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ON L^p ESTIMATES FOR SQUARE ROOTS OF SECOND ORDER ELLIPTIC OPERATORS ON $\mathbb{R}^n$

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Abstract

We prove that the square root of a uniformly complex elliptic operator $L = -\operatorname{div}(A\nabla)$ with bounded measurable coefficients in $\mathbb{R}^n$ satisfies the estimate $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$ for $\sup(1, \frac{2n}{n+4} - \varepsilon) < p < \frac{2n}{n-2} + \varepsilon$, which is new for $n \geq 5$ and $p < 2$ or for $n \geq 3$ and $p > \frac{2n}{n-2}$. One feature of our method is a Calderón-Zygmund decomposition for Sobolev functions. We make some further remarks on the topic of the converse L^p inequalities (i.e. Riesz transforms bounds), pushing the recent results of [**BK2**] and [**HM**] for $\frac{2n}{n+2} < p < 2$ when $n \geq 3$ to the range $\sup(1, \frac{2n}{n+2} - \varepsilon) < p < 2+\varepsilon'$. In particular, we obtain that $L^{1/2}$ extends to an isomorphism from $\dot{W}^{1,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for p in this range. We also generalize our method to higher order operators.

Contents

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1. Introduction

Let $A = A(x)$ be an $n \times n$ matrix of complex, L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda|\xi|^2 \leq \operatorname{Re} A\xi \cdot \overline{\xi} \text{ and } |A\xi \cdot \overline{\zeta}| \leq \Lambda|\xi||\zeta|, \tag{1.1}$$

for $\xi, \zeta \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \leq \Lambda < \infty$. We define a second order divergence form operator

$$Lf \equiv -\operatorname{div}(A\nabla f), \tag{1.2}$$

which we interpret in the sense of maximal accretive operators via a sesquilinear form.

The accretivity condition (1.1) enables one to define a square root $L^{1/2}$ (see [**K**]) again in the sense of maximal accretive operators. It is known that

$$\|L^{1/2}f\|_2 \sim \|\nabla f\|_2, \quad n \geq 1. \tag{1.3}$$

Here $\sim$ is the equivalence in the sense of norms, with implicit constants C depending only on n, λ and Λ, and $\|f\|_p = (\int_{\mathbb{R}^n} |f(x)|_H^p \, dx)^{1/p}$ denotes the usual norm for functions on $\mathbb{R}^n$ valued in a Hilbert space H.

This implies that the domain of $L^{1/2}$ is in all dimensions the Sobolev space $H^1(\mathbb{R}^n)$, which was known as Kato's conjecture. Indeed (1.3) is due to Coifman, McIntosh and Meyer [**CMcM**] when $n = 1$, to Hofmann and McIntosh [**HMc**] when $n = 2$ and to Hofmann, Lacey, McIntosh and Tchamitchian along with the author [**AHLMcT**] for all dimensions.

Although there is no explicit connections between square roots and Calderón-Zygmund operators (except if A is a constant matrix or if $n=1$) there are strong relations: the L^2-results are obtained through refinements of standard techniques in the theory (square functions, Carleson measures, $T(b)$ theorem) so one can try to compare $L^{1/2}f$ and ∇f in L^p norms, $p \neq 2$. This program was initialised in [**AT1**] for this class of complex operators. It arose from a different perspective towards applications to boundary value problems in the works of Dahlberg, Jerison, Kenig and their collaborators (see [**Ke**, problem 3.3.16]). At this time, the following results are known (obtained sometimes prior to the

Kato conjecture by making the L^2-result an assumption).

(1.4) $\|L^{1/2}f\|_p \sim \|f'\|_p, \quad n=1, \quad 1<p<\infty.$

(1.5) $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p, \quad n=2, \quad 1<p<\infty.$

(1.6) $\|\nabla f\|_p \lesssim \|L^{1/2}f\|_p, \quad n=2, \quad 1<p<2+\varepsilon.$

(1.7) $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p, \quad n=3,4, \quad 1<p<2.$

(1.8) $\|\nabla f\|_p \lesssim \|L^{1/2}f\|_p, \quad n\geq 3, \quad p_n<p<2.$

(1.9) $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p, \quad n\geq 3, \quad 2<p<p_n'.$

(1.10) $\|\nabla f\|_p \lesssim \|L^{1/2}f\|_p, \quad n=3,4 \quad 2<p<2+\varepsilon.$

Here and subsequently we set $p_n = \frac{2n}{n+2}$ for $n\geq 2$, which is the Sobolev exponent for the embedding $\|f\|_2 \leq C\|\nabla f\|_{p_n}$. Also, the inequalities are stated for f in an appropriate class for which $L^{1/2}f$ is well-defined. In view of the L^2-results, $C_0^\infty(\mathbb{R}^n)$ works.

The equivalence (1.4) is in [**AT2**]. In one dimension, the theory of singular integral operators is fully applicable but this ceases in higher dimensions. The equations (1.5) and (1.6) are from [**AT1**]: they were proved assuming (1.3) and a technical condition called the Gaussian property, which is always valid in two dimensions from [**AMcT**]. Inequalities of type (1.6) are known as L^p-boundedness for the operator $\nabla L^{-1/2}$, the (array of) Riesz transforms associated to L. Note that in (1.6), ε depends on the ellipticity constants only and the range of p's is sharp for the whole class of such operators from a counterexample of Kenig (see [**AT1**, p. 119]) as ε can be as small as one wishes. The inequality (1.7) is a consequence of [**AHLMcT**, Proposition 6.2], and an assumption on the resolvent $(1+t^2L)^{-1}$ which holds in these dimensions thanks to Sobolev inequalities (see Section 4).

The inequality (1.8) is due independently to Blunck and Kunstmann [**BK2**] and Hofmann and Martell [**HM**] taking (1.3) as starting point. In fact, we shall explain how to lower p_n to $p_n-\varepsilon$ for some $\varepsilon>0$ depending on dimension and the ellipticity constants only.

Proposition 1.

(1.11) $\|\nabla f\|_p \lesssim \|L^{1/2}f\|_p, \quad n\geq 3, \quad p_n-\varepsilon<p\leq p_n.$

The sharpness of the lower bound is not known for the whole class of operators L with (1.1). We present in a subsequent work equivalent formulations of L^p-boundedness for Riesz transforms allowing to conclude the existence of operators for which (1.11) fails for some $p>1$ and also to discuss this sharpness issue [**A**].

A cheap duality argument gives (1.9) from (1.8) applied to the adjoint L^*, and the range extends to $p < (p_n - \varepsilon)'$ thanks to Proposition 1.

Lastly, (1.10) is a consequence of (1.3), (1.7) and (1.9) combined with the perturbation result in [**AT1**, p. 131].

Here, we wish to complete this study and prove results like (1.7) in dimensions larger than or equal to 5.

Theorem 2. *We have for* $P_n = \frac{np_n}{p_n+n} = \frac{2n}{n+4}$

$$\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p, \quad n \geq 5, \quad \sup(P_n - \varepsilon, 1) < p < 2. \tag{1.12}$$

The sharpness of the lower bound is open. Note that $P_n \leq 1$ if $n \leq 4$: in fact, our proof gives in lower dimensions a weak-type $(1,1)$ estimate:

Theorem 3. *We have the weak-type estimate*

$$\|L^{1/2}f\|_{1,\infty} \lesssim \|\nabla f\|_1, \quad n \leq 4. \tag{1.13}$$

By interpolation, this provides us with alternate proofs to the strong type (p,p) in dimensions $n \leq 4$ for the range $1 < p < 2$. Boundedness on the Hardy space $\mathcal{H}^1$ is also known in these dimensions, i.e. $\|L^{1/2}f\|_{\mathcal{H}^1} \leq C\|\nabla f\|_{\mathcal{H}^1}$.

The gain from p_n in (1.11) to P_n in (1.12) comes from a somewhat magical use of Sobolev embeddings.

Let us return to large dimensions. As a consequence of the perturbation method mentioned above, we have

Corollary 4.

$$\|\nabla f\|_p \lesssim \|L^{1/2}f\|_p, \quad n \geq 5, \quad 2 < p < 2 + \varepsilon. \tag{1.14}$$

The upper bound is sharp as in dimension 2. By repeating the proof of [**AT1**, Theorem 21, p. 132], one obtains invertibility results, which were known if $n \leq 4$.

Corollary 5.

$$\|\nabla f\|_p \sim \|L^{1/2}f\|_p, \quad n \geq 2, \quad \sup(p_n - \varepsilon, 1) < p < 2 + \varepsilon'. \tag{1.15}$$

Furthermore, the operator $L^{1/2}$, a priori *defined from* $C_0^\infty(\mathbb{R}^n)$ *into* $L^2(\mathbb{R}^n)$*, extends to a bounded an invertible operator from* $\dot{W}^{1,p}(\mathbb{R}^n)$ *onto* $L^p(\mathbb{R}^n)$ *for p in the above range.*

We stress that these inequalities only require (1.1). More assumptions give additional results. They will be discussed in Section 5.

