Publ. Mat. **47** (2003), 45–69

CONTINUITY PROPERTIES FOR THE MAXIMAL OPERATOR ASSOCIATED WITH THE COMMUTATOR OF THE BOCHNER-RIESZ OPERATOR

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Abstract

In this paper, we obtain some strong and weak type continuity properties for the maximal operator associated with the commutator of the Bochner-Riesz operator on Hardy spaces, Hardy type spaces and weak Hardy type spaces.

1. Introduction

Let $b \in \mathrm{BMO}(\mathrm{R}^n)$ and T be a standard Calderón-Zygmund singular integral operator, the commutator $[b,T]$ is defined by

$$[b,T]f(x) = T((b(x)-b(\cdot))f)(x).$$

Many authors have investigated the properties for $[b,T]$. A celebrated result of Coifman, Rochberg and Weiss [**5**] states that the commutator $[b,T]$ is bounded on L^p $(1<p<\infty)$. Subsequently, Coifman and Meyer [**4**] observed that the weighted L^p $(1<p<\infty)$ boundedness for $[b,T]$ can be obtained by the weighted L^p estimate with Muckenhoupt A_p weight for T. Later, Álvarez, Bagby, Kurtz and Pérez [**2**] extended the idea of Coifman and Meyer and proved the following result: For a general linear operator T, if $1<p,q<\infty$ and T is bounded on $L^p(w)$ for all $w \in A_q$, then $[b,T]$ is bounded on $L^p(u)$ for all $u \in A_q$. In the case of $p=1$, it is a well-known fact that Calderón-Zygmund singular integral operator T is a weak type $(1,1)$ operator and a bounded operator from the standard Hardy space H^1 to L^1. Fairly recently, Pérez [**13**] observed the fact that $[b,T]$ is neither a weak type $(1,1)$ operator nor a bounded operator from H^1 to L^1. He obtained a weak type $L\log L$ inequality and

2000 *Mathematics Subject Classification.* 42B20, 35J05.

Key words. Bochner-Riesz operator, commutator, Hardy space, Hardy type space, weak Hardy type space, $\mathrm{BMO}(\mathrm{R}^n)$.

*G. Lu was partly supported by US NSF grant DMS9970352 and S. Lu was partly supported by the National 973 Project Foundation of China (G19990751).

the boundedness from a certain modified Hardy space H^1_b to L^1 for $[b, T]$. In this paper, we will consider the commutator of Bochner-Riesz operator, let λ and r be two positive numbers, the Bochner-Riesz operator T^r_λ in R^n $(n \geq 2)$ is defined in terms of Fourier transforms by

$$\widehat{T^r_\lambda f}(\xi) = \left(1 - \frac{|\xi|^2}{r^2}\right)^{\lambda}_{+} \hat{f}(\xi),$$

where $\hat{f}$ denotes the Fourier transforms of f. It can be written as a convolution operator

$$T^r_\lambda f(x) = \text{p.v.} \int_{\mathrm{R}^n} B^r_\lambda(x-y) f(y)\, dy,$$

where $B^r_\lambda(x)$ is the kernel of T^r_λ and $B^r_\lambda(x) = r^{-n} B_\lambda(\frac{x}{r})$, it is well-known that $B_\lambda(x)$ satisfies the following inequality:

$$\left| \frac{\partial^\beta}{\partial x^\beta} B_\lambda(x) \right| \leq C\,(1+|x|)^{-(\lambda + \frac{n+1}{2})},$$

for any $x \in \mathrm{R}^n$ and $r > 0$ and any multi-index $\beta \in \mathrm{Z}^n_+$.

Let $b \in \mathrm{BMO}(\mathrm{R}^n)$, the commutator generated by b and T^r_λ is defined by

$$T^r_{\lambda,b} f(x) = T^r_\lambda((b(x) - b(\cdot))f)(x), \quad f \in \mathcal{S}(\mathrm{R}^n),$$

or

$$T^r_{\lambda,b} f(x) = \text{p.v.} \int_{\mathrm{R}^n} B^r_\lambda(x-y)(b(x)-b(y)) f(y)\, dy, \quad f \in \mathcal{S}(\mathrm{R}^n).$$

The maximal operator associated with $T^r_{\lambda,b}$ is defined by

$$T^*_{\lambda,b} f(x) = \sup_{r>0} \left| T^r_{\lambda,b} f(x) \right|.$$

If $\lambda \geq \frac{n-1}{2}$, Shi and Sun [**14**] showed that the maximal Bochner-Riesz operator, $T^*_\lambda f(x) = \sup\limits_{r>0} |T^r_\lambda f(x)|$, is bounded on $L^p(w)$ provided that $1 < p < \infty$ and $w \in A_p$. Combining the above result due to Álvarez, Bagby, Kurtz and Pérez with the result due to Shi and Sun [**14**], we can easily observe that $T^*_{\lambda,b}$ is bounded on $L^p(\mathrm{R}^n)$. Hu and Lu [**7**] further discussed the L^p boundedness for $T^*_{\lambda,b}$ in the case of $\lambda > (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$ and proved the following result

Theorem A. *If* $b \in \mathrm{BMO}(\mathrm{R}^n)$*,* $1 < p < \infty$ *and* $\lambda > (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$*, then* $T^*_{\lambda,b}$ *is bounded on* $L^p(\mathrm{R}^n)$ *with bound* $C(n,p)\|b\|_*$*.*

We notice the fact that $(n-1)\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{n-1}{2}$ whenever $1 < p < \infty$ and in this case T^*_λ is not bounded on $L^p(w)$ for $w \in A_p$. It is natural to investigate the properties for $T^*_{\lambda,b}$ with $0 < p \le 1$. In Section 2 of this paper, we consider the case of $p = 1$ and obtain the boundedness from H^1 to weak L^1 for $T^*_{\lambda,b}$. In Section 3 of this paper, we get some strong and weak type boundedness estimates for $T^*_{\lambda,b}$ on a certain modified Hardy space, H^p_b, and a certain modified weak Hardy space, $H^{p,\infty}_b$, where $0 < p \le 1$. However, we do not know whether the operator $T^*_{\lambda,b}$ satisfies the weak type $L \log L$ inequality. After we submitted the paper, we learned that Jiang, Tang and Yang [**9**] proved, independently of us, the similar results in H^p_b and $H^{p,\infty}_b$ when $\frac{n}{n+1} < p \le 1$. Our range in this paper allows to have all $0 < p < 1$.

Now, let us recall some notations and definitions. Most of the notations we use are standard. Q denotes a cube with sides parallel to the axes and λQ ($\lambda > 0$) denotes the cube Q dilated by λ. For a locally integrable function f, f_Q denotes the average $f_Q = \frac{1}{|Q|}\int_Q f(y)\,dy$. Sometimes a_Q denotes an atom in certain Hardy spaces with compact support included in cube Q. For $b \in \mathrm{BMO}(\mathrm{R}^n)$, $\|b\|_*$ denotes the norm of b on $\mathrm{BMO}(\mathrm{R}^n)$.

