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ESTIMATES FOR THE BERGMAN AND SZEGÖ PROJECTIONS FOR PSEUDOCONVEX DOMAINS OF FINITE TYPE WITH LOCALLY DIAGONALIZABLE LEVI FORM

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Abstract

In this paper, we give precise isotropic and non-isotropic estimates for the Bergman and Szegö projections of a bounded pseudoconvex domain whose boundary points are all of finite type and with locally diagonalizable Levi form. Additional local results on estimates of invariant metrics are also given.

1. Introduction

This paper deals with precise mapping properties of the Bergman and Szegö projections of pseudo-convex domains of finite type in $\mathbb{C}^n$ whose Levi form are locally diagonalizable at every point of the boundary (see Section 2 for a precise definition). We obtain sharp estimates for these operators for usual L^p_k Sobolev spaces, classical Lipschitz spaces Λ_α and nonisotropic Lipschitz spaces Γ_α related to the geometry of the domain.

Our results, in the present paper, are analog to those obtained for convex domains of finite type in [**MS94**] and [**MS97**] and extend previously known results for the strictly pseudoconvex case ([**AS79**] and [**PS77**]), for the finite type domains of $\mathbb{C}^2$ ([**NRSW89**], see also [**Chr88**] and [**FK88**]) and in the case of pseudoconvex domains of finite type of $\mathbb{C}^n$ having a Levi form of rank $n-1$ ([**AC99**], see also [**Mac88**]).

Similar results where obtained for pseudoconvex domains in $\mathbb{C}^n$ whose Levi form have comparable eigenvalues (see [**Koe02**], [**Cho02b**] and [**Cho03**]).

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In 1990, C. L. Fefferman, J. J. Kohn and M. Machedon studied the class of domains we consider here and proved local Lipschitz estimate with arbitrary small loss for the $\overline{\partial}$ Neumann problem and the Szegö projection [**FKM90**] (see also [**Der99**]).

The present article is the first of two papers. Here, we prove the following results:

Theorem. *Let Ω be a bounded pseudo-convex domain of finite type in $\mathbb{C}^n$ with locally diagonalizable Levi form.*

(1) *For all p, $1 < p < +\infty$, and all $s \geq 0$, the Bergman projection of Ω maps continuously $\mathscr{L}^p_s(\Omega)$ into itself.*
(2) *For $1 < p < +\infty$ and $s \in \mathbb{N}$, the Szegö projection maps continuously $L^p_s(\partial\Omega)$ into itself.*

Theorem. *Let Ω be a bounded pseudo-convex domain of finite type in $\mathbb{C}^n$ with locally diagonalizable Levi form.*

(1) *For $0 < \alpha < +\infty$, the Bergman projection maps continuously $\Lambda_\alpha(\Omega)$ into itself.*
(2) *The Szegö projection maps continuously $\Lambda_\alpha(\partial\Omega)$ into itself, for all $\alpha \in]0, +\infty[$.*

A consequence of the proof is:

Corollary. *Under the same conditions, the Bergman projector maps continuously $L^\infty(\Omega)$ into $BMO(\Omega)$.*

Defining non-isotropic Lipschitz spaces Γ_α, for $\alpha < 1/M$ where M is the type of Ω, with the pseudo-distance associated to the geometry, we also obtain:

Theorem. *Let Ω be a bounded pseudo-convex domain of finite type in $\mathbb{C}^n$ with locally diagonalizable Levi form.*

(1) *For $\alpha < 1/M$, the Bergman projection maps continuously Λ_α into Γ_α.*
(2) *The Szegö projection maps continuously $\Lambda_\alpha(\Omega)$ into $\Gamma_\alpha(\partial\Omega)$ for $0 < \alpha < 1/M$.*

Extending the definition of Γ_α for all $\alpha > 0$, we also give similar results for $\alpha \geq 1/M$.

In the last section, we indicate some supplementary local results on invariant metrics similar to those obtained by D. W. Catlin in dimension 2 [**Cat89**] and by J. D. McNeal for convex domains [**MN01**].

In a forthcoming paper, we will improve the result of [**FKM90**] getting local (isotropic and non-isotropic) Lipschitz estimates without loss for the Szegö projection.

We essentially use the techniques developed in [**CNS92**], [**NRSW89**], [**MS94**] and [**MS97**] associated to the geometric properties of the domain and the estimates for the Bergman kernel function proved in [**CD**]. In Section 2 we quickly describe this geometry and summarize the results necessary to our purpose.

Our geometry posses some good properties similar to those of convex domains described by McNeal in [**McN94**] but they are not completely equivalent and, if we follow the ideas of [**MS94**] and [**MS97**], the proof must be modified. For example, to evaluate $|f(p) - f(q)|$, $p, q \in \Omega$, we cannot use an integration along the segment $[p, q]$ and we must use a convenient curve essentially given by the exponential map. Similarly, when we want to apply Cauchy formula, we don't have directly the existence of polydisc of good weighted size included in the domain.

Furthermore, we often need to work in some more local context, which implies that we have to modify the definitions of the concepts we use.

In Section 3, devoted to the Bergman projection, we show how to adapt the methods of [**MS94**] to the geometry of the domain we consider, in a relatively detailed way, to get the wright estimates.

Section 4 is devoted to the Szegö projection. Because, on one hand the N.I.S. operators theory is quite well known, and on the other hand, we have detailed the use of the geometry in Section 3, we will only give the main articulations of the proofs.

2. Geometry and Bergman kernel estimates

Let us first precise the class of domains we consider in this paper and introduce the basic notations.

Definition 2.1. Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ with smooth boundary. Let p be a point on the boundary of Ω. We say that the Levi form is locally diagonalizable at p if there exist a neighborhood V of p and a smooth basis $\mathscr{B}$ of sections of the complex tangent bundle $T^{0,1}$ in $V \cap \partial\Omega$ which diagonalizes the Levi form. When this property holds at every point of the boundary, we say that Ω has a locally diagonalizable Levi form.

Let ρ be a smooth defining function of Ω (i.e. $\Omega = \{\rho < 0\}$, $\nabla\rho \neq 0$ on $\partial\Omega$). Let p_0 be a point of $\partial\Omega$. Let $(L_i)_{1 \leq i \leq n-1}$ a family of smooth vector fields in a neighborhood V of p_0 which is a basis of the complex tangent space at $\partial\Omega$ in $V \cap \partial\Omega$ diagonalizing the Levi form, and let

$$L_n = N = \frac{4}{|\nabla\rho|^2} \sum_i \frac{\partial\rho}{\partial\overline{z}_i} \frac{\partial}{\partial z_i}$$

the complex normal vector field.