The proof of Theorem 2 will be completely different from the ones in smaller dimensions that gave an estimate on the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ as none of them seem to extend to this more general setting. In fact, we reprove those earlier results in a unified way. Our method is to obtain a weak type estimate for each p in the range. We shall use ideas from Blunck and Kunstmann who introduced in [**BK1**] a criterion to obtain weak type estimates for $p \neq 1$ which apply to non-integral operators, generalizing methods and result from Duong-McIntosh for $p = 1$ [**DMc**]. These results, as the original Calderón-Zygmund theorem, rely on the Calderón-Zygmund decomposition for L^p functions. The novelty here (see Section 2) is a Calderón-Zygmund decomposition for any Sobolev-L^p function, $1 \leq p \leq \infty$, namely any locally integrable function whose gradient is in L^p: it writes as the sum of a "good" part which is Lipschitz and a locally finite sum of "bad" functions which are supported in cubes with a control on the L^p-average of their gradients. In Section 3, we prove Theorem 2. Section 4 is concerned with auxilliary lemmata for elliptic operators used in Section 3. In Section 5, we make some more historical remarks on earlier L^p results. In Sections 6 and 7, we extend both the Calderón-Zygmund decomposition to $\dot{W}^{m,p}$ functions and our results to higher order operators: see there for statements. We make some final comments in Section 8.

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2. A Calderón-Zygmund lemma for Sobolev-L^p functions

Theorem 6. *Let $n \geq 1$, $1 \leq p \leq \infty$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ be such that $\|\nabla f\|_p < \infty$. Let $\alpha > 0$. Then, one can find a collection of cubes (Q_i), functions g and b_i such that*

$$f = g + \sum_i b_i \tag{2.1}$$

and the following properties hold:

$$\|\nabla g\|_\infty \leq C\alpha, \tag{2.2}$$

$$b_i \in W_0^{1,p}(Q_i) \text{ and } \int_{Q_i} |\nabla b_i|^p \leq C\alpha^p |Q_i|, \tag{2.3}$$

$$\sum_i |Q_i| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p, \tag{2.4}$$

$$\sum_i \mathbf{1}_{Q_i} \leq N, \tag{2.5}$$

where C and N depends only on dimension and p.

As usual, cubes are with sides parallel to the axes and $|E|$ is the Lebesgue measure of a set E. The space $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The point is in the fact that the functions b_i are supported in cubes as the original Calderón-Zygmund decomposition applied to ∇f would not give this. Note that the assumption on f implies by classical regularity results that f is locally integrable.

Proof: If $p=\infty$, set $g=f$. Assume next that $p<\infty$. Let $\Omega=\{x\in\mathbb{R}^n;\ M(|\nabla f|^p)(x)>\alpha^p\}$ where M is the uncentered maximal operator over cubes of $\mathbb{R}^n$. If Ω is empty, then set $g=f$. Otherwise, the maximal theorem gives us

$$|\Omega|\le C\alpha^{-p}\int_{\mathbb{R}^n}|\nabla f|^p.$$

Let F be the complement of Ω. By the Lebesgue differentiation theorem, $|\nabla f|\le\alpha$ almost everywhere on F. We also have,

Lemma 7. *One can redefine f almost nowhere on F so that for all $x\in F$, for all cube Q centered at x,*

$$|f(x)-m_Q f|\le C\alpha\ell(Q) \tag{2.6}$$

where $\ell(Q)$ is the sidelength of Q and for all $x,y\in F$,

$$|f(x)-f(y)|\le C\alpha|x-y|. \tag{2.7}$$

The constant C depends only on dimension and p.

Here $m_E f$ denotes the mean of f over E. It is well-defined if E is a cube as f is locally integrable. Let us postpone the proof of this lemma and continue the argument.

Let (Q_i) be a Whitney decomposition of Ω by dyadic cubes. Hence, Ω is the disjoint union of the Q_i's, the cubes $2Q_i$ [1] are contained in Ω and have the bounded overlap property, but the cubes $4Q_i$ intersect F. As usual, λQ is the cube co-centered with Q with sidelength λ times that of Q. Hence (2.4) and (2.5) are satisfied by the cubes $2Q_i$. Let us now define the functions b_i. Let $(\mathcal{X}_i)$ be a partition of unity on Ω associated to the covering (Q_i) so that for each i, $\mathcal{X}_i$ is a C^1 function supported in $2Q_i$ with $\|\mathcal{X}_i\|_\infty+\ell_i\|\nabla\mathcal{X}_i\|_\infty\le c(n)$, ℓ_i being the sidelength of Q_i. Pick a point $x_i\in 4Q_i\cap F$. Set

$$b_i=(f-f(x_i))\mathcal{X}_i.$$

[1] Strictly speaking, $2Q_i$ may not have the bounded overlap property but cQ_i do for some $c>1$. The value of c does not play any role and we take it as 2 for simplicity.

It is clear that b_i is supported in $2Q_i$. Let us estimate $\int_{2Q_i} |\nabla b_i|^p$. Introduce $\widetilde{Q}_i$ the cube centered at x_i with sidelength $8\ell_i$. Then $2Q_i \subset \widetilde{Q}_i$. Set $c_i = m_{2Q_i} f$ and $\tilde{c}_i = m_{\widetilde{Q}_i} f$ and write

$$b_i = (f - c_i)\mathcal{X}_i + (c_i - \tilde{c}_i)\mathcal{X}_i + (\tilde{c}_i - f(x_i))\mathcal{X}_i.$$

By (2.6) and (2.5) for the cubes $2Q_i$, $|\tilde{c}_i - f(x_i)| \le C\alpha\ell_i$, hence $\int_{2Q_i} |\tilde{c}_i - f(x_i)|^p |\nabla \mathcal{X}_i|^p \le C\alpha^p |2Q_i|$. Next, using the L^p-Poincaré inequality and the fact that $\widetilde{Q}_i \cap F$ is not empty,

$$|c_i - \tilde{c}_i| \le \frac{1}{|2Q_i|} \int_{\widetilde{Q}_i} |f - \tilde{c}_i| \le C\ell_i \left(\frac{1}{|\widetilde{Q}_i|} \int_{\widetilde{Q}_i} |\nabla f|^p \right)^{1/p} \le C\alpha\ell_i.$$

Hence, $\int_{2Q_i} |c_i - \tilde{c}_i|^p |\nabla \mathcal{X}_i|^p \le C\alpha^p |2Q_i|$. Lastly, since $\nabla\big((f - c_i)\mathcal{X}_i\big) = \mathcal{X}_i \nabla f + (f - c_i)\nabla \mathcal{X}_i$, we have again by the L^p-Poincaré inequality and the fact that the average of $|\nabla f|^p$ on $2Q_i$ is controlled by $C\alpha^p$ that

$$\int_{2Q_i} |\nabla\big((f - c_i)\mathcal{X}_i\big)|^p \le C\alpha^p |2Q_i|.$$

Thus (2.3) is proved.

Set $h(x) = \sum_i f(x_i)\nabla \mathcal{X}_i(x)$. Note that this sum is locally finite and $h(x) = 0$ for $x \in F$. Note also that $\sum_i \mathcal{X}_i(x)$ is 1 on Ω and 0 on F. Since it is also locally finite we have $\sum_i \nabla \mathcal{X}_i(x) = 0$ for $x \in \Omega$. We claim that $h(x) \le C\alpha$. Indeed, fix $x \in \Omega$. Let Q_j be the Whitney cube containing x and let I_x be the set of indices i such that $x \in 2Q_i$. We know that $\sharp I_x \le N$. Also for $i \in I_x$ we have that $C^{-1}\ell_i \le \ell_j \le C\ell_i$ and $|x_i - x_j| \le C\ell_j$ where the constant C depends only on dimension (see [**St**]). We have

$$|h(x)| = \left| \sum_{i \in I_x} (f(x_i) - f(x_j))\nabla \mathcal{X}_i(x) \right| \le C \sum_{i \in I_x} |f(x_i) - f(x_j)| \ell_i^{-1} \le CN\alpha,$$

by the previous observations.