Definition 1. Let $0 < p \le 1$ and b be a locally integrable function. Given a bounded function a, we say that a is a (p, b, ∞) atom, if (1) $\operatorname{supp} a \subset Q = Q(x_Q, r_Q)$; (2) $\|a\|_{L^\infty(\mathrm{R}^n)} \le |Q|^{-1/p}$; (3) $\int_{\mathrm{R}^n} a(x)x^\beta\,dx = \int_{\mathrm{R}^n} a(x)b(x)x^\beta\,dx = 0$, for $|\beta| \le [n(1/p - 1)]$, where $[x]$ denotes the integer part of x. A tempered distribution f is said to belong to the Hardy type space $H^p_b(\mathrm{R}^n)$ if, in the $\mathcal{S}'$-sense, it can be written as $f = \sum_{j=1}^{\infty} \lambda_j a_j$, where a_j are (p, b, ∞) atoms and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. As usual, we define on $H^p_b(\mathrm{R}^n)$ the quasinorm as

$$\|f\|_{H^p_b(\mathrm{R}^n)} = \inf_{\sum \lambda_j a_j = f} \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p}.$$

Definition 2. Let b be a locally integrable function. We say that a tempered distribution f belongs to the weak Hardy type space $H^{p,\infty}_b(\mathrm{R}^n)$, if there exists a sequence $\{f_k\}_{k=-\infty}^{\infty} \subset L^\infty(\mathrm{R}^n)$ such that

(1) $f = \sum_{k=-\infty}^{\infty} f_k$, in the $\mathcal{S}'$-sense;

(2) Each function f_k can be decomposed as $f_k = \sum_{j=1}^{\infty} b_j^k$ in $L^\infty \cap H^p$, where the functions b_j^k satisfy the following properties:

(2i) $\operatorname{supp} b_j^k \subset Q_j^k$ with $\sup_k \sum_{j=1}^{\infty} \chi_{Q_j^k} < \infty$ and $\sup_k 2^{kp} \sum_{j=1}^{\infty} |Q_j^k| < \infty$.

(2ii) There exists a constant $C = C(n,p) > 0$ such that $\|b_j^k\|_{L^\infty(\mathrm{R}^n)} \leq C2^k$, for any k and j.

(2iii) $\int_{\mathrm{R}^n} b_j^k(x) x^\beta \, dx = \int_{\mathrm{R}^n} b_j^k(x) b(x) x^\beta \, dx = 0$, for $|\beta| \leq [n(1/p - 1)]$.

We define on the space $H_b^{p,\infty}(\mathrm{R}^n)$ the following quasinorm

$$\|f\|_{H_b^{p,\infty}(\mathrm{R}^n)}^p = \inf_{\sum_k \sum_j b_j^k = f} \sup_{k \in \mathrm{Z}} 2^{kp} \sum_{j=1}^{\infty} |Q_j^k|.$$

For brevity, we will sometimes denote $C_1 = \sup_{k \in \mathrm{Z}} 2^{kp} \sum_{j=1}^{\infty} |Q_j^k|$.

2. Weak type (H^1, L^1) estimate

In this section, we establish the weak type (H^1, L^1) estimate for $T^*_{\lambda,b}$, where $H^1(\mathrm{R}^n)$ is a well-known standard Hardy space. Our main result is the following theorem.

Theorem 1. *Let $b \in \mathrm{BMO}(\mathrm{R}^n)$ and $\lambda > \frac{n-1}{2}$, then $T^*_{\lambda,b}$ is a weak type (H^1, L^1) bounded operator, i.e. there exists a constant $C > 0$, such that*

$$\Big|\{x \in \mathrm{R}^n : T^*_{\lambda,b} f(x) > \alpha\}\Big| \leq \frac{C}{\alpha} \|f\|_{H^1(\mathrm{R}^n)},$$

for any $\alpha > 0$ and any $f \in H^1(\mathrm{R}^n)$.

Remark. After the paper is accepted for publication, it has been proved using similar method that $T^*_{\lambda,b}$ is bounded from H^1 to L^1 (see [**11**]). However, we still do not know if $T^*_{\lambda,b}$ is weak type $(1,1)$.

To prove our Theorem 1, we first recall the following lemma due to M. Christ [**3**].

Lemma 1. *For any $\alpha > 0$ and any finite collection of dyadic cubes Q and associated positive scalars λ_Q, there exists a collection of pairwise disjoint dyadic cubes S such that*

(1) $\sum\limits_{Q\subset S} \lambda_Q \leq 8\alpha|S|$ *for all S;*

(2) $\sum\limits_{S} |S| \leq \alpha^{-1} \sum\limits_{Q} \lambda_Q$*;*

(3) $\| \sum\limits_{Q\not\subset S} \lambda_Q |Q|^{-1}\chi_Q\|_{L^\infty(\mathrm{R}^n)} \leq \alpha.$

Now we begin to prove Theorem 1.

It is easy to see that the result of Theorem 1 follows from the inequality

$$\Big|\{x \in \mathrm{R}^n : |T^r_{\lambda,b} f(x)| > \alpha\}\Big| \leq \frac{C}{\alpha}\|f\|_{H^1(\mathrm{R}^n)},$$

where C is independent of r, f and α.

For any given $f \in H^1(\mathrm{R}^n)$, we have the well-known atomic decomposition $f = \sum\limits_{j=1}^{\infty} \lambda_j a_j$, in the $\mathcal{S}'$-sense, where each a_j be a $(1,\infty,0)$ atom with $\|f\|_{H^1(\mathrm{R}^n)} = \inf\Big(\sum\limits_{j=1}^{\infty} |\lambda_j|\Big)$.

We may assume that f is a finite sum $\sum_Q \lambda_Q a_Q$ with $\sum_Q |\lambda_Q| \leq 2\|f\|_{H^1(\mathrm{R}^n)}$. Once Theorem 1 is proved for such f. For general f, it is the limit of this kind of f_k (in H^1 norm or almost everywhere sense) where f_k are finite sums having forms of $\sum_Q \lambda_Q a_Q$, and then Theorem 1 follows by a limiting argument. It is convenient for us to assume that each Q (the supporting cube of a_Q) in the given atomic decomposition of f is dyadic and $\lambda_Q > 0$.

For fixed $\alpha > 0$ and the finite collection of dyadic cube Q and associated positive scalars $\lambda_Q > 0$ in the given atomic decomposition of f, by Lemma 1, there exists a collection of pairwise disjoint dyadic cube S such that

(1) $\sum\limits_{Q\subset S} \lambda_Q \leq 8\alpha|S|$, for all S;

(2) $\sum\limits_{S} |S| \leq \alpha^{-1} \sum\limits_{Q} \lambda_Q$;

(3) $\Big\| \sum\limits_{Q\not\subset S} \lambda_Q |Q|^{-1}\chi_Q\Big\|_{L^\infty(\mathrm{R}^n)} \leq \alpha.$

Denote $E = \bigcup_S 2S$, then $|E| \leq \frac{C}{\alpha}\|f\|_{H^1(\mathrm{R}^n)}$.