If $\mathscr{L} = (L^1, \ldots, L^k)$ is a list of vector fields of $\mathscr{B} \cup \overline{\mathscr{B}} \cup \{L_n, \overline{L_n}\}$, we denote by $|\mathscr{L}|$ the length k of $\mathscr{L}$.

For each index i, let $c_{ii} = \langle \partial\rho; [L_i, \overline{L_i}] \rangle$. Let $\mathscr{L}(i)$ the set of all lists of vector fields $\mathscr{L} = (L^1, \ldots, L^k)$ such that $L^j = L_i$ or $L^j = \overline{L_i}$ for all j. If $\mathscr{L} \in \mathscr{L}(i)$, we denote $\mathscr{L}(c_{ii}) = L^1 \ldots L^k(c_{ii})$. Recall that if Ω is of finite type there exists $M > 0$ such that, for all i, there exists a list $\mathscr{L} \in \mathscr{L}(i)$, $|\mathscr{L}| \leq M$, such that $\mathscr{L}(c_{ii}) \neq 0$. Then we define, for $p \in V$, $\delta > 0$ and $i \leq n-1$,

$$\mathscr{F}_i(p, \delta) = \sum_{\substack{|\mathscr{L}|=0 \\ \mathscr{L} \in \mathscr{L}(i)}}^{M-2} \left| \frac{\mathscr{L}(c_{ii})(p)}{\delta} \right|^{2/|\mathscr{L}|+2},$$

and put $\mathscr{F}_n(p, \delta) = \delta^{-2}$.

Associated to these functions, we use the following notations: for $s = (s_1, \ldots, s_n)$, $s_i > 0$,

$$\mathscr{F}^s(p, \delta) = \prod_i \mathscr{F}_i^{s_i}(p, \delta),$$

and, when $s = (1, \ldots, 1)$, we simply write $\mathscr{F}$ instead of $\mathscr{F}^s$.

Moreover, if $\mathscr{L} = (L^1, \ldots, L^k)$ is a list if vector fields $L^j \in \{L_1, \overline{L}_1, \ldots, L_n, \overline{L}_n\}$, and if $l = (l_1, \ldots, l_n)$, l_i being the number of indices j such that $L^j \in \{L_i, \overline{L}_i\}$, we will also write

$$\mathscr{F}^{\mathscr{L}} = \mathscr{F}^l.$$

As usual, the notations $\lesssim$, $\gtrsim$, $\simeq$ mean that the inequality or equivalence holds up to a multiplicative constant depending only on Ω and the choice of the vector fields L_i, and $\lesssim_*$, $\gtrsim_*$, $\simeq_*$ mean that the constant depends also on $*$.

Shrinking V if necessary, the following relations hold in $V \times]0, +\infty[$:

$$\delta^{-2/M} \lesssim \mathscr{F}_i(p, \delta) \lesssim \delta^{-2}, \tag{2.1}$$

and, for $\alpha > 1$,

$$\alpha^{-2} \mathscr{F}_i(p, \delta) \leq \mathscr{F}_i(p, \alpha\delta) \leq \alpha^{-2/M} \mathscr{F}_i(p, \delta). \tag{2.2}$$

2.1. Special coordinate systems. Let (Z) be the canonical coordinate system of $\mathbb{C}^n$. With the previous notations, to each point p of V and all $\delta > 0$ one can define a change of coordinate Φ_p^δ (sometimes denoted simply Φ_p), $Z = \Phi_p^\delta(z)$, such that $\Phi_p^{-1}(0) = p$ and $L_i(p) = \frac{\partial}{\partial z_i}$, satisfying the following properties. Φ_p and Φ_p^{-1} are polynomial coordinate systems defined on $\mathbb{C}^n$ of degree less than $(2M)^{n-1}$ and:

Proposition 2.1. (1) *The coefficients of the polynomials defining Φ_p and Φ_p^{-1} are uniformly bounded for $p \in V$ and $\delta > 0$.*
(2) *The Jacobians of Φ_p and Φ_p^{-1} are uniformly bounded from below.*

Proposition 2.2. *If $a = (a_1, \dots, a_n)$, D^a denotes any derivative of the form $\frac{\partial^{|a|}}{\partial z_1^{b_1} \dots \partial z_n^{b_n} \partial \overline{z}_1^{c_1} \dots \partial \overline{z}_n^{c_n}}$ with $b_i + c_i = a_i$. Then*
(1) *For all derivative D^a, $\left|D^a(\rho \circ (\Phi_p^\delta)^{-1})(0)\right| \lesssim_{|a|} 1$, for all δ.*
(2) *For all derivative D^a, $\left|D^a(\rho \circ (\Phi_p^\delta)^{-1}(0)\right| \lesssim_{|a|} |\rho(p)| \mathscr{F}^{a/2}(p, |\rho(p)|)$, if $\delta = |\rho(p)|$.*

Proposition 2.3. *For $z \in \Phi_p(V)$, let $\tilde{z}_t = (z_1, \dots, z_n - t)$, for $t \geq 0$. If $\tilde{z}_t \in \Phi_p(V)$, then*

$$\rho \circ \Phi_p^{-1}(z) - \rho \circ \Phi_p^{-1}(\tilde{z}_t) \simeq t.$$

2.2. Polydisks and pseudo-balls. For $1 \leq i \leq n$, let

$$\mathscr{R}_i(p, \delta) = \mathscr{F}_i(p, \delta)^{-1/2}.$$

Let $\varepsilon > 0$ fixed sufficiently small. For $p \in V$ and $\delta > 0$, we define the "polydisk" centered at p of radius δ by

$$P_p(\delta) = (\Phi_p^\delta)^{-1}\left\{|z_i| \leq \varepsilon \mathscr{R}_i(p, \delta)\right\}.$$

The volume of $P_p(\delta)$ is estimated in terms of the functions $\mathscr{F}_i$:

Proposition 2.4. *For all $\alpha > 0$, if $p \in V$,*

$$\mathrm{Vol}(P_p(\delta) \cap \{|\rho| < \alpha\delta\}) \lesssim \alpha \,\mathrm{Vol}(P_p(\delta))$$

and

$$\mathrm{Vol}(P_p(\delta)) \simeq \mathscr{F}(p, \delta). \tag{2.3}$$

Note that the first inequality follows Propositions 2.3 and 2.1.

