setminus 2Q_i} \frac{1}{|x - x_{Q_i}|^n} \, d\mu(x) \times \int_{Q_i} |f| \, d\mu.$$

By Lemma 2.3, the first integral on the right hand side is bounded by some constant independent of Q_i and R_i, since there are no $(6, 6^{n+1})$-doubling cubes of the form $6^k Q_i$ between $6 Q_i$ and R_i. Therefore, (2.8) holds.

Then we have

$$\int_{\mathbb{R}^d \setminus \bigcup_k 2Q_k} |T_\varepsilon b| \, d\mu \leq \sum_i \int_{\mathbb{R}^d \setminus \bigcup_k 2Q_k} |T_\varepsilon b_i| \, d\mu$$

$$\leq C \sum_i \int_{Q_i} |f| \, d\mu \leq C \int |f| \, d\mu.$$

Therefore,

$$(2.9) \qquad \mu \left\{ x \in \mathbb{R}^d \setminus \bigcup_i 2Q_i : |T_\varepsilon b(x)| > \lambda \right\} \leq \frac{C}{\lambda} \int |f| \, d\mu.$$

The corresponding integral for the function g is easier to estimate. Taking into account that $|g| \leq C \lambda$, we get

$$(2.10) \qquad \mu \left\{ x \in \mathbb{R}^d \setminus \bigcup_i 2Q_i : |T_\varepsilon g(x)| > \lambda \right\} \leq \frac{C}{\lambda^2} \int |g|^2 \, d\mu$$

$$\leq \frac{C}{\lambda} \int |g| \, d\mu.$$

Also, we have

$$\int |g|\, d\mu \le \int_{\mathbb{R}^d \setminus \bigcup_i Q_i} |f|\, d\mu + \sum_i \int |\varphi_i|\, d\mu$$

$$\le \int |f|\, d\mu + \sum_i \int_{Q_i} |f|\, d\mu \le C \int |f|\, d\mu.$$

Now, by (2.9) and (2.10) we get (2.7).

If we want to show that T is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$, then in Lemma 2.4 and in the arguments above $f\, d\mu$ must be substituted by $d\nu$, with $\nu \in M(\mathbb{C})$, and $|f|\, d\mu$ by $d|\nu|$. Also, condition (2.3) of Lemma 2.4 should be stated as "On $\mathbb{R}^d \setminus \bigcup_i Q_i$, ν is absolutely continuous with respect to μ, that is $\nu = f\, d\nu$, and moreover $|f(x)| \le \lambda$ a.e. (μ) $x \in \mathbb{R}^d \setminus \bigcup_i Q_i$". With other minor changes, the arguments and estimates above work in this situation too. $\square$

3. Proof of Lemma 2.4

(a) Taking into account Remark 2.2, for μ-almost all $x \in \mathbb{R}^d$ such that $|f(x)| > \lambda$, there exists some cube Q_x satisfying

$$(3.1) \qquad \frac{1}{\mu(2Q_x)} \int_{Q_x} |f|\, d\mu > \frac{\lambda}{2^{d+1}}$$

and such that if Q'_x is centered at x with $l(Q'_x) > 2l(Q_x)$, then

$$\frac{1}{\mu(2Q'_x)} \int_{Q'_x} |f|\, d\mu \le \frac{\lambda}{2^{d+1}}.$$

Now we can apply Besicovich's covering theorem (see Remark 3.1 below) to get an almost disjoint subfamily of cubes $\{Q_i\}_i \subset \{Q_x\}_x$ satisfying (2.1), (2.2) and (2.3).

(b) Assume first that the family of cubes $\{Q_i\}_i$ is finite. Then we may suppose that this family of cubes is ordered in such a way that the sizes of the cubes R_i are non decreasing (i.e. $l(R_{i+1}) \ge l(R_i)$). The functions φ_i that we will construct will be of the form $\varphi_i = \alpha_i \chi_{A_i}$, with $\alpha_i \in \mathbb{R}$ and $A_i \subset R_i$. We set $A_1 = R_1$ and $\varphi_1 = \alpha_1 \chi_{R_1}$, where the constant α_1 is chosen so that $\int_{Q_1} f w_1\, d\mu = \int \varphi_1\, d\mu$.

Suppose that $\varphi_1, \ldots, \varphi_{k-1}$ have been constructed, satisfy (2.4) and

$$\sum_{i=1}^{k-1} |\varphi_i| \le B\,\lambda,$$

where B is some constant which will be fixed below.

