

## **$L^p$ BOUNDS FOR RIESZ TRANSFORMS AND SQUARE ROOTS ASSOCIATED TO SECOND ORDER ELLIPTIC OPERATORS**

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*Abstract*

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We consider the Riesz transforms  $\nabla L^{-1/2}$ , where  $L \equiv -\operatorname{div} A(x) \nabla$ , and  $A$  is an accretive,  $n \times n$  matrix with bounded measurable complex entries, defined on  $\mathbb{R}^n$ . We establish boundedness of these operators on  $L^p(\mathbb{R}^n)$ , for the range  $p_n < p \leq 2$ , where  $p_n = 2n/(n+2)$ ,  $n \geq 2$ , and we obtain a weak-type estimate at the endpoint  $p_n$ . The case  $p = 2$  was already known: it is equivalent to the solution of the square root problem of T. Kato.

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### **1. Introduction**

Let  $A = A(x) = (a_{j,k}(x))_{j,k}$  be an  $n \times n$  matrix where the coefficients  $a_{j,k}$  are complex-valued  $L^\infty(\mathbb{R}^n)$  functions. We assume that this matrix satisfies the following ellipticity (or “accretivity”) condition: there exist  $0 < \lambda \leq \Lambda < \infty$  such that

$$\lambda|\xi|^2 \leq \operatorname{Re} A \xi \cdot \bar{\xi} \quad \text{and} \quad |A \xi \cdot \bar{\zeta}| \leq \Lambda |\xi| |\zeta|,$$

for all  $\xi, \zeta \in \mathbb{C}^n$ . We have used the notation  $\xi \cdot \zeta = \xi_1 \zeta_1 + \cdots + \xi_n \zeta_n$  and therefore  $\xi \cdot \bar{\zeta}$  is the usual inner product in  $\mathbb{C}^n$ . Note that then  $A \xi \cdot \bar{\zeta} = \sum_{j,k} a_{j,k}(x) \xi_k \bar{\zeta}_j$ . Associated with this matrix we define the second order divergence form operator

$$L f = -\operatorname{div}(A \nabla f),$$

which is understood in the standard weak sense by means of a sesquilinear form.

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The accretivity condition stated before allows one to define the square root  $L^{\frac{1}{2}} = \sqrt{L}$ . Recently P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and P. Tchamitchian have obtained an affirmative answer to the so-called Kato square root problem for elliptic differential operators (see [AHLMT]). Namely, they have shown the following:

**Theorem 1.1** ([AHLMT, Theorem 1.4]). *For any operator as before the domain of  $\sqrt{L}$  coincides with the Sobolev space  $H^1(\mathbb{R}^n)$  and  $\|\sqrt{L}f\|_2 \sim \|\nabla f\|_2$ .*

In [AT], assuming the result of Theorem 1.1, the authors obtain optimal  $L^p$  bounds for  $\sqrt{L}$  and for the associated Riesz transforms  $\nabla L^{-1/2}$ , under the extra hypothesis that one has a Gaussian upper bound for the heat kernel (along with “Nash-type” local Hölder continuity). In this paper, we consider the same problem without this extra hypothesis, i.e., we study the  $L^p$  boundedness of  $\nabla L^{-1/2}$  and  $\sqrt{L}$  for general second order elliptic operators  $L$  as above. Our results are new only in the case  $n \geq 3$ . Indeed, Gaussian bounds (and Nash’s estimates) always hold for the kernel of the semigroup  $e^{-tL}$  in dimensions 1 and 2 [AMT], in which case the result of [AT] (or also that of [DM]) applies. In fact, the argument of [DM] requires only the Gaussian upper bounds, and not the Hölder continuity. We remark that in the presence of Gaussian bounds, the boundedness on  $L^q$  for  $q \leq 2$  of the Riesz transforms associated to other operators (for example, the Laplace-Beltrami operator on a manifold) have been treated (see [CD]). For the operators under consideration in the present paper, it is known that the Gaussian bounds may fail in dimensions  $n \geq 5$  [AT, pp. 32–33] (also [MNP], where the example originates). Our main result says:

**Theorem 1.2.** *Let  $p_n = \frac{2n}{n+2}$ . Then the Riesz transform  $\nabla L^{-\frac{1}{2}}$  is of weak type  $(p_n, p_n)$  and thus bounded on  $L^q(\mathbb{R}^n)$  for  $p_n < q \leq 2$ .*

*Remarks.* We learned during the preparation of this manuscript that the  $L^q$  boundedness of the Riesz transforms  $\nabla L^{-\frac{1}{2}}$  on the same range of  $q$  has also been obtained independently by S. Blunck and P. C. Kuntzmann [BK1], by essentially the same method as ours. Moreover, they have applied this technique to related matters, including  $L^p$  estimates for Riesz transforms associated to higher order elliptic operators [BK1], as well as to the existence of an  $H^\infty$  functional calculus [BK2] in  $L^p$  spaces. We are grateful to Pascal Auscher and Alan McIntosh for bringing their work to our attention.

We should point out that  $p'_n = 2^*$  where  $2^* = \frac{2n}{n-2}$  is the Sobolev exponent of 2. On the other hand, we recall that, by a standard duality argument, the  $L^2$  estimate of the Riesz transform is equivalent to

$$\|\sqrt{L^*}f\|_2 \leq C\|\nabla f\|_2,$$

where  $L^*$  is the adjoint operator of  $L$  that satisfies its same properties. In view of this fact, the  $L^2$  estimate for the Riesz transform follows from Theorem 1.1 obtained in [AHLMT]. Let us also note that the problem is invariant with respect to the taking of adjoints, i.e.,  $L$  and  $L^*$  are operators of the same nature and if one of them satisfies the required hypotheses then the other one also does.