The change of coordinates Φ_p^δ is "close" to the vector fields in the following sense.

Write $L_i = \sum b_i^j \frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial z_j} = \sum a_j^i L_i$ (in the coordinate system Φ_p^δ) then

Proposition 2.5. *For all $A > 0$, all $p \in V$, all $q \in P_p(A\delta)$, $A\delta \leq \delta_0$, and all α, β,*

$$\left| D^{\alpha\beta} a_i^j(q) \right| \leq K_{\alpha\beta} \mathscr{F}^{(\alpha+\beta)/2}(p,\delta) \mathscr{R}_i^{-1}(p,\delta) \mathscr{R}_j(p,\delta),$$

and

$$\left| D^{\alpha\beta} b_i^j(q) \right| \leq K_{\alpha\beta} \mathscr{F}^{(\alpha+\beta)/2}(p,\delta) \mathscr{R}_i^{-1}(p,\delta) \mathscr{R}_j(p,\delta),$$

where $K_{\alpha\beta}$ depends on A, α and β.

As a consequence, if $\mathscr{L} = \{L^1, \dots, L^k\}$ is a list of vector fields of length k, the vector field $L^1 \dots L^k$ can be expressed with derivatives in the coordinate system: if we write

$$L^1 \dots L^k(q) = \sum_{|s| \leq k} c_s(q) D^s,$$

then

Proposition 2.6. *For all $A > 0$, with the above notation, if $p \in V$, $|\rho(p)| \leq \delta \leq \delta_0$, for all $q \in P_p(A\delta)$, $A\delta \leq \delta_0$, $|c_s(q)| \leq K_l \mathscr{F}^{\frac{\mathscr{L}-s}{2}}(p,\delta)$.*

We now briefly recall the notations about the exponential map associated to the real and imaginary parts of the vector fields L_i: we denote by $\exp_p$ the exponential map centered on $p \in V$ associated to the $2n$ real fields $\mathcal{Y}_j$ defined by $\mathcal{Y}_{2k} = \Re\mathrm{e}(L_k)$, $\mathcal{Y}_{2k-1} = \Im\mathrm{m}(L_k)$, for $k \leq n$. Thus, V being sufficiently small, $\exp_p$ is a diffeomorphism of a neighborhood of the origin in $\mathbb{R}^{2n}$ onto a neighborhood of p in V containing a fixed ball, and, for all points p_1 and p_2 in V, there exists $u = (u_1, \dots, u_{2n})$ such that $p_2 = \exp_{p_1}(u)$. Moreover

Lemma 2.1. *For all points p_1 and p_2 in V, if $p_2 = \exp_{p_1}(u)$ then $|u_i| \lesssim |p_1 - p_2|$.*

Now we recall how the "pseudo-distance" of two points of V is defined using $\exp_p$. With the previous notations, we write, for $1 \leq k \leq n$,

$$\mathscr{R}'_{2k}(p,\delta) = \mathscr{R}'_{2k+1}(p,\delta) = \mathscr{R}_k(p,\delta),$$

and

$$\gamma(p_1,p_2) = \inf\{t \geq 0 \text{ such that } p_2 = \exp_{p_1}(u_1, \dots, u_{2n}), \text{ with } |u_i| \leq \mathscr{R}'_i(p_1,t)\}.$$

Remark 2.1. This function γ is independent of the choice of the basis of vector fields diagonalizing the Levi form in $V \cap \partial\Omega$. If γ' is associated to an other basis diagonalizing the Levi form in $V \cap \partial\Omega$, there exist two constants K_1 and K_2 (depending on the basis) such that $K_1\gamma \leq \gamma' \leq K_2\gamma$.

Let the "pseudo-ball" associated to the exponential map is defined by

$$B_{\exp}(p, \delta) = \{q = \exp_p(u_1, \ldots, u_{2n}),\ |u_i| \leq \mathscr{R}_i'(p, \delta)\},$$

and is related, for not too small δ, to the polydisks as follows:

Proposition 2.7. *There exist two constants a and A such that, for $p \in V$, $|\rho(p)| < \delta \leq \delta_0$,*

$$B_{\exp}(p, \delta) \subset P_p(a\delta) \subset B_{\exp}(p, A\delta).$$

Corollary. *If $\gamma(z, w) \leq \delta$ and $|\rho(z)| < \delta \leq \delta_0$, then $w \in P_z(a\delta)$.*

We will also use the notion of balls defined by curves. Let $p \in V$ and $\delta > 0$. We denote by $B_{\mathscr{C}}(p, \delta)$ the set of points $q \in V$ such that there exists a curve φ: $[0, 1] \to V$, piecewise $\mathscr{C}^1$ such that $\varphi(0) = p$, $\varphi(1) = q$ and $\varphi'(t) = \sum_i a_i(t)\mathcal{Y}_i(\varphi(t))$, almost everywhere, with $|a_i(t)| \leq \mathscr{R}_i'(p, \delta)$. The relation with the "balls" defined with the exponential map is given by the following proposition:

Proposition 2.8. *There exists K_0 such that, if $p \in V$, $|\rho(p)| < \delta \leq \delta_0$,*

$$B_{\exp}(p, \delta) \subset B_{\mathscr{C}}(p, \delta) \subset B_{\exp}(p, K_0\delta).$$

Corollary. *For all $B \geq 1$, if $w \in B_{\mathscr{C}}(z, B\delta)$, $|\rho(z)| \leq \delta \leq \delta_0/B$, we have $\gamma(z, w) \lesssim_B \delta$.*

Remark 2.2. γ defines a pseudo-distance on $V \cap \partial\Omega$ but not on V because of the restriction on δ in Proposition 2.7 and Proposition 2.8. But these properties will be sufficient for our purpose. If π denotes the natural projection on $\partial\Omega$, then

$$\Delta(p_1, p_2) = \gamma(\pi(p_1), \pi(p_2)) + |\rho(p_1) - \rho(p_2)|$$

is a pseudo-distance on V.