Let $R_{s_1}, \ldots, R_{s_m}$ be the subfamily of $R_1, \ldots, R_{k-1}$ such that $R_{s_j} \cap R_k \neq \varnothing$. As $l(R_{s_j}) \leq l(R_k)$ (because of the non decreasing sizes of R_i), we have $R_{s_j} \subset 3R_k$. Taking into account that for $i = 1, \ldots, k-1$

$$\int |\varphi_i| \, d\mu \leq \int_{Q_i} |f| \, d\mu$$

by (2.4), and using that R_k is $(6, 6^{n+1})$-doubling and (2.2), we get

$$\sum_j \int |\varphi_{s_j}| \, d\mu \leq \sum_j \int_{Q_{s_j}} |f| \, d\mu$$

$$\leq C \int_{3R_k} |f| \, d\mu \leq C\lambda \mu(6R_k) \leq C_2 \lambda \, \mu(R_k).$$

Therefore,

$$\mu\left\{\sum_j |\varphi_{s_j}| > 2C_2\lambda\right\} \leq \frac{\mu(R_k)}{2}.$$

So we set

$$A_k = R_k \cap \left\{\sum_j |\varphi_{s_j}| \leq 2C_2\lambda\right\},$$

and then $\mu(A_k) \geq \mu(R_k)/2$.

The constant α_k is chosen so that for $\varphi_k = \alpha_k \, \chi_{A_k}$ we have $\int \varphi_k \, d\mu = \int_{Q_k} f \, w_k \, d\mu$. Then we obtain

$$|\alpha_k| \leq \frac{1}{\mu(A_k)} \int_{Q_k} |f| \, d\mu \leq \frac{2}{\mu(R_k)} \int_{\frac{1}{2}R_k} |f| \, d\mu \leq C_3\lambda$$

(this calculation also applies to $k = 1$). Thus,

$$|\varphi_k| + \sum_j |\varphi_{s_j}| \leq (2C_2 + C_3)\,\lambda.$$

If we choose $B = 2C_2 + C_3$, (2.5) follows.

Now it is easy to check that (2.6) also holds. Indeed we have

$$\|\varphi_i\|_{L^\infty(\mu)}\,\mu(R_i) \leq C \, |\alpha_i|\,\mu(A_i) = C \left|\int_{Q_i} f \, w_i \, d\mu\right| \leq C \int_{Q_i} |f| \, d\mu.$$

Suppose now that the collection of cubes $\{Q_i\}_i$ is not finite. For each fixed N we consider the family of cubes $\{Q_i\}_{1 \leq i \leq N}$. Then, as above, we construct functions $\varphi_1^N, \ldots, \varphi_N^N$ with $\operatorname{supp}(\varphi_i^N) \subset R_i$ satisfying

$$\int \varphi_i^N \, d\mu = \int_{Q_i} f \, w_i \, d\mu,$$

$$\sum_{i=1}^N |\varphi_i^N| \leq B \lambda$$

and

$$\|\varphi_i^N\|_{L^\infty(\mu)} \, \mu(R_i) \leq C \int_{Q_i} |f| \, d\mu.$$

Notice that the sign of φ_i^N equals the sign of $\int f \, w_i \, d\mu$ and so it does not depend on N.

Then there is a subsequence $\{\varphi_1^k\}_{k \in I_1}$ which is convergent in the weak $*$ topology of $L^\infty(\mu)$ to some function $\varphi_1 \in L^\infty(\mu)$. Now we can consider a subsequence $\{\varphi_2^k\}_{k \in I_2}$ with $I_2 \subset I_1$ which is also convergent in the weak $*$ topology of $L^\infty(\mu)$ to some function $\varphi_2 \in L^\infty(\mu)$. In general, for each j we consider a subsequence $\{\varphi_j^k\}_{k \in I_j}$ with $I_j \subset I_{j-1}$ that converges in the weak $*$ topology of $L^\infty(\mu)$ to some function $\varphi_j \in L^\infty(\mu)$. It is easily checked that the functions φ_j satisfy the required properties. $\quad\square$

Remark 3.1. Recall that Besicovich's covering theorem asserts that if $\Omega \subset \mathbb{R}^d$ is a *bounded* set and for each $x \in \Omega$ there is a cube Q_x centered at x, then there exists a family of cubes $\{Q_{x_i}\}_i$ with finite overlap, that is $\sum_i \chi_{Q_i} \leq C$, which covers Ω.

In (a) of the preceeding proof we have applied Besicovich's covering theorem to $\Omega = \{x : |f(x)| > \lambda\}$. However this set may be unbounded, and the boundedness property is a necessary assumption in Besicovich's theorem (example: take $\Omega = [0, +\infty) \subset \mathbb{R}$ and consider $Q_x = [0, 2x]$ for all $x \in \Omega$).