Set $h(x) = \sum_S \sum_{Q \subset S} \lambda_Q a_Q$ and $g(x) = f(x) - h(x)$. By (3), we easily know that $\|g\|_{L^\infty(\mathrm{R}^n)} \le \alpha$. Using the $L^2(\mathrm{R}^n)$ boundedness of $T^*_{\lambda,b}$ we get that

$$\left|\left\{x \in \mathrm{R}^n \setminus E : |T^r_{\lambda,b} g(x)| > \frac{\alpha}{4}\right\}\right| \le \left|\left\{x \in \mathrm{R}^n \setminus E : |T^*_{\lambda,b} g(x)| > \frac{\alpha}{4}\right\}\right|$$
$$\le \frac{C}{\alpha^2} \|T^*_{\lambda,b} g\|^2_{L^2(\mathrm{R}^n)}$$
$$\le \frac{C}{\alpha^2} \|g\|^2_{L^2(\mathrm{R}^n)}$$
$$\le \frac{C}{\alpha} \|g\|_{L^1(\mathrm{R}^n)}$$
$$\le \frac{C}{\alpha} \|f\|_{L^1(\mathrm{R}^n)}$$
$$\le \frac{C}{\alpha} \|f\|_{H^1(\mathrm{R}^n)}.$$

Thus, we only need to prove the following inequality

$$\left|\left\{x \in \mathrm{R}^n \setminus E : |T^r_{\lambda,b} h(x)| > \frac{\alpha}{4}\right\}\right| \le \frac{C}{\alpha} \|f\|_{H^1(\mathrm{R}^n)}.$$

For fixed cube $Q = Q(x_Q, r_Q)$, by the vanishing moments of a_Q we have

$$\begin{aligned} T^r_{\lambda,b} a_Q(x) = &\int_{\mathrm{R}^n} (B^r_\lambda(x-y) - B^r_\lambda(x - x_Q))(b(x) - b_Q) a_Q(y)\, dy \\ &+ \int_{\mathrm{R}^n} B^r_\lambda(x-y)(b_Q - b(y)) a_Q(y)\, dy \\ = &\; I_Q(x) + T^r_\lambda((b_Q - b)a_Q)(x). \end{aligned}$$

Now, we first estimate $I_Q(x)$.

When $0 < r \le r_Q$ and any $x \in \mathrm{R}^n \setminus E$ and any $y \in Q$, since $x \in \mathrm{R}^n \setminus E$ implies $x \in \mathrm{R}^n \setminus 2Q$ for any Q, we have that

$$|x - y| \ge |x - x_Q| - |x_Q - y| \ge |x - x_Q| - r_Q \ge \frac{|x - x_Q|}{2},$$

this implies that

$$|B^r_\lambda(x-y)| \le C r^{-n} \left(1 + \frac{|x - x_Q|}{r}\right)^{-(\lambda + \frac{n+1}{2})} \le C r^{\lambda - \frac{n-1}{2}} |x - x_Q|^{-(\lambda + \frac{n+1}{2})}.$$

Similarly we can get

$$|B_\lambda^r(x-x_Q)| \le Cr^{\lambda-\frac{n-1}{2}}|x-x_Q|^{-(\lambda+\frac{n+1}{2})}.$$

These above estimates imply the following inequality

$$|I_Q(x)| \le Cr^{\lambda-\frac{n-1}{2}}|x-x_Q|^{-(\lambda+\frac{n+1}{2})}|b(x)-b_Q|.$$

Thus

$$\begin{aligned}
&\left|\left\{x\in \mathrm{R}^n\setminus E : \sum_S\sum_{Q\subset S}\lambda_Q|I_Q(x)|>\alpha\right\}\right| \\
&\le \frac{C}{\alpha}\sum_S\sum_{Q\subset S}\lambda_Q\int_{\mathrm{R}^n\setminus 2Q}|I_Q(x)|\,dx \\
&\le \frac{C}{\alpha}\sum_S\sum_{Q\subset S}\lambda_Q\sum_{l=1}^{\infty}\int_{2^{l+1}Q\setminus 2^lQ} r^{\lambda-\frac{n-1}{2}}|x-x_Q|^{-(\lambda+\frac{n+1}{2})}|b(x)-b_Q|\,dx \\
&\le \frac{C}{\alpha}\sum_S\sum_{Q\subset S}\lambda_Q r^{\lambda-\frac{n-1}{2}}\sum_{l=1}^{\infty}\frac{(2^{l+1}r_Q)^n}{(2^lr_Q)^{\lambda+\frac{n+1}{2}}}\frac{1}{|2^{l+1}Q|}\int_{2^{l+1}Q}|b(x)-b_Q|\,dx \\
&\le \frac{C\|b\|_*}{\alpha}\sum_S\sum_{Q\subset S}\lambda_Q\left(\frac{r}{r_Q}\right)^{\lambda-\frac{n-1}{2}}\sum_{l=1}^{\infty} l2^{-l(\lambda-\frac{n-1}{2})} \\
&\le \frac{C\|b\|_*}{\alpha}\sum_S\sum_{Q\subset S}\lambda_Q \le \frac{C\|b\|_*}{\alpha}\|f\|_{H^1(\mathrm{R}^n)}.
\end{aligned}$$

When $r>r_Q$, we can choose λ_0 satisfying $\frac{n-1}{2}<\lambda_0<\min\left(\lambda,\frac{n+1}{2}\right)$. By the mean value theorem and the boundedness of B_λ^r, we have

$$\begin{aligned}
|I_Q(x)| &\le Cr^{-n-1}|b(x)-b_Q|\int_Q|a_Q(y)||y-x_Q|\left(1+\frac{|x-x_Q|}{r}\right)^{-(\lambda+\frac{n+1}{2})}dy \\
&\le Cr^{-n-1}|b(x)-b_Q|\int_Q|a_Q(y)||y-x_Q|\left(1+\frac{|x-x_Q|}{r}\right)^{-(\lambda_0+\frac{n+1}{2})}dy \\
&\le Cr_Qr^{\lambda_0-\frac{n+1}{2}}|b(x)-b_Q||x-x_Q|^{-(\lambda_0+\frac{n+1}{2})}.
\end{aligned}$$

This implies the following estimate

$$\begin{aligned}
&\left|\left\{x \in \mathrm{R}^n \setminus E : \sum_S \sum_{Q\subset S} \lambda_Q |I_Q(x)| > \alpha\right\}\right| \\
&\leq \frac{C}{\alpha} \sum_S \sum_{Q\subset S} \lambda_Q \sum_{l=1}^{\infty} \int_{2^{l+1}Q\setminus 2^l Q} r_Q r^{\lambda_0 - \frac{n+1}{2}} |x - x_Q|^{-(\lambda_0 + \frac{n+1}{2})} |b(x) - b_Q|\, dx \\
&\leq \frac{C}{\alpha} \sum_S \sum_{Q\subset S} \lambda_Q \sum_{l=1}^{\infty} r_Q r^{\lambda_0 - \frac{n+1}{2}} \big(2^l r_Q\big)^{-(\lambda_0 + \frac{n+1}{2}) + n} \frac{1}{|2^{l+1}Q|} \int_{2^{l+1}Q} |b(x) - b_Q|\, dx \\
&\leq \frac{C\|b\|_*}{\alpha} \sum_S \sum_{Q\subset S} \lambda_Q \left(\frac{r}{r_Q}\right)^{\lambda_0 - \frac{n+1}{2}} \sum_{l=1}^{\infty} l 2^{-l(\lambda_0 - \frac{n-1}{2})} \\
&\leq \frac{C\|b\|_*}{\alpha} \sum_S \sum_{Q\subset S} \lambda_Q \leq \frac{C\|b\|_*}{\alpha} \|f\|_{H^1(\mathrm{R}^n)}.
\end{aligned}$$

We notice the fact that $T_\lambda^* f(x) \leq CMf(x)$ with $\lambda > \frac{n-1}{2}$, where M is the well-known Hardy-Littlewood maximal operator. Thus T_λ^* is of weak type $(1,1)$ and T_λ^r is weak type $(1,1)$ uniformly associated with r. We get

$$\begin{aligned}
&\left|\left\{x \in \mathrm{R}^n \setminus E : \left|\sum_S \sum_{Q\subset S} \lambda_Q T_\lambda^r((b_Q - b)a_Q)(x)\right| > \alpha\right\}\right| \\
&\qquad \leq \frac{C}{\alpha} \sum_S \sum_{Q\subset S} \lambda_Q \|(b_Q - b)a_Q\|_{L^1(\mathrm{R}^n)} \\
&\qquad \leq \frac{C}{\alpha} \sum_S \sum_{Q\subset S} \lambda_Q \frac{1}{|Q|} \int_Q |b(y) - b_Q|\, dy \\
&\qquad \leq \frac{C\|b\|_*}{\alpha} \sum_S \sum_{Q\subset S} \lambda_Q \\
&\qquad \leq \frac{C\|b\|_*}{\alpha} \|f\|_{H^1(\mathrm{R}^n)}.
\end{aligned}$$

This finishes the proof of Theorem 1. □

3. (H_b^p, L^p) type estimate

In this section we will establish the (H_b^p, L^p) type estmate for $T^*_{\lambda,b}$. Our main result is the following theorem.