The next proposition controls the variations of the weights $\mathscr{F}_i$ in the polydisks and balls previously defined:

Proposition 2.9. *For all $B > 0$, if $p \in V$, $|\rho(p)| \leq \delta \leq \delta_0/B$, and if q belongs either to $P_p(B\delta)$ or to $B_{\exp}(p, B\delta)$ or to $B_{\mathscr{C}}(p, B\delta)$, then, for all i, $\mathscr{F}_i(p, \delta) \simeq_B \mathscr{F}_i(q, \delta)$.*

2.3. Point-wise estimates of the Bergman kernel.

Proposition 2.10. *Let $K_B(z,w)$ be the Bergman kernel of Ω. For all points p, p_1, and p_2 in $V \cap \Omega$:*

(1) $K_B(p,p) \simeq \mathscr{F}(p, |\rho(p)|)$.

(2) *For all lists $\mathscr{L}$ (resp. $\overline{\mathscr{L}}$), of length less than N, composed with holomorphic (resp. anti-holomorphic) vector fields L_i (resp. $\overline{L}_i$), l_i (resp. $\overline{l}_i$) denoting the number of times L_i (resp. $\overline{L}_i$) appears in $\mathscr{L}$ (resp. $\overline{\mathscr{L}}$),*

$$\left|\mathscr{L}_z \overline{\mathscr{L}}_w K_B(p_1,p_2)\right| \lesssim_N \prod \mathscr{F}_i(p_1,\delta)^{1+\frac{l_i+\overline{l}_i}{2}},$$

where $\delta = \gamma(p_1,p_2) + |\rho(p_1)| + |\rho(p_2)|$.

(3) *In particular, for all integer $m \geq 0$,*

$$|\nabla^m K_B\,(p_1,p_2)| \lesssim_m \mathscr{F}(p_1,\delta)\delta^{-m}.$$

In the calculus in the next sections, we will use the following equivalences:

$$\begin{aligned}\gamma(p_1,p_2) + |\rho(p_1)| + |\rho(p_2)| &\simeq \gamma(p_2,p_1) + |\rho(p_1)| + |\rho(p_2)| \\ &\simeq \gamma(p_1,p_2) + |\rho(p_1)|\,. \end{aligned} \tag{2.4}$$

Remark 2.3. (1) The previous estimate does not seem to be symmetric, but, in fact, it is.

(2) In [**CD**] the non-isotropic estimates (2) of the previous proposition was used to prove the continuity of the Bergman projection from L^1 to $L^{1\infty}$. In the present paper, the isotropic estimate (3) will be sufficient to estimate the Bergman projection, but we will need the non-isotropic one for the Szegö projection.

3. The Bergman projection

3.1. Sobolev estimates. The goal of this section is to prove the following:

Theorem 3.1. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ of finite type with locally diagonalizable Levi form. For all p, $1 < p < +\infty$, and all $s \geq 0$, the Bergman projection of Ω maps continuously $\mathscr{L}^p_s(\Omega)$ into itself.*

By interpolation, it suffices to prove the result for $s = k \in \mathbb{N}$.

To each point p of $\partial\Omega$, we associate a orthogonal coordinate system (by a complex linear change) centered at p such that $\frac{\partial\rho}{\partial x_n}(p) = 1$ (x_n being the real part of the last complex coordinate) and $\frac{1}{2} \leq \frac{\partial\rho}{\partial x_n} \leq \frac{3}{2}$ in a neighborhood $W(p)$ of p.

For each integer $m \in \mathbb{N}^*$, we cover $\partial\Omega$ by a finite number of neighborhood $V_k(p_k) \subset W(p_k)$, $k = 1, \dots, N$, $p_k \in \partial\Omega$, such that, in the previous coordinate system, $V_k = V_k(p_k)$ is a polydisk centered at p_k of radius $\tilde{r}$ such that the polydisk $U_k = U_k(p_k)$ centered at p_k of radius $8m\tilde{r}$ is contained in $W(p_k)$ and in the neighborhood V related to the point p_k defined in Section 2. Let us denote $\Delta_k = \Delta_k(p_k)$ the polydisk centered at p_k of radius $2m\tilde{r}$. All properties about the geometry and the Bergman kernel, listed in the previous section are then valid in the U_k. We denote then $\mathscr{F}_i^{(k)}$ and $\gamma^{(k)}$ the corresponding functions $\mathscr{F}_i$ and γ defined in $U_k(p_k)$.

In [**MS94**, p. 181], J. D. McNeal and E. M. Stein introduced a notion of $\mathcal{B}$-type kernels. For our purpose we need a small modification of this definition:

Definition 3.1. A function $B(z,w) \in \mathscr{C}^\infty(\Omega \times \Omega)$ is called a $\mathcal{B}$-type kernel if there exist two constants C and C' such that, for $(z,w) \in U_k \times U_k$,

$$|B(z,w)| \leq C \prod_{i=1}^{n} \mathscr{F}_i^{(k)}(z,\delta_k),$$

where $\delta_k = |\rho(z)| + |\rho(w)| + \gamma_k(z,w)$, and, for $(z,w) \notin \bigcup_{k=1}^{N} U_k \times U_k$,

$$|B(z,w)| \leq C'.$$

Remark. Note that the notion of $\mathcal{B}$-type kernel does not depend neither on the choice of the basis diagonalizing the Levi form nor on the choice of m and the U_k.

Lemma 3.1. *Let $B(z,w)$ be a $\mathcal{B}$-type kernel. Let $|B|$ the operator associated to $|B(z,w)|$. Then $|B|$ maps continuously $L^p(\Omega)$ into itself for $1 < p < +\infty$.*

Remark. The proof of this lemma gives an other proof of the L^p part of Theorem 2.2 of [**CD**].

Lemma 3.1 is an easy consequence of the Hölder inequality and the following lemma:

Lemma 3.2. *For $0 < \varepsilon < 1$, there exists a constant $C_1 = C_1(\varepsilon)$ such that,*

$$|B| \left(|\rho|^{-\varepsilon} \right)(z) \leq C_1 \, |\rho|^{-\varepsilon}(z).$$

Proof: We have to prove the inequality

$$\int_\Omega |B(z,w)|\,|\rho(w)|^{-\varepsilon}\,dw \lesssim |\rho(z)|^{-\varepsilon}\,.$$

We consider, for $m=1$, a covering (V_k) of $\partial\Omega$, as defined at the beginning of the section, and the associated polydisks U_k. Suppose first $z\notin\bigcup_k V_k$. Then $|\rho(z)|$ is bounded from below and $|B(z,w)|$ is uniformly bounded and the result is clear. Suppose now $z\in V_{k_0}$. We cut the domain of integration Ω in the two pieces U_{k_0} and $\Omega\setminus U_{k_0}$. On $\Omega\setminus U_{k_0}$, Propositions 2.1 and 2.8 and (2.1) imply that $|B(z,w)|$ is uniformly bounded and it suffices to consider the integration over U_{k_0}.