By using a standard duality argument, one has that from the boundedness of  $\nabla L^{-\frac{1}{2}}$  on some  $L^q(\mathbb{R}^n)$  it follows an  $L^{q'}(\mathbb{R}^n)$  domination of the square roots by the gradient. Then, as a consequence of our main result we get the following.

**Corollary 1.3.** *Let  $2 \leq q < \frac{2n}{n-2}$ . Then,*

$$\|\sqrt{L}f\|_q \leq C\|\nabla f\|_q.$$

The paper is organized as follows. In the next section we prove some technical estimates that will be used in the sequel. In Section 3 we prove our main theorem.

### 2. $L^2$ off-diagonal estimates

Given  $E, F$  two subsets of  $\mathbb{R}^n$ , we will denote by  $\text{dist}(E, F)$  the distance between them. We will use the notation  $\vec{f}$  for  $n$ -tuples of functions. The identity operator will be written as  $\mathcal{I}$ . We begin by stating the fundamental average decay estimates satisfied by the heat kernels associated to our operator  $L$ .

**Lemma 2.1.** *Let  $E$  and  $F$  be two closed sets of  $\mathbb{R}^n$ . Then, for all  $t > 0$ ,*

$$\|e^{-tL}f\|_{L^2(F)} \leq C e^{-\frac{\text{dist}(E,F)^2}{ct}} \|f\|_{L^2(E)}, \quad \text{supp } f \subset E$$

$$\|tLe^{-tL}f\|_{L^2(F)} \leq C e^{-\frac{\text{dist}(E,F)^2}{ct}} \|f\|_{L^2(E)}, \quad \text{supp } f \subset E$$

$$\|t^{\frac{1}{2}}\nabla e^{-tL}f\|_{L^2(F)} \leq C e^{-\frac{\text{dist}(E,F)^2}{ct}} \|f\|_{L^2(E)}, \quad \text{supp } f \subset E$$

$$\|t^{\frac{1}{2}}e^{-tL} \text{div } \vec{f}\|_{L^2(F)} \leq C e^{-\frac{\text{dist}(E,F)^2}{ct}} \|\vec{f}\|_{L^2(E)}, \quad \text{supp } \vec{f} \subset E$$

where  $c > 0$  depends only on  $\lambda$  and  $\Lambda$ , and  $C > 0$  on  $n, \lambda$  and  $\Lambda$ .

An analogous result for resolvent kernels  $(\mathcal{I} + t^2L)^{-1}$  is proved in [AHLMT, Lemma 2.1], via an integration by parts argument similar to the proof of Cacciopoli's inequality. The present lemma follows from the one for resolvent kernels by functional calculus. Alternatively, one can simply modify the integration by parts argument of [AHLMT], to obtain a direct proof of the present lemma, as in the proof of the parabolic Cacciopoli inequality. We omit the details.

**Lemma 2.2.** *Let  $m \geq 1$  be an integer (eventually, we shall choose  $m$  depending only upon  $n$ ). Let  $E$  and  $F$  be two closed sets of  $\mathbb{R}^n$ . Then,*

$$\left\| t^{\frac{1}{2}} \nabla L^{-\frac{1}{2}} (\mathcal{I} - e^{-tL})^m \operatorname{div} \vec{f} \right\|_{L^2(F)} \leq C \left( \frac{\operatorname{dist}(E, F)^2}{t} \right)^{-(m+\frac{1}{2})} \|\vec{f}\|_{L^2(E)},$$

$\operatorname{supp} \vec{f} \subset E$

and

$$\left\| t^{\frac{1}{2}} \nabla \left( \nabla L^{-\frac{1}{2}} (\mathcal{I} - e^{-tL})^m \right)^* \vec{f} \right\|_{L^2(F)} \leq C \left( \frac{\operatorname{dist}(E, F)^2}{t} \right)^{-(m+\frac{1}{2})} \|\vec{f}\|_{L^2(E)},$$

$\operatorname{supp} \vec{f} \subset E$

where  $C > 0$  depends only on  $n, m, \lambda, \Lambda$ , and

$$(1) \quad \left( \nabla L^{-\frac{1}{2}} (\mathcal{I} - e^{-tL})^m \right)^* \vec{f} = - \left( L^{-\frac{1}{2}} (\mathcal{I} - e^{-tL})^m \right)^* \operatorname{div} \vec{f}.$$

To prove this result we need to establish an auxiliary lemma that says that the composition of two operators that satisfy two  $L^2$  off-diagonal estimates as in Lemma 2.1 verifies a similar inequality.

**Lemma 2.3.** *Let  $\{A_t\}_{t>0}$  and  $\{B_t\}_{t>0}$  two families of linear operators. Assume that for all closed sets  $E, F$ , for all  $f$  such that  $\operatorname{supp} f \subset E$  and for all  $t > 0$  we have the following estimates*

$$\|A_t f\|_{L^2(F)} \leq C e^{-\frac{\operatorname{dist}(E, F)^2}{ct}} \|f\|_{L^2(E)}$$

and

$$\|B_t f\|_{L^2(F)} \leq C e^{-\frac{\operatorname{dist}(E, F)^2}{ct}} \|f\|_{L^2(E)}.$$

Then, for all  $t, s > 0$  we have

$$\|A_t B_s f\|_{L^2(F)} \leq C e^{-\frac{\operatorname{dist}(E, F)^2}{c \max\{t, s\}}} \|f\|_{L^2(E)}.$$