Theorem 2. *Let $b \in \mathrm{BMO}(\mathrm{R}^n)$ and $0 < p \le 1$. If $\lambda > \frac{n}{p} - \frac{n+1}{2}$, then $T^*_{\lambda,b}$ is a bounded operator from $H_b^p(\mathrm{R}^n)$ to $L^p(\mathrm{R}^n)$.*

Proof: By the definition of space $H_b^p(\mathrm{R}^n)$, we only need to prove the following inequality for any (p, b, ∞) atom a_Q,

$$\int_{\mathrm{R}^n} |T^r_{\lambda,b} a_Q(x)|^p \, dx \le C,$$

where C is a constant independent on a_Q and r.

We easily get the following decomposition

$$\int_{\mathrm{R}^n} |T^r_{\lambda,b} a_Q(x)|^p \, dx = \int_{2Q} |T^r_{\lambda,b} a_Q(x)|^p \, dx + \int_{\mathrm{R}^n \setminus 2Q} |T^r_{\lambda,b} a_Q(x)|^p \, dx = I_1 + I_2.$$

By the $L^q(\mathrm{R}^n)$ $(1 < q < \infty)$ boundedness of $T^*_{\lambda,b}$ and the Hölder inequality, we get

$$\begin{aligned} I_1 &\le |2Q|^{1-p/q} \left(\int_{2Q} |T^*_{\lambda,b} a_Q(x)|^q \, dx \right)^{p/q} \\ &\le C|Q|^{1-p/q} \|a_Q\|^p_{L^q(\mathrm{R}^n)} \\ &\le C|Q|^{1-p/q} |Q|^{p/q-1} = C. \end{aligned}$$

Since $0 < p \le 1$, we have

$$\begin{aligned} I_2 &\le \sum_{j=1}^{\infty} \left(\int_{2^{j+1}Q \setminus 2^j Q} |b(x) - b_Q|^p |T^r_\lambda a_Q(x)|^p \, dx \right. \\ &\qquad \left. + \int_{2^{j+1}Q \setminus 2^j Q} |T^r_{\lambda,b}((b_Q - b) a_Q)(x)|^p \, dx \right) \\ &\le \sum_{j=1}^{\infty} (J_{1j} + J_{2j}) \,. \end{aligned}$$

First, we estimate each J_{1j}.

When $0<r<r_Q$, $x\in 2^{j+1}Q\setminus 2^jQ$ and $y\in Q$, $j=1,2,\dots$, we get

$$|B_\lambda^r(x-y)|\le Cr^{-n}\left(1+\frac{|x-x_Q|}{r}\right)^{-(\lambda+\frac{n+1}{2})}$$
$$\le Cr^{\lambda-\frac{n-1}{2}}|x-x_Q|^{-(\lambda+\frac{n+1}{2})}.$$

This implies that

$$|T_\lambda^r a_Q(x)|\le\int_Q|B_\lambda^r(x-y)||a_Q(y)|\,dy\le Cr^{\lambda-\frac{n-1}{2}}r_Q^{n(1-1/p)}|x-x_Q|^{-(\lambda+\frac{n+1}{2})}.$$

Thus we have

$$\begin{aligned}J_{1j}&\le C\int_{2^{j+1}Q\setminus 2^jQ}|b(x)-b_Q|^p r^{p(\lambda-\frac{n-1}{2})}|x-x_Q|^{-p(\lambda+\frac{n+1}{2})}r_Q^{n(p-1)}\,dx\\&\le Cr^{p(\lambda-\frac{n-1}{2})}(2^jr_Q)^{-p(\lambda+\frac{n+1}{2})+n}r_Q^{n(p-1)}\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(x)-b_Q|^p\,dx\\&\le Cj^p\|b\|_*^p2^{j(n-p(\lambda+\frac{n+1}{2}))}\left(\frac{r}{r_Q}\right)^{p(\lambda-\frac{n-1}{2})}\\&\le Cj^p\|b\|_*^p2^{j(n-p(\lambda+\frac{n+1}{2}))},\end{aligned}$$

in the last inequality, we use the fact that $\lambda>\frac{n}{p}-\frac{n+1}{2}$ implies that $\lambda-\frac{n-1}{2}\ge 0$ for $0<p\le 1$.

Since $0<p\le 1$, it is easy to see that there exists a nonnegative integer m such that $\frac{n}{n+m+1}<p\le\frac{n}{n+m}$. In this case, we notice the fact that $[n(1/p-1)]=m$.

When $r\ge r_Q$, if $\frac{n}{p}-\frac{n+1}{2}<\lambda<m+\frac{n+1}{2}$, by the vanishing moments of a_Q and $(m+1)$-order Taylor expansion of $B_\lambda^r(x-y)$ at $x-x_Q$, we have

$$|T_\lambda^r a_Q(x)|\le Cr^{-(n+m+1)}\int_Q|y-x_Q|^{m+1}\left(1+\frac{|x-x_Q|}{r}\right)^{-(\lambda+\frac{n+1}{2})}|a_Q(y)|\,dy$$
$$\le Cr^{\lambda-m-\frac{n+1}{2}}|x-x_Q|^{-(\lambda+\frac{n+1}{2})}r_Q^{m+1+n(1-1/p)}.$$

This implies that

$$
\begin{aligned}
J_{1j} &\le C \int_{2^{j+1}Q\setminus 2^jQ} |b(x)-b_Q|^p r^{p(\lambda-m-\frac{n+1}{2})} \\
&\qquad \times |x-x_Q|^{-p(\lambda+\frac{n+1}{2})} r_Q^{p(n+m+1)-n}\, dx \\
&\le C r^{p(\lambda-m-\frac{n+1}{2})} (2^j r_Q)^{-p(\lambda+\frac{n+1}{2})+n} r_Q^{p(n+m+1)-n} \\
&\qquad \times \frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |b(x)-b_Q|^p\, dx \\
&\le C j^p \|b\|_*^p 2^{j(n-p(\lambda+\frac{n+1}{2}))} \left(\frac{r_Q}{r}\right)^{p(m+\frac{n+1}{2}-\lambda)} \\
&\le C j^p \|b\|_*^p 2^{j(n-p(\lambda+\frac{n+1}{2}))}.
\end{aligned}
$$