It remains to obtain (2.1) and (2.2). We easily have using $\sum_i \nabla \mathcal{X}_i(x) = 0$ for $x \in \Omega$, that

$$\nabla f = (\nabla f)\mathbf{1}_F + h + \sum_i \nabla b_i, \quad \text{a.e. .}$$

Now $\sum_i b_i$ is a well-defined distribution on $\mathbb{R}^n$. Indeed, for a test function u, using the properties of the Whitney cubes,

$$\sum_i \int |b_i u| \le C \int \left(\sum_i |b_i(x)| \ell_i^{-1} \right) |u(x)|\, d(x, F)\, dx$$

and the last sum converges in L^p as a consequence of (2.4) and

Lemma 8. *Set* $p^* = \frac{np}{n-p}$ *if* $p < n$ *and* $p^* = \infty$ *otherwise, then for all real numbers* r *with* $p \le r \le p^*$,

$$\Big\| \sum_i |b_i| \ell_i^{-1} \Big\|_r^r \le C\alpha^r \sum_i |Q_i|. \tag{2.8}$$

Admit this lemma and set $g = f - \sum_i b_i$. Then $\nabla g = (\nabla f)\mathbf{1}_F + h$ in the sense of distributions and, hence, ∇g is a bounded function with $\|\nabla g\|_\infty \le C\alpha$. □

Proof of Lemma 8: By (2.5) and the Poincaré-Sobolev inequality:

$$\Big\| \sum_i |b_i| \ell_i^{-1} \Big\|_r^r \le N \sum_i \| |b_i| \ell_i^{-1} \|_r^r \le NC \sum_i \ell_i^{r\theta} \|\nabla b_i\|_p^r$$

where $\theta = \frac{n}{r} - \frac{n}{p}$. By (2.3), $\ell_i^{r\theta} \|\nabla b_i\|_p^r \le \alpha^r \ell_i^{nr/p}$, hence

$$\Big\| \sum_i |b_i| \ell_i^{-1} \Big\|_r^r \le CN\alpha^r \sum_i \ell_i^n. \qquad \square$$

Proof of Lemma 7: Let x be a point in F. Fix such cube Q with center x and let Q_k be co-centered cubes with $\ell(Q_k) = 2^k \ell(Q)$ for k a negative integer. Then, by Poincaré's inequality

$$\begin{aligned} |m_{Q_{k+1}} f - m_{Q_k} f| &\le 2^n |m_{Q_{k+1}} (f - m_{Q_{k+1}} f)| \\ &\le C 2^n \ell(Q_k) (m_{Q_{k+1}} |\nabla f|^p)^{1/p} \\ &\le C 2^k \ell(Q) \alpha \end{aligned}$$

since Q_{k+1} contains $x \in F$. It easily follows that $m_Q f$ has a limit as $|Q|$ tends to 0. If, moreover, x is in the Lebesgue set of f, then this limit is equal to $f(x)$. Redefine f on the complement of the Lebesgue set in F so that $m_Q f$ tends to $f(x)$ with Q centered at x with $|Q| \to 0$. Morevover, summing over k the previous inequality gives us (2.6). To see (2.7), let Q_x be the cube centered at x with sidelength $2|x-y|$ and Q_y be the cube centered at y with sidelength $8|x-y|$. It is easy to see that $Q_x \subset Q_y$. As before, one can see that $|m_{Q_x} f - m_{Q_y} f| \le C\alpha |x-y|$. Hence by the triangle inequality and (2.6), one obtains (2.7) readily. □

Remark. Note that this argument can be adapted on a space of homogeneous type where a notion of gradient is available (see, *e.g.*, [**HaK**]).

Remark. Lemma 7 implies that f is Lipschitz on F with Lipschitz constant α; it should be compared with that of M. Weiss (see [**C**, Lemma 1.4]) which gives a slightly stronger result but only for $p > n$.

Remark. Note that if $p > n$ then there is a sup norm estimate as follows

$$\Big\| \sum_i |b_i| \ell_i^{-\eta} \Big\|_\infty^p \leq C\alpha^p \sum_i |Q_i|,$$

with $\eta = 1 - \frac{n}{p}$.

3. Proof of Theorems 2 and 3

Let L be as in the Introduction. For $1 \leq \rho \leq \infty$, we say that L satisfies (S_ρ) if

$$\exists\, C_\rho \geq 0 \quad \forall\, t > 0 \quad \forall\, f \in L^2 \cap L^\rho(\mathbb{R}^n) \quad \|e^{-tL} f\|_\rho \leq C_\rho \|f\|_\rho. \tag{3.1}$$

Of course (S_2) holds by construction with $C_2 = 1$. The proof of the next lemma is defered to Section 4.

Lemma 9. *If $n = 1$ or $n = 2$, then (S_ρ) holds for all ρ. If $n \geq 3$, (S_ρ) holds for $p_n - \varepsilon < \rho < (p_n - \varepsilon)'$ for some $\varepsilon > 0$ depending only on dimension and the ellipticity constants.*

Theorem 2 and Theorem 3 are therefore a consequence of the next result combined with Marcinkiewicz interpolation.

Theorem 10. *Let $n \geq 1$. Assume that (S_ρ) holds for some $\rho \in [1, 2)$. Let $\rho_* = \frac{n\rho}{n+\rho}$. Then we have*

$$\|L^{1/2} f\|_{p,\infty} \lesssim \|\nabla f\|_p, \quad \text{if } 1 \leq \rho_* < p < 2, \tag{3.2}$$

$$\|L^{1/2} f\|_{1,\infty} \lesssim \|\nabla f\|_1, \quad \text{if } \rho_* < 1. \tag{3.3}$$

Proof: By Lemma 9, one can always assume that $\rho < p_n$. Let $p = 1$ if $\rho_* < 1$ and $\rho_* < p < p_n$ otherwise. Let $f \in C_0^\infty(\mathbb{R}^n)$. We have to establish the weak type estimate

$$|\{x \in \mathbb{R}^n;\ |L^{1/2} f(x)| > \alpha\}| \leq \frac{C}{\alpha^p} \int |\nabla f|^p, \tag{3.4}$$

for all $\alpha > 0$. We use the following resolution of $L^{1/2}$:

$$L^{1/2} f = c \int_0^\infty e^{-t^2 L} L f \, dt$$

with $c = \pi^{-1/2}$ which we omit from now on. It suffices to obtain the result for the truncated integrals $\int_\varepsilon^R \dots$ with bounds independent of ε, R, and then to let $\varepsilon \to 0$ and $R \to \infty$. For the truncated integrals, all the calculations are justified. We ignore this step and argue directly on $L^{1/2}$. Apply the Calderón-Zygmund decomposition of Lemma 6 to f at height α^p and write $f = g + \sum_i b_i$. By construction, $\|\nabla g\|_p \le c\|\nabla f\|_p$. Interpolating with (2.2) yields $\int |\nabla g|^2 \le c\alpha^{2-p} \int |\nabla f|^p$. Hence

$$\left|\left\{x \in \mathbb{R}^n;\, |L^{1/2}g(x)| > \frac{\alpha}{3}\right\}\right| \le \frac{C}{\alpha^2}\int |L^{1/2}g|^2 \le \frac{C}{\alpha^2}\int |\nabla g|^2 \le \frac{C}{\alpha^p}\int |\nabla f|^p$$

where we used the L^2-estimate (1.3) for square roots. To compute $L^{1/2}b_i$, let $r_i = 2^k$ if $2^k \le \ell_i = \ell(Q_i) < 2^{k+1}$ and set $T_i = \int_0^{r_i} e^{-t^2L}L\,dt$ and $U_i = \int_{r_i}^\infty e^{-t^2L}L\,dt$. It is enough to estimate $A = |\{x \in \mathbb{R}^n;\, |\sum_i T_i b_i(x)| > \alpha/3\}|$ and $B = |\{x \in \mathbb{R}^n;\, |\sum_i U_i b_i(x)| > \alpha/3\}|$. Let us bound the first term. First,

$$A \le |\cup_i 4Q_i| + \left|\left\{x \in \mathbb{R}^n \setminus \cup_i 4Q_i;\, \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right|,$$

and by (2.4), $|\cup_i 4Q_i| \le \frac{C}{\alpha^p}\int |\nabla f|^p$. To handle the other term, we need the following lemma (not optimal but sufficient for our needs), whose proof is defered to Section 4.

Lemma 11. *If (S_ρ) holds, for $\rho < q < r < \frac{2n}{n-2}$ (set $\frac{2n}{n-2} = \infty$ if $n \le 2$) then for all closed sets E and F, all $h \in L^q(\mathbb{R}^n)$ with support in E and all $t > 0$, we have*

$$\|e^{-t^2L}t^2Lh\|_{L^r(F)} \le \frac{C}{t^\gamma} e^{-\frac{cd(E,F)^2}{t^2}} \|h\|_q \tag{3.5}$$

with $\gamma = \frac{n}{q} - \frac{n}{r}$ and $d(E,F)$ the distance between E and F, where the constants C, c depend uniquely on n, λ, Λ, C_ρ, q, r.

Let $q = 2$ if $n = 1$ and $q = p^* = \frac{np}{n-p}$, the Sobolev exponent for the embedding $\dot{W}^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, if $n \geq 2$. Observe that $\rho < q \leq 2$ by our choice of p. Now,

$$\left|\left\{x \in \mathbb{R}^n \setminus \cup_i 4Q_i; \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right| \leq \frac{C}{\alpha^q} \int \left|\sum_i h_i\right|^q$$

with $h_i = \mathbf{1}_{(4Q_i)^c} |T_i b_i|$. To estimate the L^q norm, we dualize against $u \in L^{q'}(\mathbb{R}^n)$ with $\|u\|_{q'} = 1$ and follow the calculations in [**BK2**]:

$$\int |u| \sum_i h_i = \sum_i \sum_{j=2}^{\infty} A_{ij}$$

where

$$A_{ij} = \int_{F_{ij}} |T_i b_i||u|,$$

$$F_{ij} = 2^{j+1} Q_i \setminus 2^j Q_i.$$

Choose a number r with $q < r < \frac{2n}{n-2}$. By Minkowski integral inequality and Lemma 11 with $F = F_{ij}$, $E = Q_i$ and $h = b_i$

$$\begin{aligned} \|T_i b_i\|_{L^r(F_{ij})} &\leq c \int_0^{r_i} \|e^{-t^2 L} L b_i\|_{L^r(F_{ij})}\, dt \\ &\leq c \int_0^{r_i} \frac{C}{t^{\gamma+2}} e^{-\frac{c4^j r_i^2}{t^2}}\, dt \|b_i\|_q, \end{aligned}$$

where we used $r_i \sim \ell_i$. By Poincaré-Sobolev inequality and (2.3),

$$\|b_i\|_q \leq c \ell_i^{1-(\frac{n}{p}-\frac{n}{q})} \|\nabla b_i\|_p \leq c\alpha \ell_i^{1+\frac{n}{q}},$$

hence for some appropriate constants C, c,

$$\|T_i b_i\|_{L^r(F_{ij})} \leq C\alpha e^{-c4^j} \ell_i^{\frac{n}{r}}.$$

Now remark that for any $y \in Q_i$ and any $j \geq 2$,

$$\left(\int_{F_{ij}} |u|^{r'}\right)^{1/r'} \leq \left(\int_{2^{j+1}Q_i} |u|^{r'}\right)^{1/r'} \leq (2^{n(j+1)}|Q_i|)^{1/r'} \left(M(|u|^{r'})(y)\right)^{1/r'}.$$