We can solve this problem using different arguments. One possibility is to consider for each $r > 0$ the set $\Omega_r = \{x : |x| \leq r, |f(x)| > \lambda\}$ and to apply Besicovich's covering theorem to Ω_r. With the same arguments as above, we can decompose $f = g + b$, with $|g| \leq \lambda$ only on Ω_r and b as above. Then the proof of Theorem 1.1 can be modified to show that for any fixed constants $\lambda, R > 0$ one has

$$\mu\{x \in B(0, R) : |T_\varepsilon f(x)| > \lambda\} \leq C \frac{\|f\|_{L^1(\mu)}}{\lambda}.$$

However we prefer the following solution. We are interested in showing that the Calderón-Zygmund decomposition of Lemma 2.4 works also without assuming $\Omega = \{x : |f(x)| > \lambda\}$ bounded. Let us sketch the argument. Consider a cube Q_0 centered at 0 big enough so that

$$2^{d+1} \|f\|_{L^1(\mu)}/\mu(Q_0) < \lambda.$$

So for any cube Q containing Q_0 we will have

$$(3.2) \qquad 2^{d+1} \|f\|_{L^1(\mu)}/\mu(Q) < \lambda.$$

For $m \geq 0$ we set $Q_m := \left(\frac{5}{4}\right)^m Q_0$. For each m we can apply Besicovich's covering theorem to the annulus $Q_m \setminus Q_{m-1}$ (we take $Q_{-1} := \varnothing$), with cubes Q_x centered at $x \in \operatorname{supp}(\mu) \cap (Q_m \setminus Q_{m-1})$ as in (a) of the proof above, satisfying (3.1).

In this argument we have to be careful with the overlapping among the cubes belonging to coverings of different annuli. Indeed, there exist some fixed constants N and N' such that if $m \geq N'$, for $x \in \operatorname{supp}(\mu) \cap (Q_m \setminus Q_{m-1})$ we have

$$(3.3) \qquad Q_x \subset Q_{m+N} \setminus Q_{m-N}.$$

Otherwise, it is easily seen that $\ell(Q_x) > \frac{3}{4}\ell(Q_m)$, choosing N big enough. It follows that $Q_0 \subset 2Q_x$ since $\ell(Q_0) \ll \ell(Q_m)$ for N' big enough too. This cannot happen because then $2Q_x$ satisfies (3.2), which contradicts (3.1).

Because of (3.3), the covering made up of squares belonging to the Besicovich coverings of different annuli $Q_m \setminus Q_{m-1}$, $m \geq 0$, will have finite overlap.

Notice that in this argument, it is essential the fact that in (3.1) we are not dividing by $\mu(Q_x)$, but by $\mu(2Q_x)$.

References

[DM]   G. DAVID AND P. MATTILA, Removable sets for Lipschitz harmonic functions in the plane, *Rev. Mat. Iberoamericana* **16(1)** (2000), 137–215.

[GM]   J. GARCÍA CUERVA AND J. M. MARTELL, Weighted inequalities and vector-valued Calderón-Zygmund operators on non-homogeneous spaces, *Publ. Mat.* **44(2)** (2000), 613–640.

[MMNO] J. MATEU, P. MATTILA, A. NICOLAU AND J. OROBITG, *BMO* for nondoubling measures, *Duke Math. J.* **102(3)** (2000), 533–565.

[NTV1]   F. Nazarov, S. Treil and A. Volberg, Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces, *Internat. Math. Res. Notices* **15** (1997), 703–726.

[NTV2]   F. Nazarov, S. Treil and A. Volberg, Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces, *Internat. Math. Res. Notices* **9** (1998), 463–487.

[NTV3]   F. Nazarov, S. Treil and A. Volberg, *Tb*-theorem on non-homogeneous spaces, Preprint (1999).

[NTV4]   F. Nazarov, S. Treil and A. Volberg, Accretive *Tb*-systems on non-homogeneous spaces, Preprint (1999).

[OP]   J. Orobitg and C. Pérez, A_p weights for non doubling measures in $\mathbb{R}^n$ and applications, *Trans. Amer. Math. Soc.* (to appear).

[To1]   X. Tolsa, L^2-boundedness of the Cauchy integral operator for continuous measures, *Duke Math. J.* **98(2)** (1999), 269–304.

[To2]   X. Tolsa, Cotlar's inequality without the doubling condition and existence of principal values for the Cauchy integral of measures, *J. Reine Angew. Math.* **502** (1998), 199–235.

[To3]   X. Tolsa, A $T(1)$ theorem for non doubling measures with atoms, *Proc. London Math. Soc.* **82** (2001), 195–228.

[To4]   X. Tolsa, *BMO*, H^1 and Calderón-zygmund operators for non doubling measures, *Math. Ann.* (to appear).

[Ve]   J. Verdera, On the $T(1)$-theorem for the Cauchy integral, *Ark. Mat.* **38** (2000), 183–199.

Department of Mathematics
Chalmers
412 96 Göteborg
Sweden

Current address:
Département de Mathématique
Université de Paris-Sud
Bât. 425
91405 Orsay Cedex
France
E-mail address: `xavier.tolsa@math.u-psud.fr`