When $r \ge r_Q$, if $\lambda \ge m + \frac{n+1}{2}$, Also by the vanishing moments of a_Q and $(m+1)$-order Taylor expansion of $B_\lambda^r(x-y)$ at $x-x_Q$, we have

$$
\begin{aligned}
|T_\lambda^r a_Q(x)| &\le C r^{-(n+m+1)} \int_Q |y-x_Q|^{m+1} \left(1+\frac{|x-x_Q|}{r}\right)^{-(\lambda+\frac{n+1}{2})} |a_Q(y)|\, dy \\
&\le C r^{-(n+m+1)} \int_Q |y-x_Q|^{m+1} \left(1+\frac{|x-x_Q|}{r}\right)^{-(n+m+1)} |a_Q(y)|\, dy \\
&\le C |x-x_Q|^{-(n+m+1)} r_Q^{m+1+n(1-1/p)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
J_{1j} &\le C \int_{2^{j+1}Q\setminus 2^jQ} |b(x)-b_Q|^p |x-x_Q|^{-p(n+m+1)} r_Q^{p(n+m+1)-n}\, dx \\
&\le C j^p \|b\|_*^p 2^{j(n-p(n+m+1))}.
\end{aligned}
$$

Because of these above estimates and the condition that $\frac{n}{n+m+1} < p \le \frac{n}{n+m}$ and $\lambda > \frac{n}{p} - \frac{n+1}{2}$, we obtain

$$
\sum_{j=1}^{\infty} J_{1j} \le C \|b\|_*^p.
$$

Now we start to estimate J_{2j}.

When $0 < r < r_Q$, we also get

$$\begin{aligned}\left|T_\lambda^r((b_Q - b)a_Q)(x)\right| &\le \int_Q |B_\lambda^r(x-y)(b_Q - b(y))a_Q(y)|\,dy\\ &\le Cr^{\lambda-\frac{n-1}{2}}|x-x_Q|^{-(\lambda+\frac{n+1}{2})} r_Q^{-n/p}\int_Q |b(y)-b_Q|\,dy\\ &\le Cr^{\lambda-\frac{n-1}{2}}|x-x_Q|^{-(\lambda+\frac{n+1}{2})} r_Q^{n(1-1/p)}\|b\|_*,\end{aligned}$$

and

$$\begin{aligned}J_{2j} &\le Cr^{p(\lambda-\frac{n-1}{2})} r_Q^{n(p-1)}\|b\|_*^p \int_{2^{j+1}Q\setminus 2^jQ} |x-x_Q|^{-p(\lambda+\frac{n+1}{2})}\,dx\\ &\le Cr^{p(\lambda-\frac{n-1}{2})} r_Q^{n(p-1)}\|b\|_*^p (2^{j+1}r_Q)^{n-p(\lambda+\frac{n+1}{2})}\\ &\le C\|b\|_*^p \left(\frac{r}{r_Q}\right)^{p(\lambda-\frac{n-1}{2})} 2^{j(n-p(\lambda+\frac{n+1}{2}))}\\ &\le C\|b\|_*^p 2^{j(n-p(\lambda+\frac{n+1}{2}))}.\end{aligned}$$

When $r \ge r_Q$, if $\frac{n}{p} - \frac{n+1}{2} < \lambda < m + \frac{n+1}{2}$. By the vanishing moments of a_Q and Taylor formula, we also have

$$\begin{aligned}|T_\lambda^r((b_Q - b)a_Q)(x)| &\le Cr^{-(n+m+1)}\int_Q |y-x_Q|^{m+1}\left(1+\frac{|x-x_Q|}{r}\right)^{-(\lambda+\frac{n+1}{2})}\\ &\qquad\times |b(y)-b_Q||a_Q(y)|\,dy\\ &\le C\|b\|_* r^{\lambda-m-\frac{n+1}{2}} r_Q^{m+1+n(1-1/p)}|x-x_Q|^{-(\lambda+\frac{n+1}{2})},\end{aligned}$$

and

$$\begin{aligned}J_{2j} &\le C\|b\|_*^p r^{p(\lambda-m-\frac{n+1}{2})} r_Q^{p(n+m+1)-n}\int_{2^{j+1}Q\setminus 2^jQ}|x-x_Q|^{-p(\lambda+\frac{n+1}{2})}\,dx\\ &\le C\|b\|_*^p r^{p(\lambda-m-\frac{n+1}{2})} r_Q^{p(n+m+1)-n}(2^jr_Q)^{n-p(\lambda-\frac{n+1}{2})}\\ &\le C\|b\|_*^p 2^{j(n-p(\lambda+\frac{n+1}{2}))}.\end{aligned}$$

When $r \geq r_Q$ and $\lambda \geq m + \frac{n+1}{2}$, we also get

$$\begin{aligned}|T_\lambda^r((b_Q-b)a_Q)(x)| &\leq Cr^{-(n+m+1)}\int_Q |y-x_Q|^{m+1}\left(1+\frac{|x-x_Q|}{r}\right)^{-(\lambda+\frac{n+1}{2})} \\ &\qquad \times |b(y)-b_Q||a_Q(y)|\,dy \\ &\leq Cr^{-(n+m+1)}\int_Q |y-x_Q|^{m+1}\left(1+\frac{|x-x_Q|}{r}\right)^{-(n+m+1)} \\ &\qquad \times |b(y)-b_Q||a_Q(y)|\,dy \\ &\leq Cr_Q^{m+1+n(1-1/p)}|x-x_Q|^{-(n+m+1)}\|b\|_*.\end{aligned}$$

This implies that

$$\begin{aligned}J_{2j} &\leq C\|b\|_*^p r_Q^{p(n+m+1)-n}\int_{2^{j+1}Q\setminus 2^jQ}|x-x_Q|^{-p(n+m+1)}\,dx \\ &\leq C\|b\|_*^p 2^{j(n-p(n+m+1))}.\end{aligned}$$

Because of above estimates and the condition $\frac{n}{n+m+1} < p \leq \frac{n}{n+m}$ and $\lambda > \frac{n}{p} - \frac{n+1}{2}$, we obtain

$$\sum_{j=1}^{\infty} J_{2j} \leq C\|b\|_*^p.$$

This completes the proof of Theorem 2. □

4. $(H_b^{p,\infty}, L^{p,\infty})$ type estimate

In this section, we obtain the $(H_b^{p,\infty}, L^{p,\infty})$ type estimate for $T_{\lambda,b}^*$, where $L^{p,\infty}$ is a well-known weak L^p space.

Theorem 3. *Let $b \in \mathrm{BMO}(\mathrm{R}^n)$ and $0 < p \leq 1$. If $\lambda > \frac{n}{p} - \frac{n+1}{2}$, then $T_{\lambda,b}^*$ is a bounded operator from $H_b^{p,\infty}(\mathrm{R}^n)$ to $L^{p,\infty}(\mathrm{R}^n)$.*

Before proving Theorem 3, we need to recall the so-called superposition principle on the weak-type estimates.

Lemma 2. *Let $0 < p < 1$, if a series of measurable functions $\{f_k\}$ satisfy*

$$\left|\{x \in \mathrm{R}^n : |f_k(x)| > \alpha\}\right| \le \alpha^{-p}, \quad \forall\, k \in \mathbf{N},$$

and $\sum\limits_{k=1}^{\infty} |c_k|^p < \infty$*, then*

$$\left|\left\{x \in \mathrm{R}^n : \left|\sum_{k=1}^{\infty} c_k f_k(x)\right| > \alpha\right\}\right| \le \frac{2-p}{1-p}\alpha^{-p}\left(\sum_{k=1}^{\infty}|c_k|^p\right).$$

Now we begin to prove Theorem 3.