Applying Hölder inequality, one obtains

$$A_{ij} \leq C\alpha 2^{nj/r'} e^{-c4^j} \ell_i^n \left(M(|u|^{r'})(y)\right)^{1/r'}.$$

Averaging over Q_i yields

$$A_{ij} \leq C\alpha 2^{nj/r'} e^{-c4^j} \int_{Q_i} \Big(M(|u|^{r'})(y) \Big)^{1/r'} \, dy.$$

Summing over $j \geq 2$ and i, we have

$$\int |u| \sum_i h_i \leq C\alpha \int \sum_i \mathbf{1}_{Q_i}(y) \Big(M(|u|^{r'})(y) \Big)^{1/r'} \, dy.$$

Applying Hölder's inequality with exponent q, q' and the maximal theorem since $q' > r'$, one obtains

$$\int |u| \sum_i h_i \leq C\alpha \Big\| \sum_i \mathbf{1}_{Q_i} \Big\|_q .$$

Hence

$$\left| \left\{ x \in \mathbb{R}^n \setminus \cup_i 4Q_i ; \left| \sum_i T_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right| \leq C \Big\| \sum_i \mathbf{1}_{Q_i} \Big\|_q^q \leq \frac{C}{\alpha^p} \int |\nabla f|^p$$

by (2.5) and (2.4).

It remains to handling the term B. Using functional calculus for L one can compute U_i as $r_i^{-1}\psi(r_i^2 L)$ with ψ the holomorphic function on the sector $|\arg z| < \pi/2$ given by

$$\psi(z) = \int_1^\infty e^{-t^2 z} z \, dt. \tag{3.6}$$

It is easy to show that $|\psi(z)| \leq C|z|^{1/2} e^{-c|z|^2}$, uniformly on subsectors $|\arg z| \leq \mu < \pi/2$.

We invoke the following lemma, whose proof is also defered to Section 4.

Lemma 12. *If (S_ρ) holds then for $\rho < q \leq 2$*

$$\Big\| \sum_{k\in\mathbb{Z}} \psi(4^k L)\beta_k \Big\|_q \leq C \Big\| \Big(\sum_{k\in\mathbb{Z}} |\beta_k|^2 \Big)^{1/2} \Big\|_q , \tag{3.7}$$

whenever the right hand side is finite. The constant C depends on n, λ, Λ, C_ρ, q.

To apply this lemma, observe that the definitions of r_i and U_i yield

$$\sum_i U_i b_i = \sum_{k\in\mathbb{Z}} \psi(4^k L)\beta_k$$

with

$$\beta_k = \sum_{i, r_i = 2^k} \frac{b_i}{r_i}.$$

Using the bounded overlap property (2.5), one has that

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q^q \leq C \int \sum_i \frac{|b_i|^q}{r_i^q}.$$

By Lemma 8, together with $\ell_i \sim r_i$,

$$\int \sum_i \frac{|b_i|^q}{r_i^q} \leq C \alpha^q \sum_i |Q_i|.$$

Hence, by (2.4)

$$\left| \left\{ x \in \mathbb{R}^n; \left| \sum_i U_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right| \leq C \sum_i |Q_i| \leq \frac{C}{\alpha^p} \int |\nabla f|^p. \qquad \square$$

4. Proof of technical lemmata

Let L be as in the Introduction. There exists $\omega \in [0, \pi/2)$ depending only on the ellipticity constants such that L is of type ω (see, *e.g.* [**AT1**]) on $L^2(\mathbb{R}^n)$. This implies that for some holomorphic functions f in the open sectors $\Sigma_\mu = \{z \in \mathbb{C}^*; |\arg z| < \mu\}$, $f(L)$ can be defined as a bounded operator on $L^2(\mathbb{R}^n)$. Since L is maximal-accretive, it has an $H^\infty(\Sigma_{\pi/2})$-functional calculus on $L^2(\mathbb{R}^n)$. The semigroup (e^{-tL}) has an analytic extension to a complex semigroup (e^{-zL}) of contractions on $L^2(\mathbb{R}^n)$ for $z \in \Sigma_{\pi/2-\omega}$.

Lemma 13. *There is an $r \in (1, 2)$ depending on dimension and the ellipticity constants only, such that L extends to a bounded and invertible operator from $\dot{W}^{1,p}(\mathbb{R}^n)$ onto $\dot{W}^{-1,p}(\mathbb{R}^n)$ and $I + L$ extends to a bounded operator from $W^{1,p}(\mathbb{R}^n)$ onto $W^{-1,p}(\mathbb{R}^n)$ for $|\frac{1}{2} - \frac{1}{p}| < |\frac{1}{2} - \frac{1}{r}|$.*

Recall that the homogenous Sobolev space $\dot{W}^{1,p}(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ for $\|\nabla f\|_p$ when $1 < p < \infty$ and $\dot{W}^{-1,p'}(\mathbb{R}^n)$ is its dual space. This lemma is in [**AT1**]. It can be seen by two methods: either by a direct comparison between L and $-\Delta$ the Laplacian operator after renormalisation of the coefficients of L and r can be estimated in terms of $\|A - I\|_\infty$ and the norms of the classical Riesz transforms $\partial_j(-\Delta)^{1/2}$ acting on $L^p(\mathbb{R}^n)$; or by an abstract interpolation method due to do I. Sneiberg relying on the Schwarz lemma for holomorphic functions.

Lemma 14. *Let $r_* = \frac{nr}{n+r}$. Let $\rho \le 2$ be such that $\rho > r_*$ if $r_* \ge 1$ or $\rho = 1$ if $r_* < 1$. For $\mu \in (\omega, \pi/2)$ and $z \in \Sigma_{\pi/2-\mu}$,*

$$(4.1)\quad \|e^{-zL}f\|_2 + \|e^{-zL^*}f\|_2 \le C\sqrt{|z|}^{-(\frac{n}{\rho}-\frac{n}{2})}\|f\|_\rho, \quad f \in L^2 \cap L^\rho(\mathbb{R}^n)$$

where C depending only on dimension, ellipticity, ρ and μ.

Proof: Assume first that $|z| = 1$. By Lemma 13 and the Sobolev embedding theorem, in a finite number of steps $(1+L)^{-k}$ extends to a bounded map from $L^\rho(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Note that k is depends only on r, hence ellipticity, and dimension. Let $f \in L^2 \cap L^\rho(\mathbb{R}^n)$. Since f is in $L^2(\mathbb{R}^n)$, the equality

$$e^{-zL}f = e^{-zL}(I+L)^k(I+L)^{-k}f$$

is justified. As $e^{-zL}(I+L)^k$ extends to bounded operator on $L^2(\mathbb{R}^n)$, we have obtained that $\|e^{-zL}f\|_{\rho'} \le C\|f\|_\rho$, with a constant C that depends only on ellipicity, dimension, ρ and μ.

If $|z| \ne 1$, then the affine change of variable in $\mathbb{R}^n$ defined by $g(x) = f(|z|^{1/2}x)$ gives us $e^{-zL}f(x) = (e^{-\arg z\, L_{|z|}}g)(|z|^{-1/2}x)$ with $L_{|z|}$ the second order operator with coefficients $A(|z|^{1/2}x)$. Since $L_{|z|}$ has same ellipticity constants as L, the previous bound applies and yields the desired estimate.

The same argument applies to L^*. □

Let us now recall the following well-known results although not explicitely stated as such in the literature.