*Proof:* Set  $\rho = \text{dist}(E, F)$  and  $G = \{x : \text{dist}(x, F) < \rho/2\}$ . Then, it is clear that  $\text{dist}(E, \overline{G}) \geq \rho/2$  where we have written  $\overline{G}$  for the topological closure of the set  $G$ . Besides, by definition  $\text{dist}(\mathbb{R}^n \setminus G, F) \geq \rho/2$ . Then,

$$\begin{aligned} \|A_t(B_s f \cdot \chi_G)\|_{L^2(F)} &\leq \|A_t(B_s f \cdot \chi_G)\|_{L^2(\mathbb{R}^n)} \\ &\leq C \|B_s f\|_{L^2(\overline{G})} \\ &\leq C e^{-\frac{\text{dist}(\overline{G}, E)^2}{cs}} \|f\|_{L^2(E)} \\ &\leq C e^{-\frac{\rho^2}{cs}} \|f\|_{L^2(E)}. \end{aligned}$$

Note that in the second inequality we have used that  $A_t$  is uniformly bounded on  $L^2(\mathbb{R}^n)$ , fact that follows from the hypotheses on  $A_t$  by taking  $E = F = \mathbb{R}^n$ . The third inequality is just the  $L^2$  off-diagonal estimate assumed on  $B_s$ . On the other hand,

$$\begin{aligned} \|A_t(B_s f \cdot \chi_{\mathbb{R}^n \setminus G})\|_{L^2(F)} &\leq C e^{-\frac{\text{dist}(\mathbb{R}^n \setminus G, F)^2}{ct}} \|B_s f\|_{L^2(\mathbb{R}^n \setminus G)} \\ &\leq C e^{-\frac{\rho^2}{ct}} \|B_s f\|_{L^2(\mathbb{R}^n)} \\ &\leq C e^{-\frac{\rho^2}{ct}} \|f\|_{L^2(E)}, \end{aligned}$$

where the first inequality is a consequence of the  $L^2$  off-diagonal estimate for  $A_t$  and the last one holds because, as before,  $B_s$  is uniformly bounded on  $L^2(\mathbb{R}^n)$ . If we combine the two estimates we get:

$$\begin{aligned} \|A_t B_s f\|_{L^2(F)} &\leq \|A_t(B_s f \cdot \chi_G)\|_{L^2(F)} + \|A_t(B_s f \cdot \chi_{\mathbb{R}^n \setminus G})\|_{L^2(F)} \\ &\leq C \left( e^{-\frac{\rho^2}{cs}} + e^{-\frac{\rho^2}{ct}} \right) \|f\|_{L^2(E)} \\ &\leq C e^{-\frac{\text{dist}(E, F)^2}{c \max\{t, s\}}} \|f\|_{L^2(E)}. \end{aligned} \quad \square$$

Now we can give a proof of Lemma 2.2.

*Proof of Lemma 2.2:* We prove the first estimate since the other one follows by duality. We use the following representation of the Riesz transform:

$$\nabla L^{-\frac{1}{2}} h = \frac{1}{2\sqrt{\pi}} \int_0^\infty \nabla e^{-sL} h \frac{ds}{\sqrt{s}} = \frac{\sqrt{m+2}}{2\sqrt{\pi}} \int_0^\infty \nabla e^{-(m+2)sL} h \sqrt{s} \frac{ds}{s}$$

which leads to

$$\begin{aligned} \nabla L^{-\frac{1}{2}}(\mathcal{I}-e^{-tL})^m \operatorname{div} \vec{f} &= C \int_0^\infty \nabla e^{-sL} e^{-msL} (\mathcal{I}-e^{-tL})^m e^{-sL} \operatorname{div} \vec{f} \sqrt{s} \frac{ds}{s} \\ &= C \left( \int_0^t \cdots + \int_t^\infty \cdots \right) \\ &= C(I_t + II_t). \end{aligned}$$

Notice that we have used the commutation property of the semigroup. Now we study each operator separately. For the first one we have

$$(\mathcal{I} - e^{-tL})^m = \sum_{k=0}^m \binom{m}{k} (-1)^k e^{-ktL} = \mathcal{I} + \sum_{k=1}^m c_k e^{-ktL}$$

and then

$$\begin{aligned} I_t &= \int_0^t \nabla e^{-sL} e^{-msL} e^{-sL} \operatorname{div} \vec{f} \sqrt{s} \frac{ds}{s} \\ &\quad + \sum_{k=1}^m c_k \int_0^t \nabla e^{-\frac{kt}{2}L} e^{-(m+2)sL} e^{-\frac{kt}{2}L} \operatorname{div} \vec{f} \sqrt{s} \frac{ds}{s} \\ &= I_{t,0} + \sum_{k=1}^m c_k I_{t,k}. \end{aligned}$$

Then,

$$\begin{aligned} \|I_{t,0}\|_{L^2(F)} &\leq \int_0^t \|\nabla e^{-sL} e^{-msL} e^{-sL} \operatorname{div} \vec{f}\|_{L^2(F)} \sqrt{s} \frac{ds}{s} \\ &= \int_0^t \|(s^{\frac{1}{2}} \nabla e^{-sL}) \circ (e^{-msL}) \circ (s^{\frac{1}{2}} e^{-sL} \operatorname{div}) \vec{f}\|_{L^2(F)} s^{-\frac{1}{2}} \frac{ds}{s} \\ &\leq C \|\vec{f}\|_{L^2(E)} \int_0^t e^{-\frac{\operatorname{dist}(E,F)^2}{cm s}} s^{-\frac{1}{2}} \frac{ds}{s} \\ &= C \|\vec{f}\|_{L^2(E)} t^{-\frac{1}{2}} \int_1^\infty e^{-\frac{\operatorname{dist}(E,F)^2}{ct}} s^{\frac{1}{2}} \frac{ds}{s}, \end{aligned}$$