Given $f \in H_b^{p,\infty}(\mathrm{R}^n)$, let $\sum\limits_k f_k = \sum\limits_k \sum\limits_{j\ge1} b_j^k$ be an atomic decomposition for f as Definition 2.

Fix $N = 1, 2, \dots$, and consider $\sum\limits_{k=-N}^{N} f_k$. By a usual limiting argument it is enough to prove that there exists $C = C(b, \lambda) > 0$ such that

$$\sup_{\alpha>0} \alpha^p \left|\left\{x \in \mathrm{R}^n : \left|T^r_{\lambda,b}\left(\sum_{k=-N}^{N} f_k\right)(x)\right| > \alpha\right\}\right| \le CC_1,$$

for any $N = 1, 2, \dots$.

Given $\alpha > 0$, let $k_0 \in \mathbf{Z}$ such that $2^{k_0} \le \alpha < 2^{k_0+1}$, then we write

$$\sum_{k=-N}^{N} f_k = \sum_{k=-N}^{k_0} f_k + \sum_{k=k_0+1}^{N} f_k = F_1 + F_2.$$

Let us first observe that $F_1 \in L^q(\mathrm{R}^n)$ for any $1 < q < \infty$, in fact

$$|F_1(x)| \le \sum_{k=-N}^{k_0}\sum_{j=1}^{\infty}|b_j^k(x)| \le C\sum_{k=-N}^{k_0} 2^k \sum_{j=1}^{\infty}\chi_{Q_j^k} \le C\sum_{k=-N}^{k_0} 2^k \chi_{\cup_{j=1}^{\infty} Q_j^k}.$$

This implies that

$$\begin{aligned}
\|F_1\|_{L^q(\mathrm{R}^n)} &\le C \sum_{k=-N}^{k_0} 2^k \left| \bigcup_{j=1}^{\infty} Q_j^k \right|^{1/q} \\
&\le C \sum_{k=-N}^{k_0} 2^k \left(\sum_{j=1}^{\infty} |Q_j^k| \right)^{1/q} \\
&\le C C_1^{1/q} \sum_{k=-N}^{k_0} 2^{k(1-p/q)} \\
&\le C C_1^{1/q} 2^{k_0(1-p/q)} \\
&\le C C_1^{1/q} \alpha^{1-p/q}.
\end{aligned}$$

Thus we have

$$\alpha^p \Big| \{x \in \mathrm{R}^n : |T^r_{\lambda,b}(F_1)(x)| > \alpha\} \Big| \le C\alpha^{p-q} \|F_1\|^q_{L^q(\mathrm{R}^n)} \le CC_1.$$

Now let $A_k Q_j^k$ be the cube with the same center as Q_j^k and sides A_k times longer, where A_k being a positive number to be chosen later and depending on k and p. Denote $B_{k_0,N} = \bigcup_{k=k_0+1}^{N} \bigcup_{j=1}^{\infty} A_k Q_j^k$, and Q_j^k be a cube with center x_j^k and side-length r_j^k, we write

$$\begin{aligned}
\alpha^p \Big| \{x \in \mathrm{R}^n : |T^r_{\lambda,b}(F_2)(x)| > \alpha\} \Big| &\le C\alpha^p \Big| \{x \in B_{k_0,N} : |T^r_{\lambda,b}(F_2)(x)| > \alpha\} \Big| \\
&\quad + C\alpha^p \Big| \{x \in \mathrm{R}^n \setminus B_{k_0,N} : |T^r_{\lambda,b}(F_2)(x)| > \alpha\} \Big| \\
&= K_1 + K_2.
\end{aligned}$$

Let us first estimate K_2. If c is any complex number, we can write

$$\begin{aligned}
K_2 &\le \alpha^p \Big| \{x \in \mathrm{R}^n \setminus B_{k_0,N} : |b(x) - c||T^r_\lambda(F_2)(x)| > \alpha/2\} \Big| \\
&\quad + \alpha^p \Big| \{x \in \mathrm{R}^n \setminus B_{k_0,N} : |T^r_\lambda((b-c)F_2)(x)| > \alpha/2\} \Big| \\
&= K_{21} + K_{22}.
\end{aligned}$$

Similar to proof of Theorem 2, there exists a nonnegative integer m such that $\frac{n}{n+m+1} < p \leq \frac{n}{n+m}$ for fixed $0 < p \leq 1$. Now we estimate K_{21}

$$\begin{aligned} K_{21} &\leq C \int_{\mathrm{R}^n \setminus B_{k_0,N}} |b(x) - c|^p |T_\lambda^r(F_2)(x)|^p \, dx \\ &\leq C \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \int_{\mathrm{R}^n \setminus A_k Q_j^k} |b(x) - c|^p |T_\lambda^r(b_j^k)(x)|^p \, dx. \end{aligned}$$

When $\frac{n}{p} - \frac{n+1}{2} < \lambda < m + \frac{n+1}{2}$, if $0 < r < r_j^k$ for fixed k and j, we have

$$\begin{aligned} |T_\lambda^r(b_j^k)(x)| &= \left| \int_{Q_j^k} B_\lambda^r(x-y) b_j^k(y) \, dy \right| \\ &\leq C r^{-n} \int_{Q_j^k} \left(1 + \frac{|x - x_j^k|}{r} \right)^{-(\lambda + \frac{n+1}{2})} |b_j^k(y)| \, dy \\ &\leq C r^{\lambda - \frac{n-1}{2}} |x - x_j^k|^{-(\lambda + \frac{n+1}{2})} 2^k |Q_j^k| \\ &\leq C (r_j^k)^{\lambda - \frac{n-1}{2}} |x - x_j^k|^{-(\lambda + \frac{n+1}{2})} 2^k |Q_j^k|, \end{aligned}$$

and if $r \geq r_j^k$, by the vanishing moments of b_j^k and the $(m+1)$-order Taylor formula of $B_\lambda^r(x-y)$ at $x - x_Q$, we have

$$\begin{aligned} |T_\lambda^r(b_j^k)(x)| &\leq C r^{-(n+m+1)} \int_{Q_j^k} |y - x_j^k|^{m+1} \left(1 + \frac{|x - x_j^k|}{r} \right)^{-(\lambda + \frac{n+1}{2})} |b_j^k(y)| \, dy \\ &\leq C r^{\lambda - m - \frac{n+1}{2}} (r_j^k)^{m+1} |x - x_j^k|^{-(\lambda + \frac{n+1}{2})} 2^k |Q_j^k| \\ &\leq C (r_j^k)^{\lambda - \frac{n-1}{2}} |x - x_j^k|^{-(\lambda + \frac{n+1}{2})} 2^k |Q_j^k|. \end{aligned}$$

These imply that

$$K_{21} \le C \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \int_{2^{l+1}A_kQ_j^k \setminus 2^lA_kQ_j^k} |b(x)-c|^p (r_j^k)^{p(\lambda-\frac{n-1}{2})} \times |x-x_j^k|^{-p(\lambda+\frac{n+1}{2})} 2^{kp}|Q_j^k|^p\,dx$$

$$\le C \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} 2^{kp}|Q_j^k| 2^{l(n-p(\lambda+\frac{n+1}{2}))} A_k^{n-p(\lambda+\frac{n+1}{2})} \times \frac{1}{|2^{l+1}A_kQ_j^k|}\int_{2^{l+1}A_kQ_j^k} |b(x)-c|^p\,dx$$

$$\le CB_1 \sum_{k=k_0+1}^{N} 2^{kp}\left(\sum_{j=1}^{\infty}|Q_j^k|\right) A_k^{n-p(\lambda+\frac{n+1}{2})} \sum_{l=1}^{\infty} 2^{l(n-p(\lambda+\frac{n+1}{2}))}$$

$$\le CC_1B_1 \sum_{k=k_0+1}^{N} A_k^{n-p(\lambda+\frac{n+1}{2})},$$

where $B_1 = \sup\limits_{Q} \frac{1}{|Q|}\int_Q |b(x)-c|^p\,dx$.