Proposition 15. *Let $p \in [1,2)$ and $n \ge 1$.*

1. *If (S_p) holds then $e^{-tL}\colon L^p \to L^2$ with norm bounded by $Ct^{-\gamma_p/2}$, $\gamma_p = \frac{n}{p} - \frac{n}{2}$.*
2. *If $e^{-tL}\colon L^p \to L^2$ with norm bounded by $Ct^{-\gamma_p/2}$, then for all $q \in (p,2)$ there is a sector Σ_ν for which we have the following $L^q - L^2$ off-diagonal bounds: for all closed sets E and F, all $h \in L^q(\mathbb{R}^n)$ with support in E and all $z \in \Sigma_\nu$ for some $\nu > 0$, we have*

$$(4.2)\qquad \|e^{-zL}h\|_{L^2(F)} \le \frac{C}{|z|^{\gamma_q/2}}e^{-\frac{cd(E,F)^2}{|z|}}\|h\|_q.$$

3. *If $L^p - L^2$ off-diagonal bounds above hold then L satisfies (S_p).*

Proof: The proof of 1 is classical from Nash type inequalities. Briefly, we start from the Gagliardo-Nirenberg inequality

$$\|f\|_2^2 \le C\|\nabla f\|_2^{2\alpha}\|f\|_p^{2\beta}$$

with $\alpha+\beta=1$ and $(1+\gamma_p)\alpha=\gamma_p$. By ellipticity

$$\|\nabla e^{-tL}f\|_2^2 \le \lambda^{-1}\Re\langle A\nabla e^{-tL}f, \nabla e^{-tL}f\rangle \le (2\lambda)^{-1}\frac{d}{dt}\|e^{-tL}f\|_2^2.$$

If $f\in L^2\cap L^p$ with $\|f\|_p=1$, set $\varphi(t)=\|e^{-tL}f\|_2^2$. It is a non-increasing function. Using (S_p), one has $\varphi(t)^{1/\alpha}\le C\varphi'(t)$. Integrating between t and $2t$ one finds easily that $\varphi(t)\le Ct^{-\frac{\alpha}{\alpha-1}}$, which is the desired estimate.

The proof of 2 consists in interpolating by the complex method the L^p-L^2 boundeness assumption with the L^2-L^2 off diagonal bounds [**Da**], [**AHLMcT**]: for all closed sets E and F, all $h\in L^2(\mathbb{R}^n)$ with support in E and all $z\in\Sigma_\mu$, $\mu\le\frac{\pi}{2}-\omega$, we have

$$\|e^{-zL}h\|_{L^2(F)} \le Ce^{-\frac{cd(E,F)^2}{|z|}}\|h\|_2. \tag{4.3}$$

The proof of 3 can be seen by adapting the one of Theorem 25 in [**Da**] to our situation. □

We now prove the lemmata stated in Section 3.

Proof of Lemma 9: The case of dimensions $n=1,2$ is in [**AMcT**]. For $n\ge 3$, it suffices to combine Lemma 14 and Proposition 15. □

Proof of Lemma 11: The assumption is that (S_ρ) holds. And we know (S_p) holds at least for $|\frac{1}{2}-\frac{1}{p}|<\frac{1}{n}$. It follows from Proposition 15 that

$$\|e^{-zL}h\|_{L^r(F)} \le \frac{C}{\sqrt{|z|^{\frac{n}{q}-\frac{n}{r}}}}e^{-\frac{cd(E,F)^2}{|z|}}\|h\|_q \tag{4.4}$$

for any q, r with $\rho<q\le r<\frac{2n}{n-2}$ and $|\arg z|\le\nu$ for some $\nu>0$ whenever h is supported in E. Then (3.5) follows by analyticity of the semigroup on L^2. □

Proof of Lemma 12: Dualizing, (3.7) is equivalent to the square function estimate,

$$\left\|\left(\sum_{k\in\mathbb{Z}}|\psi(4^kL^*)u|^2\right)^{1/2}\right\|_{q'} \le C\|u\|_{q'} \tag{4.5}$$

where L^* is the adjoint of L. Now, it is proved in [**BK1**, Theorem 1.1], that under the off-diagonal estimates (4.4) imply that L has a bounded holomorphic functional calculus on L^q for $\rho<q<2$ (see also Proposition 2.3 of this same paper). By duality, L^* has a bounded holomorphic functional calculus on $L^{q'}$. But it is proved in [**CDMcY**] that this implies (4.5) for a class of functions ψ containing ours (see also [**LeM**] for an informative discussion on this). This proves Lemma 12. □

Proof of Proposition 1: Lemma 9 gives us a range $(p_n - \varepsilon, 2)$ of ρ for which (S_ρ) holds. Hence we obtain (4.4) and it suffices to apply [**BK2**, Theorem 1.1] to obtain L^p-boundedness the Riesz transforms for $p \in (p_n - \varepsilon, 2)$, hence for $p \in (p_n - \varepsilon, p_n]$. □

5. Comments on earlier L^p results

Let us begin with earlier results on the Riesz transform L^p-boundedness.

They hold in the range of $1 < p < 2$ if the Gaussian property holds (see [**AT1**]), which means that the kernel of the semigroup e^{-tL} satisfies a global (in time) Gaussian upper bound together with Hölder regularity in the space variables. In that case, the Riesz transform is bounded on the Hardy space $\mathcal{H}^1$. This applies for example to any **real** elliptic operator, symmetric or not. The L^p result holds under the weaker hypothesis that the Gaussian upper bound hold, and in that case it is weak type $(1, 1)$.

Then, the next results are for a range $\rho < p < \infty$ for some $\rho \geq 1$. Proposition 2.3 of [**BK2**] (see also the argument of Proposition 1 above) implies that if (S_ρ) is satisfied then weak type (p, p), hence strong type (p, p) holds in this range. In particular, the result applies to $\rho = 1$ which gives an improvement over [**DMc**] L^p-boundedness result (whereas it does not yield weak type $(1, 1)$ as in [**DMc**]).

Indeed, the Gaussian upper bound (assumed in [**DMc**]) implies (S_1) but the converse is not known in general. It is true if $C_1 = 1$, *i.e.* the semigroup is contracting on L^1, as by [**ABBO**] this is equivalent to the fact that L is real. But we are not aware if (S_1) with $C_1 > 1$ implies the Gaussian upper bound.

The L^p-boundedness of Riesz transforms for $p > 2$ cannot hold in general even for real and symmetric operators. There is always an improvement from L^2-boundedness to $L^{2+\varepsilon}$-boundedness with an ε depending on the operator. We study related problems in a subsequent work [**A**]. Let us mention the results of [**ERS**] where it is proved that L^p boundedness holds for all $p \in (1, \infty)$ when the coefficients are continuous and periodic with the same period.

Let us now turn to the reverse inequality, that is $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$. As we said, it follows by duality from the $L^{p'}$-boundedness of the Riesz transform associated to L^* (again, we shall say more on this in our forthcoming work). This gives one results for $p > 2$, but not for $p < 2$.

In [**AT1**], it is proved that it holds for all $p \in (1, \infty)$ provided the Gaussian property holds. In fact, there is a quite remarkable factorization of $L^{1/2}$ as $T\nabla$ where T is a Calderón-Zygmund operator. Again, this applies to any real elliptic operator, symmetric or not. The next available result is from [**AHLMcT**, Proposition 6.2]. There, a Hardy space $\mathcal{H}^1$ estimate is obtained, hence L^p estimates for $1 < p < 2$ by interpolation, under the hypothesis of uniform L^ρ estimates for the resolvent $(I+t^2L)^{-1}$ for some $\rho < \frac{n}{n-1}$. As mentioned in the Introduction, this covers all dimensions up to and including 4. But the method of proof seems limited to such an hypothesis.

As we see the hypothesis there is on the resolvent and here on the semigroup. They are essentially equivalent, up to changing the ρ's. The hypothesis on the resolvent is implied by (S_ρ) for the same ρ, using the Laplace transform to compute resolvents from semigroups. Next by analyticity and complex interpolation, uniform L^ρ estimates for the resolvent $(I + t^2L)^{-1}$ for some $\rho < 2$ implies $(S_{\rho'})$ for any $\rho < \rho' < 2$. Thus, Theorem 10 is an extension of Proposition 6.2 of [**AHLMcT**].

6. Calderón-Zygmund lemma for $W^{m,p}$-functions

Lemma 16. *Let $n \geq 1$, $1 \leq p \leq \infty$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ be such that $\|\nabla^m f\|_p < \infty$. Let $\alpha > 0$. Then, one can find a collection of cubes (Q_i), functions g and b_i such that*

$$f = g + \sum_i b_i \tag{6.1}$$

and the following properties hold:

$$\|\nabla^m g\|_\infty \leq C\alpha, \tag{6.2}$$

$$b_i \in W_0^{m,p}(Q_i) \text{ and } \int_{Q_i} |\nabla^m b_i|^p \leq C\alpha^p |Q_i|, \tag{6.3}$$

$$\sum_i |Q_i| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla^m f|^p, \tag{6.4}$$

$$\sum_i \mathbf{1}_{Q_i} \leq N, \tag{6.5}$$

where C and N depends only on dimension and p.

Here $\nabla^m f$ is the array of all partial derivatives $D^\gamma f$ of f with order m, $\|\nabla^m f\|_p$ is the L^p-norm of its length, the latter being computed with any convenient norm on a finite dimensional space. The space $W_0^{m,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. Note that the assumption on f implies by classical regularity results that f is locally integrable.

Proof: If $p=\infty$, set $g=f$. Assume next that $p<\infty$. Let $\Omega=\{x\in\mathbb{R}^n;\ M(|\nabla^m f|^p)(x)>\alpha^p\}$ where M is the uncentered maximal operator over cubes of $\mathbb{R}^n$. If Ω is empty, then set $g=f$. Otherwise, the maximal theorem gives us

$$|\Omega|\le C\alpha^{-p}\int_{\mathbb{R}^n}|\nabla^m f|^p.$$

Let F be the complement of Ω.