Let

$$E_0=\{w\in U_{k_0}\cap\Omega,\ \text{such that } \gamma_{k_0}(z,w)\le|\rho(z)|\}$$

and, for $k\ge1$,

$$E_k=\left\{w\in U_{k_0}\cap\Omega,\ \text{such that } 2^{k-1}\,|\rho(z)|\le\gamma_{k_0}(z,w)\le 2^k\,|\rho(z)|\right\},$$

and let us estimate

$$\int_{E_k}|B(z,w)|\,|\rho(w)|^{-\varepsilon}\,dw.$$

Let $E_k^l=E_k\cap\{w \text{ such that } |\rho(w)|\le 2^{k-l}\,|\rho(z)|\}$.

By the corollary of Proposition 2.7 $E_k\subset B_{\exp}(z,2^k|\rho(z)|)\subset P_z(a2^k|\rho(z)|)$. Moreover, by Propositions 2.7 and 2.4 and (2.2)

$$\operatorname{Vol}(E_k^l)\lesssim 2^{-l}\prod_{i=1}^n\mathscr{F}_i^{-1}(z,a2^k\,|\rho(z)|)\lesssim 2^{-l}\prod_{i=1}^n\mathscr{F}_i^{-1}(z,2^k\,|\rho(z)|),$$

and, $B(z,w)$ being a $\mathcal{B}$-type kernel, (2.2) implies

$$\int_{E_k^l\setminus E_k^{l+1}}|B(z,w)|\,|\rho(w)|^{-\varepsilon}\,dw\lesssim 2^{-l}(2^{k-l}\,|\rho(z)|)^{-\varepsilon}$$

$$\le(2^{1-\varepsilon})^{-l}2^{-k\varepsilon}\,|\rho(z)|^{-\varepsilon}\,,$$

which finishes the proof of the lemma, because $0<\varepsilon<1$. □

Following [**MS94**, Lemma 2, p. 184], we now define operators related to derivatives of the Bergman kernel and show that they are $\mathcal{B}$-type kernels. The fact that we are not on a convex domain implies that we need to modify slightly the definitions of those operators. These operators are associated to a covering of $\partial\Omega$ and defined by integrals. To give sense to the considered integrals, we must restrict the domain of integration ($[0,+\infty[$ in [**MS94**]) to intervals depending on the order m of derivation. This induces also that the covering of $\partial\Omega$ must have properties depending on m. So, for m a non zero integer, we consider a covering of $\partial\Omega$ associated to m by polydisks V_k (as defined at the beginning of the section) and the associated Δ_k and $\tilde{r}$. Then

Lemma 3.3. *Let $k \in \{1, \dots, N\}$. Let m' be an integer less or equal than m. Let $X_1, \dots, X_{m'}$ be $\mathscr{C}^\infty$ vector fields on V_k acting on the first variable z of the Bergman kernel $K_B(z, w)$, and let ψ a function in $\mathscr{D}(\Delta_k)$. Then the operator whose kernel K is defined, in the coordinate system associated to $V(p_k)$, by*

$$K(z,w) = \int_0^{5\tilde{r}} \cdots \int_0^{5\tilde{r}} X_1 \dots X_{m'} K_B\left(z, w_1, \dots, w_{n-1}, w_n - \sum_{i=1}^{m'} t_i\right) \times \psi\left(w_1, \dots, w_n - \sum_{i=1}^{m'} t_i\right) dt_1 \dots dt_{m'}$$

is defined on $\Omega \times \Omega$ and is a $\mathcal{B}$-type kernel.

Proof: First, let us show that the kernel K is well defined. Let $w \in \Omega$, and denote

$$t = \sum_{i=1}^{m'} t_i \text{ and } \tilde{w}_t = (w_1, \dots, w_{n-1}, w_n - t)\,.$$

Suppose $\tilde{w}_t \in \Delta_k$. The inequality $\sum t_i \leq 5m\tilde{r}$ and the definition of U_k imply $w \in U_k$, and, consequently, by the choice of the coordinate in $V(p_k)$, for $s < t$, $\frac{\partial}{\partial x_n}\rho(\tilde{w}_s) \simeq 1$ and $\rho(\tilde{w}_t) \leq \rho(w) < 0$. Thus $\tilde{w}_t \in \Omega$ and K is well defined.

More precisely, if $\psi(\tilde{w}_t) \neq 0$, $d(\tilde{w}_t, \mathbb{C}^n \setminus U_k) \geq 6m\tilde{r}$, and thus, if $z \notin U_k$, the distance from w to z and $\gamma_k(z, w)$ are uniformly bounded from below (Propositions 2.1 and 2.8) and $|X_1 \dots X_{m'} K_B(z, w_{\tilde{t}})|$ is uniformly bounded (Proposition 2.10).

By (2.4) and (2.2), it suffices then to prove that, for z and w in U_k (omitting the subscript k for $\mathscr{F}_i^{(k)}$ and γ_k),

$$|K(z,w)| \leq C \prod_{i=1}^{n} \mathscr{F}_i(z, |\rho(z)| + \gamma(z,w)).$$

By a change of variables,

$$|K(z,w)| \lesssim \int_0^{2m'\tilde{r}} t^{m'-1} \left|X_1 \dots X_{m'} K_B(z, \tilde{w}_t)\right| |\psi(\tilde{w}_t)| \, dt,$$

and the estimates of the derivatives of the Bergman kernel (Proposition 2.10) give

$$|K(z,w)| \lesssim \int_0^{2m'\tilde{r}} t^{m'-1} \frac{\prod_{i=1}^n \mathscr{F}_i(z,\delta(t))}{\delta_t^{m'}} |\psi(\tilde{w}_t)|\, dt, \tag{3.1}$$

where $\delta(t) = |\rho(z)| + |\rho(\tilde{w}_t)| + \gamma(z,\tilde{w}_t)$.

Now, we estimate $\gamma(z,\tilde{w}_t)$ in terms of $\gamma(z,w)$. By definition of the function γ, there exist real numbers u_i, $|u_i| \le \mathscr{R}'_i(z,\gamma(z,\tilde{w}_t)$ such that $\tilde{w}_t = \exp_z(u_1,\dots,u_{2n})$. Moreover, there exits, by Lemma 2.1, numbers v_i, $|v_i| \le K_1 |\tilde{w}_t - w|$, such that $w = \exp_{w_t}(v_1,\dots,v_{2n})$. As $\dfrac{\partial\tilde{\rho}}{\partial x_n} \sim 1$, we also have $|v_i| \le K_2 (|\rho(\tilde{w}_t)| - |\rho(w)|)$.