When $\lambda \ge m+\frac{n+1}{2}$, if $0<r<r_j^k$ for a fixed k and j, we also have following estimate,

$$|T_\lambda^r(b_j^k)(x)| \le Cr^{\lambda-\frac{n-1}{2}}|x-x_j^k|^{-(\lambda+\frac{n+1}{2})}2^k|Q_j^k|,$$

and if $r \ge r_{k,j}$ for a fixed k and j,

$$|T_\lambda^r(b_j^k)(x)| \le Cr^{-(n+m+1)}\int_{Q_j^k}|y-x_j^k|^{m+1}\left(1+\frac{|x-x_j^k|}{r}\right)^{-(\lambda+\frac{n+1}{2})}|b_j^k(y)|\,dy$$

$$\le Cr^{-(n+m+1)}\int_{Q_j^k}|y-x_j^k|^{m+1}\left(1+\frac{|x-x_j^k|}{r}\right)^{-(n+m+1)}|b_j^k(y)|\,dy$$

$$\le C(r_j^k)^{m+1}|x-x_j^k|^{-(n+m+1)}2^k|Q_j^k|.$$

Thus we can get

$$K_{21} \le C \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \int_{2^{l+1}A_kQ_j^k \setminus 2^lA_kQ_j^k} |b(x)-c|^p (r_j^k)^{p(\lambda-\frac{n-1}{2})} \times |x-x_j^k|^{-p(\lambda+\frac{n+1}{2})} 2^{kp}|Q_j^k|^p\,dx$$
$$+ C \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \int_{2^{l+1}A_kQ_j^k \setminus 2^lA_kQ_j^k} |b(x)-c|^p (r_j^k)^{p(m+1)} \times |x-x_j^k|^{-p(n+m+1)} 2^{kp}|Q_j^k|^p\,dx$$
$$= P_1 + P_2.$$

Because

$$P_2 \le C \sum_{k=k_0+1}^{N} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \left(2^lA_kr_j^k\right)^{-p(n+m+1)} 2^{kp}|Q_j^k|^p (r_j^k)^{p(m+1)} \times \int_{2^{l+1}A_kQ_j^k \setminus 2^lA_kQ_j^k} |b(x)-c|^p\,dx$$
$$\le C \sum_{k=k_0+1}^{N} 2^{kp} \sum_{j=1}^{\infty} |Q_j^k| A_k^{n-p(n+m+1)} \sum_{l=1}^{\infty} 2^{l(n-p(n+m+1))} \frac{1}{|2^{l+1}A_kQ_j^k|} \times \int_{2^{l+1}A_kQ_j^k} |b(x)-c|^p\,dx$$
$$\le CC_1B_1 \sum_{k_0+1}^{N} A_k^{n-p(n+m+1)},$$

and

$$P_1 \le CC_1B_1 \sum_{k=k_0+1}^{N} A_k^{n-p(\lambda+\frac{n+1}{2})}.$$

We obtain that

$$K_{21} \le CC_1B_1 \left(\sum_{k=k_0+1}^{N} A_k^{n-p(\lambda+\frac{n+1}{2})} + \sum_{k=k_0+1}^{N} A_k^{n-p(n+m+1)} \right).$$

Let $A_k = 2^{(k-k_0)\max\left(\frac{1}{n+m+1}, \frac{1}{\lambda+\frac{n+1}{2}}\right)}$, then

$$\begin{aligned} K_{21} &\leq CC_1 B_1 \\ &= CC_1 \sup_Q \frac{1}{|Q|} \int_Q |b(x) - c|^p \, dx \\ &\leq CC_1 \left(\sup_Q \frac{1}{|Q|} \int_Q |b(x) - c| \, dx \right)^p . \end{aligned}$$

By taking the infimum with $c \in \mathbf{C}$ in the right-hand side of above inequality, we obtain that

$$K_{21} \leq CC_1 \|b\|_*^p.$$

Now let us estimate K_{22}.

Let

$$\Gamma_1 = \{(k, j) : k_0 + 1 \leq k \leq N, \; j \geq 1 \text{ and } 0 < r < r_j^k\}$$

and

$$\Gamma_2 = \{(k, j) : k_0 + 1 \leq k \leq N, \; j \geq 1, \; r \geq r_j^k\}.$$

Write

$$\begin{aligned} K_{22} \leq \alpha^p &\left| \left\{ x \in \mathrm{R}^n \setminus B_{k_0,N} : \sum_{(k,j)\in\Gamma_1} |T_\lambda^r((b-c)b_j^k)(x)| > \frac{\alpha}{4} \right\} \right| \\ &+ \alpha^p \left| \left\{ x \in \mathrm{R}^n \setminus B_{k_0,N} : \sum_{(k,j)\in\Gamma_2} |T_\lambda^r((b-c)b_j^k)(x)| > \frac{\alpha}{4} \right\} \right| \\ &= M_1 + M_2. \end{aligned}$$

For fix $(k,j)\in\Gamma_1$, we have

$$\begin{aligned}|T_\lambda^r((b-c)b_j^k)(x)| &\le \int_{Q_j^k}|B_\lambda^r(x-y)(b(y)-c)b_j^k(y)|\,dy\\ &\le Cr^{-n}\int_{Q_j^k}\left(1+\frac{|x-x_j^k|}{r}\right)^{-(\lambda+\frac{n+1}{2})}|b(y)-c||b_j^k(y)|\,dy\\ &\le Cr^{\lambda-\frac{n-1}{2}}|x-x_j^k|^{-(\lambda+\frac{n+1}{2})}2^k\int_{Q_j^k}|b(y)-c|\,dy\\ &\le C(r_j^k)^{\lambda-\frac{n-1}{2}}|x-x_j^k|^{-(\lambda+\frac{n+1}{2})}2^kB_2|Q_j^k|\\ &\le \frac{C2^kB_2(r_j^k)^{\lambda+\frac{n+1}{2}}}{|x-x_j^k|^{\lambda+\frac{n+1}{2}}},\end{aligned}$$

where $B_2=\sup\limits_Q\frac{1}{|Q|}\int_Q|b(y)-c|\,dy$.