Let (Q_i) be a Whitney decomposition of Ω by dyadic cubes. Hence, Ω is the disjoint union of the Q_i's, the cubes $2Q_i$ are contained in Ω and have the bounded overlap property, but the cubes $4Q_i$ intersect F. Hence (6.4) and (6.5) are satisfied by the cubes $2Q_i$. Let us now define the functions b_i. Let $(\mathcal{X}_i)$ be a partition of unity on Ω associated to the covering (Q_i) so that for each i, $\mathcal{X}_i$ is a C^m function supported in $2Q_i$ with $\ell_i^{|\gamma|}\|D^\gamma\mathcal{X}_i\|_\infty\le c(n,m)$ for all multiindices γ with $|\gamma|\le m$, ℓ_i being the sidelength of $2Q_i$.

If Q is a cube, let $P_Q f$ be the Poincaré polynomial of f with degree less than or equal to $m-1$ associated to Q. Set

$$b_i=(f-P_{2Q_i}f)\mathcal{X}_i,$$

and

$$g=f-\sum_i b_i.$$

It remains to establish the desired properties on b_i and g.

First, we recall some properties of $P_Q f$, following [**Mor**] (see [**GM**] for a short presentation). It is uniquely defined by the relations $\int_Q D^\gamma(f-P)=0$ for all multiindices γ with $|\gamma|\le m-1$. For Q fixed, it is linearly dependent on f. We have the uniform estimates

$$\sup_Q|D^\gamma P_Q f|\le C(n,m)\ell^{-n}\sum_{\beta>\gamma,\,|\beta|\le m-1}\int_Q|D^\beta f| \tag{6.6}$$

and the Poincaré inequalities

$$\int_Q|D^\gamma(f-P_Q f)|^p\le C(n,m,p)\ell^{p(m-|\gamma|)}\int_Q|\nabla^m f|^p \tag{6.7}$$

for all γ with $|\gamma|\le m-1$, where $C(n,m)$, $C(n,m,p)$ are universal constants independent of f, Q and ℓ is the sidelength of Q.

It is clear that b_i is supported in $2Q_i$. Let us estimate $\int_{2Q_i} |\nabla^m b_i|^p$. It suffices to apply the Leibniz rule and to invoke the Poincaré inequalities to each term. This readily yields (6.3).

The next step is to verify that $\sum_i b_i$ is a distribution. Indeed, for a test function u, using the properties of the Whitney cubes,

$$\sum_i \int |b_i u| \leq C \int \left(\sum_i |b_i(x)| \ell_i^{-m} \right) |u(x)|\, d(x,F)^m\, dx$$

and the last sum converges in L^p as a consequence of (6.4) and

Lemma 17. *Set $p^* = \frac{np}{n-mp}$ if $mp < n$ and $p^* = \infty$ otherwise. Then for all real numbers r with $p \leq r \leq p^*$,*

$$\left\| \sum_i |b_i| \ell_i^{-m} \right\|_r^r \leq C\alpha^r \sum_i |Q_i|. \tag{6.8}$$

Proof of Lemma 17: By (6.5) and the Poincaré-Sobolev inequality:

$$\left\| \sum_i |b_i| \ell_i^{-m} \right\|_r^r \leq N \sum_i \| |b_i| \ell_i^{-m} \|_r^r \leq NC \sum_i \ell_i^{r\theta} \|\nabla^m b_i\|_p^r$$

where $\theta = \frac{n}{r} - \frac{n}{p}$. By (6.3), $\ell_i^{r\theta} \|\nabla^m b_i\|_p^r \leq \alpha^r \ell_i^{nr/p}$, hence

$$\left\| \sum_i |b_i| \ell_i^{-m} \right\|_r^r \leq CN\alpha^r \sum_i \ell_i^n.$$

Hence $g = f - \sum_i b_i$ is well-defined. Let us compute $\nabla^m g$. Recall that $\sum_i \mathcal{X}_i(x)$ is 1 on Ω and 0 on F. Since it is also locally finite we have $\sum_i D^\gamma \mathcal{X}_i(x) = 0$ for $x \in \Omega$ for all γ with $1 \leq |\gamma| \leq m$. Hence, if $|\gamma| = m$, this and the Leibniz rule yield in the sense of distributions on $\mathbb{R}^n$,

$$\sum_i D^\gamma b_i = D^\gamma f \sum_i \mathcal{X}_i + \sum_i \sum_{\beta<\gamma,\, |\beta|\geq 1} c_{\beta,\gamma} D^\beta (P_i f) D^{\gamma-\beta} \mathcal{X}_i.$$

Fix a multiindex β and set $h = h_{\beta,\gamma} = \sum_i D^\beta (P_i f) D^{\gamma-\beta} \mathcal{X}_i$. We show that $h \leq C\alpha$. Admitting this, we obtain $D^\gamma g$ is a bounded function with

$$D^\gamma g = (D^\gamma f) \mathbf{1}_F - \sum_{\beta<\gamma,\, |\beta|\geq 1} c_{\beta,\gamma} h_{\beta,\gamma},$$

almost everywhere and this gives us (6.2).

Note that the sum defining h is locally finite on Ω and $h(x) = 0$ for $x \in F$. Let $x \in \Omega$ and Q_j be the Whitney cube containing x and let I_x be the set of indices i such that $x \in 2Q_i$. We know that $\sharp I_x \leq N$. For $i \in I_x$ we have that $C^{-1} \ell_i \leq \ell_j \leq C \ell_i$ and $|z - y| \leq C \ell_j$ for $z \in Q_i$

and $y \in Q_j$, where the constant C depends only on dimension (see [**St**]). Let x_j be a point in $F \cap 4Q_j$ and let $\tilde{Q}_j$ be the smallest cube centered at x_j containing all of the cubes $2Q_i$ for $i \in I_x$. It is easy to see that its lengthside $\tilde{\ell}_j$ is comparable to ℓ_j. As $\gamma - \beta \neq 0$, we may write

$$h(x) = \sum_{i \in I_x} D^\beta (P_{2Q_i} f - P_{\tilde{Q}_j} f)(x) D^{\gamma-\beta} \mathcal{X}_i(x)$$

so that the conclusion will follow from

$$|D^\beta (P_{2Q_i} f - P_{\tilde{Q}_j} f)(x)| \leq C(\tilde{\ell}_j)^{m-|\beta|} \left(\int_{\tilde{Q}_j} |\nabla^m f|^p \right)^{1/p} \tag{6.9}$$

with a constant C independent of x, f, as

$$\int_{\tilde{Q}_j} |\nabla^m f|^p \leq M(|\nabla^m f|^p)(x_j)|\tilde{Q}_j| \leq \alpha^p |\tilde{Q}_j|.$$

First remark that by construction and uniqueness $P_{2Q_i}(P_{\tilde{Q}_j} f) = P_{\tilde{Q}_j} f$. Hence the left hand side of (6.9) is dominated by

$$\sup_{2Q_i} |D^\beta (P_{2Q_i}(f - P_{\tilde{Q}_j} f))|.$$

By (6.6), this is controlled by constant times sums of

$$\ell_i^{-n} \int_{2Q_i} |D^{\beta'}(f - P_{\tilde{Q}_j} f)|$$

with $\beta' > \beta$ and $|\beta'| \leq m - 1$. Using that $2Q_i \subset \tilde{Q}_j$, $\ell_i \sim \tilde{\ell}_j$, and the Poincaré inequalities (6.7), this last expression is dominated by the right hand side of (6.9) as desired. □

7. Results for higher order operators

Consider an homogeneous elliptic operator L of order m, $m \in \mathbb{N}$, $m \geq 2$, defined by

$$Lf = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (a_{\alpha\beta} \partial^\beta f), \tag{7.1}$$

where the coefficients $a_{\alpha\beta}$ are complex-valued L^∞ functions on $\mathbb{R}^n$, and we assume

$$\left| \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta}(x) \partial^\beta f(x) \partial^\alpha \bar{g}(x)\, dx \right| \leq \Lambda \|\nabla^m f\|_2 \|\nabla^m g\|_2 \tag{7.2}$$

and the Gårding inequality

$$\text{(7.3)} \qquad \operatorname{Re} \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta}(x)\partial^\beta f(x)\partial^\alpha \bar{f}(x)\,dx \geq \lambda \|\nabla^m f\|_2^2$$

for some $\lambda > 0$ and $\Lambda < +\infty$ independent of $f, g \in H^m(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n)$. Again, L is constructed as before as a maximal-accretive operator. It has a square root. The main result of [**AHMcT**] is

$$\text{(7.4)} \qquad \|L^{1/2} f\|_2 \sim \|\nabla^m f\|_2, \quad n \geq 1.$$

If $2m \geq n \geq 1$, it is a consequence of the method of [**AT1**] for second order operators that the following L^p a priori inequalities hold:

$$\text{(7.5)} \qquad \|\nabla^m f\|_p \lesssim \|L^{1/2} f\|_p, \quad 1 < p < 2 + \varepsilon$$

$$\text{(7.6)} \qquad \|L^{1/2} f\|_p \lesssim \|\nabla^m f\|_p, \quad 1 < p < \infty.$$

This is sharp in the range of p's. Further, if $n = 1$, (7.5) and (7.6) are true for $1 < p < \infty$.