where we have used Lemma 2.3 and noted that every operator satisfies the corresponding  $L^2$  off-diagonal inequality due to Lemma 2.1. Now we

bound the integral:

$$\begin{aligned} \int_1^\infty e^{-\frac{\text{dist}(E,F)^2}{ct}} s^{\frac{1}{2}} \frac{ds}{s} &\leq C \int_1^\infty \left( \frac{\text{dist}(E,F)^2}{ct} s \right)^{-(m+\frac{1}{2})} s^{\frac{1}{2}} \frac{ds}{s} \\ &= C \left( \frac{\text{dist}(E,F)^2}{t} \right)^{-(m+\frac{1}{2})} \end{aligned}$$

since  $m \geq 1$ . Then we have obtained

$$\|I_{t,0}\|_{L^2(F)} \leq C t^{-\frac{1}{2}} \left( \frac{\text{dist}(E,F)^2}{t} \right)^{-(m+\frac{1}{2})} \|\vec{f}\|_{L^2(E)}.$$

Now, fix  $1 \leq k \leq m$ , then

$$\begin{aligned} \|I_{t,k}\|_{L^2(F)} &\leq \int_0^t \|\nabla e^{-\frac{kt}{2}L} e^{-(m+2)sL} e^{-\frac{kt}{2}L} \text{div } \vec{f}\|_{L^2(F)} \sqrt{s} \frac{ds}{s} \\ &= \int_0^t \left\| \left( \sqrt{\frac{kt}{2}} \nabla e^{-\frac{kt}{2}L} \right) \circ (e^{-(m+2)sL}) \right. \\ &\quad \left. \circ \left( \sqrt{\frac{kt}{2}} e^{-\frac{kt}{2}L} \text{div} \right) \vec{f} \right\|_{L^2(F)} \frac{2}{kt} \sqrt{s} \frac{ds}{s} \\ &\leq C \|\vec{f}\|_{L^2(E)} t^{-1} \int_0^t e^{-\frac{\text{dist}(E,F)^2}{c \max\{kt/2, (m+2)s\}}} \sqrt{s} \frac{ds}{s} \\ &\leq C \|\vec{f}\|_{L^2(E)} t^{-1} e^{-\frac{\text{dist}(E,F)^2}{ct}} \int_0^t \sqrt{s} \frac{ds}{s} \\ &\leq C t^{-\frac{1}{2}} e^{-\frac{\text{dist}(E,F)^2}{ct}} \|\vec{f}\|_{L^2(E)} \\ &\leq C t^{-\frac{1}{2}} \left( \frac{\text{dist}(E,F)^2}{t} \right)^{-(m+\frac{1}{2})} \|\vec{f}\|_{L^2(E)}. \end{aligned}$$

Let us observe that because of Lemma 2.1 in the composition of the operators above each of them verifies an  $L^2$  off-diagonal estimate. This fact allowed us to employ Lemma 2.3. We also used that  $1 \leq k \leq m$ .

Collecting this estimate and the one proved for  $I_{t,0}$  we get

$$\begin{aligned} \|I_t\|_{L^2(F)} &\leq \|I_{t,0}\|_{L^2(F)} + \sum_{k=1}^m |c_k| \|I_{t,k}\|_{L^2(F)} \\ &\leq C t^{-\frac{1}{2}} \left( \frac{\text{dist}(E, F)^2}{t} \right)^{-(m+\frac{1}{2})} \|\vec{f}\|_{L^2(E)}. \end{aligned}$$

Next, we proceed with the estimate of  $II_t$ .

$$\begin{aligned} &\|II_t\|_{L^2(F)} \\ &\leq C \int_t^\infty \|(\sqrt{s} \nabla e^{-sL}) \circ (e^{-sL} - e^{-(s+t)L})^m \circ (\sqrt{s} e^{-sL} \text{div}) \vec{f}\|_{L^2(F)} s^{-\frac{1}{2}} \frac{ds}{s}. \end{aligned}$$

We know that the first and the last operators satisfy an  $L^2$  off-diagonal estimate. We are going to show not only that the one in the middle verifies a similar estimate but also that we gain some extra decay  $(t/s)^m$ . Namely, let  $E, F$  be two closed sets and  $g$  such that  $\text{supp } g \subset E$ , then

$$\begin{aligned} \|(e^{-sL} - e^{-(s+t)L})g\|_{L^2(F)} &= \left\| - \int_0^t \frac{d}{dr} e^{-(s+r)L} g \, dr \right\|_{L^2(F)} \\ &\leq \int_0^t \|(s+r)L e^{-(s+r)L} g\|_{L^2(F)} \frac{dr}{s+r} \\ &\leq C \|g\|_{L^2(E)} \int_0^t e^{-\frac{\text{dist}(E,F)^2}{c(s+r)}} \frac{dr}{s+r} \\ &\leq C \frac{t}{s} e^{-\frac{\text{dist}(E,F)^2}{cs}} \|g\|_{L^2(E)}, \end{aligned}$$

where in the last step we used that  $t \leq s \leq s+r \leq s+t \leq 2s$ , and Lemma 2.1. Then we have proved that

$$\left\| \frac{s}{t} (e^{-sL} - e^{-(s+t)L})g \right\|_{L^2(F)} \leq C e^{-\frac{\text{dist}(E,F)^2}{cs}} \|g\|_{L^2(E)}$$

uniformly on  $t$ . Then this operator composed with itself  $m$  times will satisfy the same inequality in view of Lemma 2.3. We can use this



estimate to bound  $\|II_t\|_{L^2(F)}$  as follows:

$$\begin{aligned} \|II_t\|_{L^2(F)} &\leq C\|\vec{f}\|_{L^2(E)} \int_t^\infty e^{-\frac{\text{dist}(E,F)^2}{cs}} \left(\frac{t}{s}\right)^m s^{-\frac{1}{2}} \frac{ds}{s} \\ &= C\|\vec{f}\|_{L^2(E)} t^{-\frac{1}{2}} \int_0^1 e^{-\frac{\text{dist}(E,F)^2}{ct}} s^{m+\frac{1}{2}} \frac{ds}{s} \\ &\leq C\|\vec{f}\|_{L^2(E)} t^{-\frac{1}{2}} \int_0^\infty e^{-\frac{\text{dist}(E,F)^2}{ct}} s^{m+\frac{1}{2}} \frac{ds}{s} \\ &= C\|\vec{f}\|_{L^2(E)} t^{-\frac{1}{2}} \left(\frac{\text{dist}(E,F)^2}{t}\right)^{-(m+\frac{1}{2})} \int_0^\infty e^{-s} s^{m+\frac{1}{2}} \frac{ds}{s} \\ &\leq C t^{-\frac{1}{2}} \left(\frac{\text{dist}(E,F)^2}{t}\right)^{-(m+\frac{1}{2})} \|\vec{f}\|_{L^2(E)}. \end{aligned}$$

Collecting the estimates for  $I_t$  and  $II_t$  we get

$$\begin{aligned} \|t^{\frac{1}{2}}\nabla L^{-\frac{1}{2}}(\mathcal{I} - e^{-tL})^m \text{div } \vec{f}\|_{L^2(F)} &\leq C t^{\frac{1}{2}} (\|I_t\|_{L^2(F)} + \|II_t\|_{L^2(F)}) \\ &\leq C \left(\frac{\text{dist}(E,F)^2}{t}\right)^{-(m+\frac{1}{2})} \|\vec{f}\|_{L^2(E)}. \end{aligned}$$

□

### 3. Proof of Theorem 1.2

The proof of this result is inspired by [DM], where a general method is developed to prove weak-type (1, 1) estimates for singular integral operators without assuming explicit regularity on the space variables of the kernel. That method can be applied to derive the boundedness of the Riesz transforms if one assumes sufficient pointwise decay of the heat kernel. It is to be noted that in [DM], there is an extra assumption involving a pointwise estimate for the gradient of the heat kernel, but this hypothesis is superfluous, as observed in [CD]. Here, a further step is taken: we derive the bounds for the Riesz transforms from off-diagonal estimates on the heat semigroup that hold in an average, as opposed to a pointwise, sense. These average off-diagonal estimates hold for all operators of the type under consideration (this is the content of Lemma 2.1), in contrast to pointwise estimates which may fail.

We remark also that pointwise decay of the heat kernel allows one to get some weighted norm inequalities for weights in the Muckenhoupt classes. This has been achieved in [Ma1], [Ma2] by means of a good- $\lambda$  inequality for some new sharp maximal function.

We set  $T = \nabla L^{-\frac{1}{2}}$  and, for simplicity of notation, we set

$$p \equiv p_n = 2n/(n + 2),$$

and we use this convention through the rest of the paper. We want to prove

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} |f(x)|^p dx.$$

Let us fix  $\alpha > 0$  and without loss of generality we can assume that  $f \in L^p(\mathbb{R}^n)$  is nonnegative. Let us write  $M$  for the Hardy-Littlewood maximal function. We use the Calderón-Zygmund decomposition for  $f(x)^p$  at height  $\alpha^p$ . Then there exists a collection of pairwise disjoint cubes  $\{Q_j\}_j$  such that

$$\{x \in \mathbb{R}^n : M(f^p)(x)^{\frac{1}{p}} > \alpha\} = \bigcup_j Q_j,$$

and they satisfy the following property

$$(2) \quad \alpha \leq \left( \frac{1}{|Q_j|} \int_{Q_j} f(x)^p dx \right)^{\frac{1}{p}} \leq C \alpha.$$

Then, we write  $f = g + b = g + \sum_j b_j$  where

$$g(x) = f(x) \chi_{\mathbb{R}^n \setminus \cup_j Q_j} + \sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} f(y) dy \right) \chi_{Q_j}(x),$$

$$b_j(x) = \left( f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y) dy \right) \chi_{Q_j}(x).$$

Estimate (2), the fact that  $p > 1$  and standard arguments yield that  $0 \leq g(x) \leq C \alpha$  for almost every  $x \in \mathbb{R}^n$ . Besides,

$$(3) \quad \int_{Q_j} b_j(x) dx = 0 \quad \text{and} \quad \left( \frac{1}{|Q_j|} \int_{Q_j} |b_j(x)|^p dx \right)^{\frac{1}{p}} \leq C \alpha.$$

Then,

$$|\{x : |Tf(x)| > 3 \alpha\}| \leq |\{x : |Tg(x)| > \alpha\}| + |\{x : |Tb(x)| > 2 \alpha\}| = I + II.$$

We analyze every term separately. For  $I$  we use that  $T$  is bounded on  $L^2(\mathbb{R}^n)$  (that is a consequence of [AHLMT, Theorem 1.4]) and the properties of  $g$  to get

$$\begin{aligned} I &\leq \frac{1}{\alpha^2} \int_{\mathbb{R}^n} |Tg(x)|^2 dx \\ &\leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} g(x)^2 dx \\ &\leq \frac{C}{\alpha^2} \alpha^{2-p} \int_{\mathbb{R}^n} g(x)^p dx \\ &= \frac{C}{\alpha^p} \left[ \int_{\mathbb{R}^n \setminus \cup_j Q_j} f(x)^p dx + \sum_j |Q_j| \left( \frac{1}{|Q_j|} \int_{Q_j} f(x) dx \right)^p \right]. \end{aligned}$$