Because of the facts that

$$\left|\left\{x\in\mathrm{R}^n\setminus B_{k_0,N}:\frac{1}{|x-x_j^k|^{\lambda+\frac{n+1}{2}}}>\frac{\alpha}{4}\right\}\right|\le C\alpha^{-\frac{n}{\lambda+\frac{n+1}{2}}}$$

and

$$\begin{aligned}\sum_{(k,j)\in\Gamma_1}\left(C2^kB_2(r_j^k)^{\lambda+\frac{n+1}{2}}\right)^{\frac{n}{\lambda+\frac{n+1}{2}}} &= \sum_{(k,j)\in\Gamma_1}\left(C2^kB_2\right)^{\frac{n}{\lambda+\frac{n+1}{2}}}(r_j^k)^n\\ &\le C\sum_{k=k_0+1}^{N}B_2^{\frac{n}{\lambda+\frac{n+1}{2}}}2^{\frac{kn}{\lambda+\frac{n+1}{2}}}\sum_{j=1}^{\infty}|Q_j^k|\\ &\le CC_1B_2^{\frac{n}{\lambda+\frac{n+1}{2}}}\sum_{k=k_0+1}^{N}2^{k(\frac{n}{\lambda+\frac{n+1}{2}}-p)}\\ &\le CC_1B_2^{\frac{n}{\lambda+\frac{n+1}{2}}}2^{k_0(\frac{n}{\lambda+\frac{n+1}{2}}-p)},\end{aligned}$$

by Lemma 2, we obtain

$$M_1 \leq \alpha^p C C_1 B_2^{\frac{n}{\lambda+\frac{n+1}{2}}} 2^{k_0(\frac{n}{\lambda+\frac{n+1}{2}}-p)} \alpha^{-\frac{n}{\lambda+\frac{n+1}{2}}}$$
$$\leq C C_1 B_2^{\frac{n}{\lambda+\frac{n+1}{2}}}.$$

For $(k,j) \in \Gamma_2$, if $\frac{n}{p} - \frac{n-1}{2} < \lambda < m + \frac{n+1}{2}$, we have the following estimate by the vanishing moments of b_j^k and Taylor formula,

$$\left|T_\lambda^r((b-c)b_j^k)(x)\right| \leq C r^{-(n+m+1)} \int_{Q_j^k} |y - x_j^k|^{m+1} \left(1 + \frac{|x-x_j^k|}{r}\right)^{-(\lambda+\frac{n+1}{2})}$$
$$\times |b(y)-c||b_j^k(y)|\,dy$$
$$\leq C r^{\lambda-m-\frac{n+1}{2}} (r_j^k)^{m+1} 2^k |x-x_j^k|^{-(\lambda+\frac{n+1}{2})} \int_{Q_j^k} |b(y)-c|\,dy$$
$$\leq \frac{C 2^k B_2 (r_j^k)^{\lambda+\frac{n+1}{2}}}{|x-x_j^k|^{\lambda+\frac{n+1}{2}}}.$$

Similar to the estimate of M_1, we get

$$M_2 \leq C C_1 B_2^{\frac{n}{\lambda+\frac{n+1}{2}}}.$$

For $(k,j)\in\Gamma_2$, if $\lambda\ge m+\frac{n+1}{2}$, we also obtain

$$\begin{aligned}\left|T_\lambda^r((b-c)b_j^k)(x)\right| &\le Cr^{-(n+m+1)}\int_{Q_j^k}|y-x_j^k|^{m+1}\left(1+\frac{|x-x_j^k|}{r}\right)^{-(\lambda+\frac{n+1}{2})}\\ &\qquad\times|b(y)-c||b_j^k(y)|\,dy\\ &\le Cr^{-(n+m+1)}\int_{Q_j^k}|y-x_j^k|^{m+1}\left(1+\frac{|x-x_j^k|}{r}\right)^{-(n+m+1)}\\ &\qquad\times|b(y)-c||b_j^k(y)|\,dy\\ &\le C(r_j^k)^{m+1}|x-x_j^k|^{-(n+m+1)}2^k\int_{Q_j^k}|b(y)-c|\,dy\\ &\le\frac{CB_2 2^k(r_j^k)^{n+m+1}}{|x-x_j^k|^{n+m+1}}.\end{aligned}$$

Since that

$$\left|\left\{x\in\mathrm{R}^n\setminus B_{k_0,N}:\frac{1}{|x-x_j^k|^{n+m+1}}>\frac{\alpha}{4}\right\}\right|\le C\alpha^{-\frac{n}{n+m+1}},$$

and

$$\begin{aligned}\sum_{(k,j)\in\Gamma_2}\left(C2^kB_2(r_j^k)^{n+m+1}\right)^{\frac{n}{n+m+1}} &= C\sum_{(k,j)\in\Gamma_2}B_2^{\frac{n}{n+m+1}}2^{\frac{kn}{n+m+1}}|Q_j^k|\\ &\le CC_1B_2^{\frac{n}{n+m+1}}\sum_{k=k_0+1}^{N}2^{k(\frac{n}{n+m+1}-p)}\\ &\le CC_1B_2^{\frac{n}{n+m+1}}2^{k_0(\frac{n}{n+m+1}-p)},\end{aligned}$$

using Lemma 2 we obtain

$$\begin{aligned}M_2&\le\alpha^pCC_1B_2^{\frac{n}{n+m+1}}2^{k_0(\frac{n}{n+m+1}-p)}\alpha^{-\frac{n}{n+m+1}}\\ &\le CC_1B_2^{\frac{n}{n+m+1}}.\end{aligned}$$

Thus we get that

$$K_{22} \le CC_1 \left(B_2^{\frac{n}{\lambda+\frac{n+1}{2}}} + B_2^{\frac{n}{n+m+1}} \right)$$

$$= CC_1 \left(\left(\sup_Q \frac{1}{|Q|} \int_Q |b(y) - c| \, dy \right)^{\frac{n}{\lambda+\frac{n+1}{2}}} + \left(\sup_Q \frac{1}{|Q|} \int_Q |b(y) - c| \, dy \right)^{\frac{n}{n+m+1}} \right).$$

By taking the infimum with $c \in \mathbf{C}$ in each term on the right hand of above inequality, we obtain that

$$K_{22} \le CC_1 \left(\|b\|_*^{\frac{n}{\lambda+\frac{n+1}{2}}} + \|b\|_*^{\frac{n}{n+m+1}} \right).$$

Finally let us estimate K_1,

$$\begin{aligned} K_1 &\le \alpha^p |B_{k_0,N}| \le \alpha^p \sum_{k=k_0+1}^{N} A_k^n \sum_{j=1}^{\infty} |Q_j^k| \\ &\le CC_1 \alpha^p \sum_{k=k_0+1}^{N} A_k^n 2^{-kp} \\ &\le CC_1 \alpha^p 2^{-k_0 p} \sum_{k=k_0+1}^{N} A_k^n 2^{-(k-k_0)p} \\ &\le CC_1 \sum_{k=k_0+1}^{N} 2^{(k-k_0)(\max(\frac{n}{n+m+1}, \frac{n}{\lambda+\frac{n+1}{2}})-p)} \\ &\le CC_1. \end{aligned}$$

Following above estimates, we have that

$$\alpha^p \left| \left\{ x \in \mathrm{R}^n : \left| T_{\lambda,b}^r \left(\sum_{k=-N}^{N} f_k \right)(x) \right| > \lambda \right\} \right| \le CC_1 \left(1 + \|b\|_*^p + \|b\|_*^{\frac{n}{n+m+1}} + \|b\|_*^{\frac{n}{\lambda+\frac{n}{n+1}}} \right),$$

where C is dependent on r, N and the atomic decomposition of f.

Finally, taking the limit as $N \to \infty$, the infimum of C_1 over all possible representations $\sum_k \sum_j b_j^k = f$ and the supremum for $r > 0$ in the left-hand side of above inequality, we complete the proof of Theorem 3. □

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Primera versió rebuda el 26 d'octubre de 2001,
darrera versió rebuda el 22 d'octubre de 2002.