Let us now restrict ourselves to the case $2m < n$. First the above inequalities hold if the semigroup satisfies the Gaussian property. It is likely that the method in [**DMc**] extends to give us the Riesz transform weak type $(1,1)$ estimate if only a Gaussian upper bound holds. In their recent mentioned work [**BK2**], Blunck and Kunstmann establish that

$$\text{(7.7)} \qquad \|\nabla^m f\|_p \lesssim \|L^{1/2} f\|_p, \quad p(n,m) < p < 2$$

with $p(n,m)$ the Sobolev exponent for the Sobolev embedding $W^{m,p} \subset L^2$: $p(n,m) = \frac{2n}{2m+n}$. Note that $p(n,m) \leq 1$ is exactly the condition $n \leq 2m$ so that their result recovers the part $p < 2$ of (7.5). In fact their result, together with the relation between (S_ρ) and $L^p - L^q$ off-diagonal estimates (which are somehow abstract and apply as well to higher order operators), states as follows: if (S_ρ) holds for some $\rho \in [1,2)$ then (7.7) is valid for $\rho < p < 2$. Next it is true that $L^p - L^q$ off-diagonal estimates holds for $p = p(n,m)$ and $q = p(n,m)'$, hence (S_ρ) for any $\rho \in [p(n,m), p(n,m)']$ [**Da**, Theorem 25]. In fact, as for second order operators, we observe that (S_ρ) always holds in an extended range $p(n,m) - \varepsilon < \rho < (p(n,m) - \varepsilon)'$ for some ε depending only on dimension and the ellipticity constants, hence the Riesz transform L^p-boundedness is valid in the range $p(n,m) - \varepsilon < \rho < 2$.

We are interested in the reverse inequality and we obtain the following result.

Theorem 18. *Let $n \geq 1$ and $m \geq 2$. We have for* $P(n,m) = \frac{2n}{n+4m}$

$$\|L^{1/2}f\|_p \lesssim \|\nabla^m f\|_p, \quad n \geq 1, \quad \sup(P(n,m)-\varepsilon, 1) < p < 2. \tag{7.8}$$

Furthermore, if $P(n,m) \leq 1$ that is $1 \leq n \leq 4m$, then we have the weak type $(1,1)$ estimate

$$\|L^{1/2}f\|_{1,\infty} \lesssim \|\nabla^m f\|_1. \tag{7.9}$$

We observe that the proof works for $2m \geq n$ and $2m < n$ as well. Note also that $P(n,m)$ is the Sobolev exponent for the embedding $W^{m,p} \subset L^{p(n,m)}$ if $P(n,m) \geq 1$. The consequences are the same as for second order operators extending what was known when $2m \geq n$.

Corollary 19.

$$\|\nabla f\|_p \sim \|L^{1/2}f\|_p, \quad n \geq 2, \quad \sup(p(n,m)-\varepsilon, 1) < p < 2+\varepsilon'. \tag{7.10}$$

Furthermore, the operator $L^{1/2}$, a priori *defined on $C_0^\infty(\mathbb{R}^n)$ with values in $L^2(\mathbb{R}^n)$, extends to a bounded an invertible operator from $\dot{W}^{m,p}(\mathbb{R}^n)$ onto $L^p(\mathbb{R}^n)$ for p in the above range.*

The method of proof of Theorem 18 parallels that for second order operators. First, we observe that (S_ρ) is valid when $p(n,m)-\varepsilon < \rho < (p(n,m)-\varepsilon)'$. So it suffices to show that if (S_ρ) holds for some $\rho \in [1,2)$, then (7.8) holds for $\inf(\rho_*, 1) < p < 2$ where $\rho_* = \frac{\rho n}{\rho m + n}$ (this is the Sobolev exponent p for $W^{m,p} \subset L^\rho$ when $\rho_* \geq 1$) and furthermore (7.9) holds when $\rho_* < 1$.

Next, one can always assume that $\rho < p(n,m)$. Let $p = 1$ if $\rho_* < 1$ and $\rho_* < p < p(n,m)$ otherwise. Let $f \in C_0^\infty(\mathbb{R}^n)$. By interpolating with the L^2-result, it suffices to establish the weak type estimate

$$|\{x \in \mathbb{R}^n;\, |L^{1/2}f(x)| > \alpha\}| \leq \frac{C}{\alpha^p} \int |\nabla^m f|^p, \tag{7.11}$$

for all $\alpha > 0$. To take care of the parabolic homogeneity, we resolve $L^{1/2}$ by

$$L^{1/2}f = c\int_0^\infty e^{-t^{2m}L} Lf \, d(t^m)$$

with $c = \pi^{-1/2}$ which we omit from now on. Again, a rigorous argument would be to truncate the integral away from 0 and ∞. Apply the Calderón-Zygmund decomposition of Lemma 16 to f at height α^p and write $f = g + \sum_i b_i$. By construction, $\|\nabla^m g\|_p \leq c\|\nabla^m f\|_p$. Interpolating with (6.2) yields $\int |\nabla^m g|^2 \le c\alpha^{2-p} \int |\nabla^m f|^p$. Hence

$$\left|\left\{x \in \mathbb{R}^n;\, |L^{1/2}g(x)| > \frac{\alpha}{3}\right\}\right| \le \frac{C}{\alpha^2}\int |L^{1/2}g|^2 \le \frac{C}{\alpha^2}\int |\nabla^m g|^2 \le \frac{C}{\alpha^p}\int |\nabla^m f|^p$$

where we used the L^2-estimate for square roots. To compute $L^{1/2}b_i$, let $r_i = 2^k$ if $2^k \le \ell_i = \ell(Q_i) < 2^{k+1}$ and set $T_i = \int_0^{r_i} e^{-t^{2m}L} L\, d(t^m)$ and $U_i = \int_{r_i}^{\infty} e^{-t^{2m}L} L\, d(t^m)$. It is enough to estimate $A = |\{x \in \mathbb{R}^n;\, |\sum_i T_i b_i(x)| > \alpha/3\}|$ and $B = |\{x \in \mathbb{R}^n;\, |\sum_i U_i b_i(x)| > \alpha/3\}|$. Let us bound the first term. First,

$$A \le |\cup_i 4Q_i| + \left|\left\{x \in \mathbb{R}^n \setminus \cup_i 4Q_i;\, \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right|,$$

and by (2.4), $|\cup_i 4Q_i| \le \frac{C}{\alpha^p}\int |\nabla^m f|^p$. To handle the other term, we invoke the off-diagonal estimates for higher order operators which can be obtained as for second order operators using the same arguments and [**Da**]:

Lemma 20. *If (S_ρ) holds then for $\rho < q < r < \frac{2n}{n-2m}$ (if $n \le 2m$, then set $\frac{2n}{n-2m} = \infty$) for all closed sets E and F, all $h \in L^q(\mathbb{R}^n)$ with support in E and all $t > 0$, we have*

$$\|e^{-t^{2m}L} t^{2m} L h\|_{L^r(F)} \le \frac{C}{t^\gamma} G\left(\frac{cd(E,F)}{t}\right)\|h\|_q \tag{7.12}$$

with $\gamma = \frac{n}{q} - \frac{n}{r}$ and $d(E,F)$ the distance between E and F, where the constants C, c depend uniquely on n, λ, Λ, C_ρ, q, r, and $G(u) = \exp(-u^{\frac{2m}{2m-1}})$.

Let $q = \inf(2, p^*)$ where $p^* = \frac{np}{n-mp}$ is the Sobolev exponent for the embedding $\dot{W}^{m,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$. Observe that $\rho < q \le 2$ by our choice of p. Now,

$$\left|\left\{x \in \mathbb{R}^n \setminus \cup_i 4Q_i;\, \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right| \le \frac{C}{\alpha^q}\int \left|\sum_i h_i\right|^q$$

with $h_i = \mathbf{1}_{(4Q_i)^c}|T_i b_i|$. Let $u \in L^{q'}(\mathbb{R}^n)$ with $\|u\|_{q'} = 1$, then

$$\int |u| \sum_i h_i = \sum_i \sum_{j=2}^{\infty} A_{ij}$$

where

$$A_{ij} = \int_{F_{ij}} |T_i b_i||u|,$$
$$F_{ij} = 2^{j+1}Q_i \setminus 2^j Q_i.$$

Choose a number r with $q < r < \frac{2n}{n-2m}$, then by Minkowski integral inequality and Lemma 20 with q and r, $F = F_{ij}$, $E = Q_i$ and $h = b_i$

$$\begin{aligned}\|T_i b_i\|_{L^r(F_{ij})} &\le \int_0^{r_i} \|e^{-t^{2m}L} L b_i\|_{L^r(F_{ij})}\, d(t^m)\\ &\le \int_0^{r_i} \frac{C}{t^{\gamma+2m}} G\left(\frac{c2^j r_i}{t}\right) d(t^m)\|b_i\|_q\\ &\le CG(c2^j) r_i^{\gamma+m}\|b_i\|_q,\end{aligned}$$

where we used $r_i \sim \ell_i$. By Poincaré-Sobolev inequality and (6.3),

$$\|b_i\|_q \le c\ell_i^{m-(\frac{n}{p}-\frac{n}{q})}\|\nabla^m b_i\|_p \le c\alpha\ell_i^{m+\frac{n}{q}},$$

hence

$$\|T_i b_i\|_{L^r(F_{ij})} \le C\alpha G(c2^j)\ell_i^{\frac{n}{r}},$$

for some appropriate constants C, c. Now remark that for any $y \in Q_i$ and any $j \ge 2$,

$$\left(\int_{F_{ij}} |u|^{r'}\right)^{1/r'} \le \left(\int_{2^{j+1}Q_i} |u|^{r'}\right)^{1/r'} \le (2^{n(j+1)}|Q_i|)^{1/r'}\Big(M(|u|^{r'})(y)\Big)^{1/r'}.$$