Thus there exists a curve φ, piecewise $\mathscr{C}^1$, such that $\varphi(0)=z$, $\varphi(1) = w$ and $\varphi'(s) = \sum_i a_i(s)\mathcal{Y}'_i(\varphi(s))$, for almost all s, with

$$\begin{aligned} |a_i(s)| &\le 2\mathscr{R}'_i(z,\gamma(z,\tilde{w}_t) + K_2 (|\rho(\tilde{w}_t)| - |\rho(w)|) \\ &\le \mathscr{R}'_i\left(z, 2^M \left(\gamma(z,\tilde{w}_t) + K_2 (|\rho(\tilde{w}_t)| - |\rho(w)|)\right)\right), \end{aligned}$$

by (2.2), and, the function $\delta \mapsto \mathscr{R}'_i(z,\delta)$ being increasing, we have

$$|a_i(s)| \le \mathscr{R}'_i\left(z, 2^M(\gamma(z,\tilde{w}_t) + K_2(|\rho(\tilde{w}_t)| - |\rho(w)| + |\rho(z)|))\right),$$

and we can apply the equivalence between the balls defined by the exponential map and the balls defined by curves (corollary of Proposition 2.8) which implies

$$\gamma(z,w) \lesssim \gamma(z,\tilde{w}_t) + (|\rho(\tilde{w}_t)| - |\rho(w)| + |\rho(z)|).$$

Then $\delta(t) \gtrsim \gamma(z,w) + |\rho(w)|$, and, because $\delta(t) \ge |\rho(z)|$ and $|\rho(\tilde{w}_t)| \gtrsim |\rho(w)| + t$, we get

$$\delta(t) \gtrsim |\rho(z)| + |\rho(w)| + \gamma(z,w) + t = \delta(0) + t.$$

Using the fact that the functions $\delta \mapsto \mathscr{F}_i(w,\delta)$ are decreasing, we obtain from (3.1)

$$\begin{aligned} |K(z,w)| &\lesssim \int_0^\infty \frac{t^{m'-1}}{(\delta(0)+t)^{m'+2}} \prod_{i=1}^{n-1} \mathscr{F}_i(z,\delta(0)+t)\, dt \\ &\le \delta(0)^2 \prod_{i=1}^n \mathscr{F}_i(z,\delta(0)) \int_0^\infty \frac{t^{m'-1}}{(\delta(0)+t)^{m'+2}}\, dt \\ &\lesssim \prod_{i=1}^n \mathscr{F}_i(z,\delta(0)), \end{aligned}$$

which completes the proof of the lemma. □

Lemma 3.4. *Under the conditions of the preceding lemma, for* $k \in \{1, \dots, N\}$*, there exists a* $\mathscr{C}^\infty$ *vector field* T *defined on* U_k*,* $T\rho = 0$ *on* $\partial\Omega \cap U_k$*, such that, if* D *is a derivative of order* m*, for all integer* m'*,* $0 < m' \le m$*, we have, for* $z \in U_k$*,* $w \in V_k$ *and* $\psi_k \in \mathscr{D}(V_k)$*,*

$$D_z K_B(z,w)\psi_k(w) = \int_0^{5\tilde r} \cdots \int_0^{5\tilde r} \sum_{l=0}^{m'} T_w^l \left(D_z K_B \left(z, w_1, \dots, w_n - \sum_{i=1}^{m'} t_i \right)\right) \times \Phi_{l,m'} \left(w_1, \dots, w_n - \sum_{i=1}^{m'} t_i \right) dt_1 \dots dt_{m'},$$

where $\Phi_{l,m'} \in \mathscr{D}(\Delta_k)$*,* D_z *acts on the* z *variable and* T_w *is the vector field* T *acting on the* w *variable.*

Proof: As we saw at the beginning of the proof of the previous lemma, we can write, for $t_2 \in [0, 5\tilde r]$ (denoting $w' = (w_1, \dots, w_{n-1})$),

$$D_z K_B(z, w', w_n - t_2)\psi_k(w', w_n - t_2) = \int_0^{5\tilde r} \frac{d}{dt_1} \left[D_z K_B(z, w', w_n - t_1 - t_2)\psi_k(w', w_n - t_1 - t_2) \right] dt_1.$$

Let us denote $x_n = \Re\mathrm{e}(w_n)$. Then

$$\frac{d}{dt_1} \left[D_z K_B(z, w', w_n - t_1 - t_2)\psi_k(w', w_n - t_1 - t_2) \right] = -D_z K_B(z, w', w_n - t_1 - t_2) \frac{\partial}{\partial x_n} \psi_k(w', w_n - t_1 - t_2) + \frac{\partial}{\partial \overline{w}_n} \left(D_z K_B(z, w', w_n - t_1 - t_2) \right) \psi_k(w', w_n - t_1 - t_2),$$

thus, if $T = \frac{\partial}{\partial \overline{w}_n} - \frac{\partial\rho/\partial\overline{w}_n}{\partial\rho/\partial w_n} \frac{\partial}{\partial w_n}$, we obtain (the Bergman kernel being antiholomorphic in the second variable)

$$D_z K_B(z, w', w_n - t_2)\psi_k(w', w_n - t_2) = \int_0^{5\tilde r} T_w \left(D_z K_B(z, w', w_n - t_1 - t_2) \right) \psi_k(w', w_n - t_1 - t_2)\, dt_1 + \int_0^{5\tilde r} D_z K_B(z, w', w_n - t_1 - t_2) \left(-\frac{\partial}{\partial x_n} \psi_k(w', w_n - t_1 - t_2) \right) dt_1.$$

The lemma is then obtained easily by induction using the same method and noting that

$$\frac{\partial}{\partial \overline{w}_n}\left(T_w^l D_z K_B\left(z, w_1, \ldots, w_n - \sum_{i=1}^{p} t_i\right)\right) = T_w^{l+1} D_z K_B\left(z, w_1, \ldots, w_n - \sum_{i=1}^{p} t_i\right). \quad \square$$

Let D be a derivative of order m. We want to estimate $D_z(P_B f)(z)$ for $f \in \mathscr{L}_s^p(\Omega)$. Denote by Δ_0 the interior of $\Omega \setminus \bigcup_{k=1}^N V_k$, and let ψ_k be a partition of unity associated to the Δ_k.