The second term in the previous inequality can be estimated as follows

$$\begin{aligned} \sum_j |Q_j| \left( \frac{1}{|Q_j|} \int_{Q_j} f(x) dx \right)^p &= \sum_j \int_{Q_j} \left( \frac{1}{|Q_j|} \int_{Q_j} f(x) dx \right)^p dy \\ &\leq \sum_j \int_{Q_j} Mf(y)^p dy \\ &\leq \int_{\mathbb{R}^n} Mf(y)^p dy \\ &\leq C \int_{\mathbb{R}^n} f(x)^p dx. \end{aligned}$$

This inequality allows us to conclude that

$$I \leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} f(x)^p dx.$$

Now we proceed with  $II$ . Let us fix an integer  $m > \frac{n-2}{4}$ . We write  $t_j = \ell(Q_j)^2$ , where  $\ell(Q_j)$  stands for the side length of the cube  $Q_j$ . We will use the notation  $Q_j^* = 2Q_j$ , where in general we write  $\rho Q$  for the  $\rho$ -dilated  $Q$ , that is, for the cube with same center as  $Q$  and with side

length  $\rho \ell(Q)$ . Let  $E^* = \mathbb{R}^n \setminus \cup_j Q_j^*$ . Then,

$$\begin{aligned} Tb &= \sum_j Tb_j \\ &= \sum_j T(\mathcal{I} - e^{-t_j L})^m b_j + \sum_j (T - T(\mathcal{I} - e^{-t_j L})^m) b_j \\ &= \sum_j D_j b_j + \sum_j (T - D_j) b_j \end{aligned}$$

where we write  $D_j = T(\mathcal{I} - e^{-t_j L})^m$ . Thus,

$$\begin{aligned} II &\leq \left| \left\{ x : \left| \sum_j D_j b_j(x) \right| > \alpha \right\} \right| + \left| \left\{ x : \left| \sum_j (T - D_j) b_j(x) \right| > \alpha \right\} \right| \\ &\leq \left| \bigcup_j Q_j^* \right| + \left| \left\{ x \in E^* : \left| \sum_j D_j b_j(x) \right| > \alpha \right\} \right| \\ &\quad + \left| \left\{ x : \left| \sum_j (T - D_j) b_j(x) \right| > \alpha \right\} \right| \\ &= II_1 + II_2 + II_3. \end{aligned}$$

The first term can be estimated as follows

$$\begin{aligned} II_1 &\leq \sum_j |Q_j^*| \\ &\leq 2^n \sum_j |Q_j| \\ &= 2^n |\{x : M(f^p)(x) > \alpha^p\}| \\ &\leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} f(x)^p dx. \end{aligned}$$

Let us study  $II_3$ . We first use that  $T$  is bounded on  $L^2(\mathbb{R}^n)$  to get

$$\begin{aligned}
 II_3 &= \left| \left\{ x : \left| T \left( \sum_j (\mathcal{I} - (\mathcal{I} - e^{-t_j L})^m) b_j \right) (x) \right| > \alpha \right\} \right| \\
 &\leq \frac{C}{\alpha^2} \left\| \sum_j (\mathcal{I} - (\mathcal{I} - e^{-t_j L})^m) b_j \right\|_2^2.
 \end{aligned}$$

Besides,

$$\mathcal{I} - (\mathcal{I} - e^{-t_j L})^m = \mathcal{I} - \sum_{k=0}^m \binom{m}{k} e^{-k t_j L} = - \sum_{k=1}^m \binom{m}{k} e^{-k t_j L}$$

and hence

$$(4) \quad II_3 \leq \frac{C}{\alpha^2} \sum_{k=1}^m \left\| \sum_j e^{-k t_j L} b_j \right\|_2^2.$$

We fix  $1 \leq k \leq m$ . Then,

$$(5) \quad \left\| \sum_j e^{-k t_j L} b_j \right\|_2 = \sup_h \left| \int_{\mathbb{R}^n} \sum_j e^{-k t_j L} b_j(x) h(x) dx \right|$$

where the supremum is taken over all functions  $h \in L^2(\mathbb{R}^n)$  with  $\|h\|_2 = 1$ . Take such a function  $h$ . We set

$$S(0, j) = 2Q_j; \quad S(l, j) = 2^{l+1}Q_j \setminus 2^lQ_j, \quad l = 1, 2, \dots,$$

and  $h_{(l,j)}(x) = h(x)\chi_{S(l,j)}(x)$ . In this way, we obtain

$$\begin{aligned}
 (6) \quad & \left| \int_{\mathbb{R}^n} \sum_j e^{-k t_j L} b_j(x) h(x) dx \right| \\
 &= \left| \sum_j \sum_{l=0}^{\infty} \int_{\mathbb{R}^n} e^{-k t_j L} b_j(x) h_{(l,j)}(x) dx \right| \\
 &= \left| \sum_j \sum_{l=0}^{\infty} \int_{Q_j} b_j(x) (e^{-k t_j L})^* h_{(l,j)}(x) dx \right| \\
 &= \left| \sum_j \sum_{l=0}^{\infty} \int_{Q_j} b_j(x) \left( (e^{-k t_j L})^* h_{(l,j)}(x) - ((e^{-k t_j L})^* h_{(l,j)})_{Q_j} \right) dx \right| \\
 &\leq \sum_j \sum_{l=0}^{\infty} \|b_j\|_{L^p(Q_j)} \left\| (e^{-k t_j L})^* h_{(l,j)} - ((e^{-k t_j L})^* h_{(l,j)})_{Q_j} \right\|_{L^{p'}(Q_j)} \\
 &\leq C \alpha \sum_j \sum_{l=0}^{\infty} |Q_j|^{\frac{1}{p}} \left\| (e^{-k t_j L})^* h_{(l,j)} - ((e^{-k t_j L})^* h_{(l,j)})_{Q_j} \right\|_{L^{p'}(Q_j)},
 \end{aligned}$$