Applying Hölder inequality, one obtains

$$A_{ij} \le C\alpha 2^{nj/r'} G(c2^j)\ell_i^n \Big(M(|u|^{r'})(y)\Big)^{1/r'}.$$

Averaging over Q_i yields

$$A_{ij} \le C\alpha 2^{nj/r'} G(c2^j) \int_{Q_i} \Big(M(|u|^{r'})(y)\Big)^{1/r'} dy.$$

Summing over $j \ge 2$ and i, we have

$$\int |u| \sum_i h_i \le C\alpha \int \sum_i \mathbf{1}_{Q_i}(y)\Big(M(|u|^{r'})(y)\Big)^{1/r'} dy.$$

Applying Hölder inequality with exponent q, q' and the maximal theorem since $q' > r'$, one obtains

$$\int |u| \sum_i h_i \leq C\alpha \left\| \sum_i \mathbf{1}_{Q_i} \right\|_q .$$

Hence

$$\left| \left\{ x \in \mathbb{R}^n \setminus \cup_i 4Q_i ; \left| \sum_i T_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right| \leq C \left\| \sum_i \mathbf{1}_{Q_i} \right\|_q^q \leq \frac{C}{\alpha^p} \int |\nabla^m f|^p$$

by (6.4) and (6.5).

It remains to handling the term B. Again, one has $U_i = r_i^{-m} \psi(r_i^{2m} L)$ with ψ given by (3.6). We invoke the following lemma which can be proved exactly as for second order operators using recent results in [**BK1**].

Lemma 21. *If (S_ρ) holds then for $\rho < q \leq 2$*

$$\left\| \sum_{k\in\mathbb{Z}} \psi(2^{km} L)\beta_k \right\|_q \leq C \left\| \left(\sum_{k\in\mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q , \tag{7.13}$$

whenever the right hand side is finite. The constant C depends on n, λ, Λ, C_ρ, q.

To apply this lemma, observe that the definitions of r_i and U_i yield

$$\sum_i U_i b_i = \sum_{k\in\mathbb{Z}} \psi(2^{km} L)\beta_k$$

with

$$\beta_k = \sum_{i, r_i = 2^k} \frac{b_i}{r_i} .$$

Using the bounded overlap property (6.5), one has that

$$\left\| \left(\sum_{k\in\mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q^q \leq C \int \sum_i \frac{|b_i|^q}{r_i^q} .$$

By Lemma 17, together with $\ell_i \sim r_i$,

$$\int \sum_i \frac{|b_i|^q}{r_i^q} \leq C\alpha^q \sum_i |Q_i| .$$

Hence, by (6.4)

$$\left|\left\{x \in \mathbb{R}^n;\ \left|\sum_i U_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right| \le C \sum_i |Q_i| \le \frac{C}{\alpha^p} \int |\nabla^m f|^p.$$

8. Concluding remarks

For higher order operators, (7.3) is often replaced by

$$\text{(8.1)}\qquad \operatorname{Re} \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta}(x)\partial^\beta f(x)\partial^\alpha \bar{f}(x)\,dx \ge \lambda \|\nabla^m f\|_2^2 - \kappa \|f\|_2^2$$

for some $\lambda > 0$ and $\kappa \ge 0$ independent of $f, g \in H^m(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n)$. Now $L + \kappa$ is constructed as before as a maximal-accretive operator. It has a square root. By [**AHMcT**], if $\kappa' > \kappa$,

$$\text{(8.2)}\qquad \|(L+\kappa')^{1/2} f\|_2 \sim \|\nabla^m f\|_2 + \|f\|_2.$$

The methods in [**BK1**] for the functional calculus, in [**BK2**] or [**HM**] for the Riesz transforms and in here can be adapted to this situation. The Calderón-Zygmund decomposition is performed under the condition that $f \in W^{m,p}(\mathbb{R}^n)$. To go back to an homogeneous situation, we set $Df = (f, \nabla^m f)$ and argue with respect to the maximal function of $|Df|^p$. We leave to the reader the care of stating the corresponding results, the condition (S_ρ) being used only for small times $t < 1$.

Finally, all these methods can be applied to elliptic systems where ellipticity is in the sense of the validity of the Gårding inequality. And the results are similar.

References

[A] P. Auscher, On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates, Preprint 4, Université Paris XI-Orsay (2004).

[ABBO] P. Auscher, L. Barthélemy, P. Bénilan and El M. Ouhabaz, Absence de la L^∞-contractivité pour les semi-groupes associés aux opérateurs elliptiques complexes sous forme divergence, *Potential Anal.* **12(2)** (2000), 169–189.

[AHLMcT] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian, The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$, *Ann. of Math. (2)* **156(2)** (2002), 633–654.

[AHMcT] P. Auscher, S. Hofmann, A. McIntosh and Ph. Tchamitchian, The Kato square root problem for higher order elliptic operators and systems on $\mathbb{R}^n$, dedicated to the memory of Tosio Kato, *J. Evol. Equ.* **1(4)** (2001), 361–385.

[AMcT] P. Auscher, A. McIntosh and Ph. Tchamitchian, Heat kernels of second order complex elliptic operators and applications, *J. Funct. Anal.* **152(1)** (1998), 22–73.

[AT1] P. Auscher and Ph. Tchamitchian, Square root problem for divergence operators and related topics, *Astérisque* **249** (1998), 172 pp.

[AT2] P. Auscher and Ph. Tchamitchian, Calcul fontionnel précisé pour des opérateurs elliptiques complexes en dimension un (et applications à certaines équations elliptiques complexes en dimension deux), *Ann. Inst. Fourier (Grenoble)* **45(3)** (1995), 721–778.

[BK1] S. Blunck and P. C. Kunstmann, Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus, *Rev. Mat. Iberoamericana* **19(3)** (2003), 919–942.

[BK2] S. Blunck and P. C. Kunstmann, Weak-type (p, p) estimates for Riesz transforms, *Rev. Mat. Iberoamericana* (to appear).

[C] C. P. Calderón, On commutators of singular integrals, *Studia Math.* **53(2)** (1975), 139–174.

[CMcM] R. R. Coifman, A. McIntosh and Y. Meyer, L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes, *Ann. of Math. (2)* **116(2)** (1982), 361–387.

[CDMcY] M. Cowling, I. Doust, A. McIntosh and A. Yagi, Banach space operators with a bounded H^∞ functional calculus, *J. Austral. Math. Soc. Ser. A* **60(1)** (1996), 51–89.

[Da] E. B. Davies, Uniformly elliptic operators with measurable coefficients, *J. Funct. Anal.* **132(1)** (1995), 141–169.

[DMc] X. T. Duong and A. McIntosh, Singular integral operators with non-smooth kernels on irregular domains, *Rev. Mat. Iberoamericana* **15(2)** (1999), 233–265.

[ERS] A. F. M. ter Elst, D. W. Robinson and A. Sikora, On second-order periodic elliptic operators in divergence form, *Math. Z.* **238(3)** (2001), 569–637.

[GM] M. Giaquinta and G. Modica, Regularity results for some classes of higher order nonlinear elliptic systems, *J. Reine Angew. Math.* **311/312** (1979), 145–169.

[HaK] P. Hajłasz and P. Koskela, Sobolev met Poincaré, *Mem. Amer. Math. Soc.* **145(688)** (2000), 101 pp.

[HMc] S. Hofmann and A. McIntosh, The solution of the Kato problem in two dimensions, in: *"Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations"* (El Escorial, 2000), *Publ. Mat.* **Vol. extra** (2002), 143–160.

[HM] S. Hofmann and J. M. Martell, L^p bounds for Riesz transforms and square roots associated to second order elliptic operators, *Publ. Mat.* **47(2)** (2003), 497–515.

[K] T. Kato, Fractional powers of dissipative operators, *J. Math. Soc. Japan* **13** (1961), 246–274.

[Ke] C. E. Kenig, *"Harmonic analysis techniques for second order elliptic boundary value problems"*, CBMS Regional Conference Series in Mathematics **83**, published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.

[LeM] C. Le Merdy, Square functions associated to sectorial operators, *Bull. Soc. Math. France* (to appear).

[Mor] C. B. Morrey, Jr., *"Multiple integrals in the calculus of variations"*, Die Grundlehren der mathematischen Wissenschaften **130**, Springer-Verlag New York, Inc., New York, 1966.

[St] E. M. Stein, *"Singular integrals and differentiability properties of functions"*, Princeton Mathematical Series **30**, Princeton University Press, Princeton, N.J., 1970.

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