Using the same notation for w' as before, we deduce

Corollary. *For all m', $0 < m' \leq m$, and for each $k \geq 1$, there exist functions $\Psi_i = \Psi_i^{(k)}$, $0 \leq i \leq m'$, with compact support in Δ_k such that, for $f \in \mathscr{C}^\infty(\overline{\Omega})$,*

$$\begin{aligned} DP_B(f\psi_k)(z) &= DP_B(f\psi_0)(z) \\ &+ \sum_{k=1}^{N} \int_{\Delta_k \cap \Omega} \int_0^{5\tilde{r}} \cdots \int_0^{5\tilde{r}} D_z K_B\left(z, w', w_n - \sum_{i=1}^{m'} t_i\right) \\ &\times \left[\Psi_{m'}(w', w_n - \sum_{i=1}^{m'} t_i) f(w) + \Psi_{m'-1}(w', w_n - \sum_{i=1}^{m'} t_i) Tf(w) + \cdots \right. \\ &\left. \cdots + \Psi_0(w', w_n - \sum_{i=1}^{m'} t_i) T^{m'} f(w)\right] dt_1 \ldots dt_{m'}\, dw. \end{aligned}$$

Proof: We have

$$D_z(P_B f \psi_k)(z) = \int_{\Delta_k \cap \Omega} D_z(K_B(z,w)\psi_k(w) f(w))\, dw.$$

An integration by parts gives

$$\int_{\Delta_k\cap\Omega} T_w^l\left(D_zK_B\left(z,w',w_n-\sum_{i=1}^{m'}t_i\right)\right)\Phi_{l,m'}\left(w',w_n-\sum_{i=1}^{m'}t_i\right)f(w)\,dw$$

$$= \int_{\Delta_k\cap\Omega} T_w^{l-1}\left(D_zK_B\left(z,w',w_n-\sum_{i=1}^{m'}t_i\right)\right)$$

$$\times\left[\left(-T_w\Phi_{l,m'}\left(w',w_n-\sum_{i=1}^{m'}t_i\right)+\Phi_{l,m'}\left(w',w_n-\sum_{i=1}^{m'}t_i\right)\operatorname{div}T(w)\right)f(w)\right.$$

$$\left.-\ \Phi_{l,m'}\left(w',w_n-\sum_{i=1}^{m'}t_i\right)T_wf(w)\right]\,dw,$$

where $\operatorname{div}T$ is the divergence of T.

By induction, we obtain

$$\int_{\Delta_k\cap\Omega}\sum_{l=0}^{m'}T_w^l\left(D_zK_B\left(z,w',w_n-\sum_{i=1}^{m}t_i\right)\right)\Phi_{l,m'}\left(w',w_n-\sum_{i=1}^{m'}t_i\right)f(w)\,dw$$

$$= \int_{\Delta_k\cap\Omega} D_zK_B\left(z,w',w_n-\sum_{i=1}^{m'}t_i\right)$$

$$\times\left[\Psi_m(w',w_n-\sum_{i=1}^{m'}t_i)f(w)+\cdots+\Psi_0(w',w_n-\sum_{i=1}^{m'}t_i)T_w^{m'}f(w)\right]\,dw,$$

which gives the corollary, by Lemma 3.4 and Fubini's theorem. □

Remark. For $w\in\Delta_0$, $DK_B(z,w)$ is uniformly bounded and thus $|DP_B(f\psi_0)(z)|\lesssim\|f\psi_0\|$.

Theorem 3.1 is now a trivial consequence of the previous corollary, Lemma 3.3 and Lemma 3.1.

3.2. Isotropic Hölder estimates. This section is also strongly inspired by [**MS94**] and we will use the same notations.

For convenience, we briefly recall here the definition of the spaces $\Lambda_\alpha(\Omega)$, $0<\alpha<+\infty$.

First, for $0 < \alpha < 1$, a function f belongs to $\Lambda_\alpha(\mathbb{C}^n)$ if it is bounded and if $|f(x) - f(y)| \lesssim |x - y|^\alpha$, $x, y \in \mathbb{C}^n$. For $\alpha > 1$, $\alpha \notin \mathbb{N}$, f belongs to $\Lambda_\alpha(\mathbb{C}^n)$ if it is bounded and if, $D^\beta f \in \Lambda_{\alpha-|\beta|}(\mathbb{C}^n)$ for β of length equal to the integral part of α. A function f belongs to $\Lambda_1(\mathbb{C}^n)$ if it is bounded and, for all x and h, $|f(x+h) + f(x-h) - 2f(x)| \lesssim |h|$, and f belongs to $\Lambda_k(\mathbb{C}^n)$, $k \in \mathbb{N}_*$, if $D^\beta f \in \Lambda_1(\mathbb{C}^n)$ for all β of length $k-1$.

A function f belongs then to $\Lambda_\alpha(\Omega)$ if there exists a function $F \in \Lambda_\alpha(\mathbb{C}^n)$ whose restriction to Ω is equal to f. In other words, the Banach space $\Lambda_\alpha(\Omega)$ is defined as a quotient space.

We will prove the following result:

Theorem 3.2. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ of finite type with locally diagonalizable Levi form. For $0 < \alpha < +\infty$, the Bergman projection maps continuously $\Lambda_\alpha(\Omega)$ into itself.*

The proof is based on a result essentially due to Hardy and Littlewood:

Proposition 3.1. *Let f be a bounded $\mathscr{C}^\infty$ function on Ω. If there exists an integer $m > \alpha$ such that $|\nabla^m f(z)| \lesssim |\rho(z)|^{-m+\alpha}$ on Ω, then $f \in \Lambda_\alpha(\Omega)$.*

To see that $P_B f$ satisfies this sufficient condition when $f \in \Lambda_\alpha(\Omega)$, we use the following characterisation:

Proposition 3.2. *For a function f defined on Ω the three following properties are equivalent:*

(1) $f \in \Lambda_\alpha(\Omega)$.
(2) *There are functions f_k, such that $f = \sum_0^\infty f_k$ and*
   (a) $\|f_k\|_{L^\infty(\Omega)} \lesssim_f 2^{-k\alpha}$,
   (b) *for all integer $m > \alpha$, $\|\nabla^m f_k\|_{L^\infty(\Omega)} \lesssim_{f,m} 2^{mk} 2^{-k\alpha}$.*
(3) *For all integers k and m, $m > \alpha$, there exist two functions g_k and b_k such that $f = g_k + b_k$ and*
   (a) $\|b_k\|_{L^\infty(\Omega)} \lesssim_f 2^{-k\alpha}$,
   (b) $\|\nabla^m g_k\|_{L^\infty(\Omega)} \lesssim_{f,m} 2^{mk} 2^{-k\alpha}$.