where  $(e^{-k t_j L})^*$  is the adjoint operator of  $e^{-k t_j L}$ . In the third equality we have used that  $b_j$  has vanishing integral and  $((e^{-k t_j L})^* h_{(l,j)})_{Q_j}$  denotes the average of  $(e^{-k t_j L})^* h_{(l,j)}$  over the cube  $Q_j$ . The last step above is just the second property in (3). Now we are going to apply Poincaré-Sobolev inequality. Note that  $p' = \frac{2n}{n-2} = 2^*$  which denotes the Sobolev exponent of 2. Then,

$$\begin{aligned}
 & \left\| (e^{-k t_j L})^* h_{(l,j)} - ((e^{-k t_j L})^* h_{(l,j)})_{Q_j} \right\|_{L^{p'}(Q_j)} \\
 & \leq C \left\| \nabla (e^{-k t_j L})^* h_{(l,j)} \right\|_{L^2(Q_j)} \\
 & \leq C t_j^{-\frac{1}{2}} e^{-\frac{\text{dist}(S(l,j), Q_j)^2}{c k t_j}} \|h_{(l,j)}\|_{L^2(S(l,j))},
 \end{aligned}$$

where the last estimate follows from the fourth conclusion in Lemma 2.1 by duality, and we have used that  $k \geq 1$ . If  $l = 0$  we have that

$\text{dist}(S(l, j), Q_j) = 0$ . For  $l \geq 1$ , we get  $\text{dist}(S(l, j), Q_j) \geq 2^{l-2}\ell(Q_j)$ . Thus,

$$e^{-\frac{\text{dist}(S(l, j), Q_j)^2}{c k t_j}} \leq e^{-\frac{4^{l-2}\ell(Q_j)^2}{c k t_j}} \leq e^{-c 4^l}$$

since  $t_j = \ell(Q_j)^2$  and  $1 \leq k \leq m$ . We have eventually proved that

$$\left\| (e^{-k t_j})^* h_{(l, j)} - ((e^{-k t_j})^* h_{(l, j)})_{Q_j} \right\|_{L^{p'}(Q_j)} \leq C t_j^{-\frac{1}{2}} e^{-c 4^l} \|h\|_{L^2(S(l, j))}.$$

Plugging this estimate into (6) and using again that  $t_j = \ell(Q_j)^2$  we get

$$\begin{aligned} (7) \quad & \left| \int_{\mathbb{R}^n} \sum_j e^{-k t_j} L b_j(x) h(x) dx \right| \\ & \leq C \alpha \sum_j \sum_{l=0}^{\infty} |Q_j|^{\frac{1}{p}} t_j^{-\frac{1}{2}} e^{-c 4^l} \|h\|_{L^2(S(l, j))} \\ & \leq C \alpha \sum_j \sum_{l=0}^{\infty} |Q_j|^{\frac{1}{2}} e^{-c 4^l} |2^{l+1} Q_j|^{\frac{1}{2}} \left( \frac{1}{|2^{l+1} Q_j|} \int_{2^{l+1} Q_j} |h(y)|^2 dy \right)^{\frac{1}{2}} \\ & \leq C \alpha \sum_j |Q_j| \text{ess inf}_{y \in Q_j} M(|h|^2)(y)^{\frac{1}{2}} \sum_{l=0}^{\infty} e^{-c 4^l} 2^{\frac{l n}{2}} \\ & \leq C \alpha \sum_j \int_{Q_j} \text{ess inf}_{y \in Q_j} M(|h|^2)(y)^{\frac{1}{2}} dx \\ & \leq C \alpha \int_{\cup_j Q_j} M(|h|^2)(x)^{\frac{1}{2}} dx \\ & \leq C \alpha \left| \bigcup_j Q_j \right|^{\frac{1}{2}} \| |h|^2 \|_1^{\frac{1}{2}} \\ & = C \alpha |\{x \in \mathbb{R}^n : M(f^p)(x)^{\frac{1}{p}} > \alpha\}|^{\frac{1}{2}}. \end{aligned}$$

Notice that in the last inequality we have used that  $\|h\|_2 = 1$  and the previous one follows from Kolmogorov's lemma since the Hardy-Littlewood maximal function is of weak type  $(1, 1)$  (see, for example, [Du, p. 102]). Collect this estimate, (4) and (5) to conclude that

$$II_3 \leq C |\{x \in \mathbb{R}^n : M(f^p)(x)^{\frac{1}{p}} > \alpha\}| \leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} f(x)^p dx$$

and we get the desired estimate for  $II_3$ . Now we are concerned with  $II_2$ . By Chebychev's inequality we get

$$\begin{aligned}
 (II_2)^{\frac{1}{2}} &= \left| \left\{ x \in E^* : \left| \sum_j D_j b_j(x) \right| > \alpha \right\} \right|^{\frac{1}{2}} \\
 (8) \quad &\leq \frac{1}{\alpha} \left\| \sum_j D_j b_j \right\|_{L^2(E^*)} \\
 &= \frac{1}{\alpha} \sup_{\vec{h}} \left| \int_{\mathbb{R}^n} \left\langle \sum_j D_j b_j(x), \vec{h}(x) \right\rangle dx \right|,
 \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{C}^n$  and the supremum is taken over all  $\mathbb{C}^n$ -valued functions  $\vec{h} \in L^2(E^*)$  with  $\|\vec{h}\|_{L^2(E^*)} = 1$ . Let us fix such a function  $\vec{h}$ . For every  $j$ , we define as before  $\vec{h}_{(l,j)}(x) = \vec{h}(x)\chi_{S(l,j)}(x)$ . Let us recall that  $E^* = \mathbb{R}^n \setminus \cup_j Q_j^*$ . In this way, and since  $\text{supp } \vec{h} \subset E^* \subset (2Q_j)^c$  we have

$$\begin{aligned}
 &\left| \int_{\mathbb{R}^n} \left\langle \sum_j D_j b_j(x), \vec{h}(x) \right\rangle dx \right| \\
 &= \left| \sum_j \sum_{l=1}^{\infty} \int_{\mathbb{R}^n} \langle D_j b_j(x), \vec{h}_{(l,j)}(x) \rangle dx \right| \\
 (9) \quad &= \left| \sum_j \sum_{l=1}^{\infty} \int_{Q_j} b_j(x) D_j^* \vec{h}_{(l,j)}(x) dx \right| \\
 &= \left| \sum_j \sum_{l=1}^{\infty} \int_{\mathbb{R}^n} b_j(x) \left( D_j^* \vec{h}_{(l,j)}(x) - (D_j^* \vec{h}_{(l,j)})_{Q_j} \right) dx \right| \\
 &\leq \sum_j \sum_{l=1}^{\infty} \|b_j\|_{L^p(Q_j)} \left\| D_j^* \vec{h}_{(l,j)} - (D_j^* \vec{h}_{(l,j)})_{Q_j} \right\|_{L^{p'}(Q_j)} \\
 &\leq C \alpha \sum_j \sum_{l=1}^{\infty} |Q_j|^{\frac{1}{p}} \left\| D_j^* \vec{h}_{(l,j)} - (D_j^* \vec{h}_{(l,j)})_{Q_j} \right\|_{L^{p'}(Q_j)},
 \end{aligned}$$



where  $D_j^*$  is the operator given in (1) with  $t = t_j$ , and we have used both properties in (3). We apply Poincaré-Sobolev inequality and Lemma 2.2 to get

$$\begin{aligned} & \left\| D_j^* \vec{h}_{(l,j)} - (D_j^* \vec{h}_{(l,j)})_{Q_j} \right\|_{L^{p'}(Q_j)} \\ & \leq C \left\| \nabla D_j^* \vec{h}_{(l,j)} \right\|_{L^2(Q_j)} \\ & = C \left\| \nabla (T(\mathcal{I} - e^{-t_j L})^m)^* \vec{h}_{(l,j)} \right\|_{L^2(Q_j)} \\ & \leq C t_j^{-\frac{1}{2}} \left( \frac{\text{dist}(S(l,j), Q_j)^2}{t_j} \right)^{-(m+\frac{1}{2})} \|\vec{h}_{(l,j)}\|_{L^2(S(l,j))} \\ & \leq C \ell(Q_j)^{-1} 2^{-2(m+\frac{1}{2})l} \|\vec{h}\|_{L^2(S(l,j))}, \end{aligned}$$

since  $t_j = \ell(Q_j)^2$  and for  $l \geq 1$  we have  $\text{dist}(S(l,j), Q_j) \geq 2^{l-2}\ell(Q_j)$ . We now plug this estimate into (9) and it follows that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \sum_j \langle D_j b_j(x), \vec{h}(x) \rangle dx \right| \\ & \leq C \alpha \sum_j \sum_{l=1}^{\infty} |Q_j|^{\frac{1}{p}} \ell(Q_j)^{-1} 2^{-2(m+\frac{1}{2})l} \|\vec{h}\|_{L^2(S(l,j))} \\ & \leq C \alpha \sum_j \sum_{l=1}^{\infty} |Q_j|^{\frac{1}{2}} 2^{-2(m+\frac{1}{2})l} |2^{l+1} Q_j|^{\frac{1}{2}} \left( \frac{1}{|2^{l+1} Q_j|} \int_{2^{l+1} Q_j} |\vec{h}(y)|^2 dy \right)^{\frac{1}{2}} \\ & \leq C \alpha \sum_j |Q_j| \text{ess inf}_{y \in Q_j} M(|\vec{h}|^2)(y)^{\frac{1}{2}} \sum_{l=1}^{\infty} 2^{-l(2(m+\frac{1}{2})-\frac{n}{2})} \\ & \leq C \alpha |\{x \in \mathbb{R}^n : M(f^p)(x)^{\frac{1}{p}} > \alpha\}|^{\frac{1}{2}}, \end{aligned}$$

where in the last step we used that  $m > \frac{n-2}{4}$  and so  $2(m + \frac{1}{2}) - \frac{n}{2} > 0$ , and we have proceeded as in (7). Then we can use this estimate, (8) and

that  $M$  is of weak type  $(1, 1)$  to get

$$II_2 \leq C |\{x \in \mathbb{R}^n : M(f^p)(x)^{\frac{1}{p}} > \alpha\}| \leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} f(x)^p dx.$$

The proof of the fact that  $T$  is of weak type  $(p, p)$  is now completed by collecting the estimates that we have obtained for  $I$ ,  $II_1$ ,  $II_2$  and  $II_3$ .

*Note added in proof:* P. Auscher has recently observed that the exponent  $p_n$  may be replaced by  $p_n - \epsilon$ ,  $\epsilon = \epsilon(n, \lambda, \Lambda)$ ,  $n \geq 3$  [A].

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