Choose an integer $m > \alpha + 1$ and use the decomposition of f given by property (3) of Proposition 3.2 with k such that $|\rho(z)| \simeq 2^{-k}$. Then

$$\nabla^m P_B f(z) = \nabla^m P_B b_k(z) + \nabla^m P_B g_k(z).$$

To estimate these quantities, we first prove the two following lemmas:

Lemma 3.5. *For all $m \geq 1$, there exits a constant C depending only on Ω and m such that*

$$\int_\Omega |\nabla_z^m K_B(z, w)| \, dw \leq C \, |\rho(z)|^{-m} .$$

Lemma 3.6. *Let D be a derivation of order $m \geq 2$. There exist operators $\mathscr{B}_0, \dots, \mathscr{B}_{m-1}$ defined by kernels $B_0, \dots, B_{m-1}$ and vector fields $X_1, \dots, X_{m-1}$, of order 1, such that:*

(1) *for $i = 1, \dots, m-1$, $\int_\Omega |B_i(z,w)|\, dw \lesssim |\rho(z)|^{-1}$,*
(2) *for all $\mathscr{C}^\infty$ function f,*

$$\int_\Omega D_z K_B(z,w) f(w)\, dw$$
$$= \mathscr{B}_{m-1} f(z) + \mathscr{B}_{m-2}(X_1 f)(z) + \cdots + \mathscr{B}_0(X_1 \dots X_{m-1} f)(z).$$

Proof of Lemma 3.5: We use the covering $\{V_k\}$ of $\partial\Omega$, associated to m, defined in the previous section. If $z \notin \bigcup_{k \geq 1} V_k$ then $|\rho(z)|$ is bounded from below and $\nabla_z K_B(z,w)$ is bounded. Suppose now $z \in V_{k_0}$ and write

$$\int_\Omega |\nabla_z^m K_B(z,w)|\, dw = \int_{\Omega \setminus U_{k_0}} |\nabla_z^m K_B(z,w)|\, dw + \int_{U_{k_0} \cap \Omega} |\nabla_z^m K_B(z,w)|\, dw.$$

In the first integral the distance from z to w is uniformly bounded from below and, thus, $\nabla_z^m K_B(z,w)$ is uniformly bounded. It suffices then to estimate the second integral. As in the proof of Lemma 3.2 we use the following partition of U_{k_0}:

$$E_0 = \{w \in U_{k_0} \cap \Omega, \text{ such that } \gamma(z,w) \leq |\rho(z)|\}$$

and, for $k \geq 1$,

$$E_k = \left\{w \in U_{k_0} \cap \Omega, \text{ such that } 2^{k-1} |\rho(z)| < \gamma(z,w) \leq 2^k |\rho(z)|\right\}.$$

By the estimates on the derivatives of the Bergman kernel (Proposition 2.10) and the relations between $\mathscr{F}_i(z, \alpha\delta)$ and $\mathscr{F}_i(z,\delta)$ ((2.2)), we get

$$\int_{E_k} |\nabla_z^m K_B(z,w)|\, dw \lesssim_m \prod_{i=1}^n \mathscr{F}_i(z, 2^k |\rho(z)|) \frac{1}{(2^k |\rho(z)|)^m} \operatorname{Vol}(E_k),$$

and the proof is finished recalling (as it has been established in the proof of Lemma 3.2) that

$$\operatorname{Vol}(E_k) \lesssim_m \prod_{i=1}^n \mathscr{F}_i(z, 2^k |\rho(z)|)^{-1}. \qquad \square$$

Proof of Lemma 3.6: We use the partition of unity defined in Lemma 3.4, and, by the corollary of that lemma for $m' = m - 1$

$$D_z P_B f(z) = \int_\Omega D_z K_B(z,w) f(w) \psi_0(w)\, dw$$
$$+ \sum_{k=1}^{n} \int_\Omega \left(\int_0^{5\tilde{r}} \cdots \int_0^{5\tilde{r}} D_z K_B \left(z, w', w_n - \sum_{i=1}^{m-1} t_i \right) \sum_{l=0}^{m-1} \right.$$
$$\left. \times \left(\Psi_{m-1-l}^{(k)} \left(w', w_n - \sum_{i=1}^{m-1} t_i \right) T^l f \left(w', w_n - \sum_{i=1}^{m-1} t_i \right) \right) dt_1 \dots dt_{m-1} \right) dw.$$

$\int_\Omega |D_z K_B(z,w) f(w)|\, \psi_0(w)\, dw$ is uniformly bounded (see remark after the corollary) and the functions $\Psi_i^{(k)}$ being uniformly bounded it then suffices to prove that

$$\int_{\Omega\cap\Delta_k} \int_0^{5\tilde{r}} \cdots \int_0^{5\tilde{r}} \left| D_z K_B \left(z, w', w_n - \sum_{i=1}^{m-1} t_i \right) \right| dt_1 \dots dt_{m-1}\, dw \lesssim_m |\rho(z)|^{-1}.$$

As in Lemma 3.5, this is trivial if $z \notin U_k$. Suppose $z \in U_k$ and $w \in \Omega \cap \Delta_k$. Then $(w', w_n - \sum t_i)$ belongs also to U_k and we use the estimates of the derivatives of the Bergman kernel (Proposition 2.10) and the estimate of $\delta(t)$ obtained in the proof of Lemma 3.3

$$\delta(t) = |\rho(z)| + |\rho(\tilde{w}_t)| + \gamma(z, \tilde{w}_t) \gtrsim |\rho(z)| + |\rho(w)| + \gamma(z,w) + \sum_{i=1}^{m-1} t_i.$$

The properties of functions $\mathscr{F}_i$ (2.2) implies

$$\left| D_z K_B \left(z, w', w_n - \sum_{i=1}^{m-1} t_i \right) \right|$$
$$\lesssim_m \frac{\prod_{i=1}^n \mathscr{F}_i(z, |\rho(z)| + |\rho(w)| + \gamma(z,w))}{(|\rho(z)| + |\rho(w)| + \gamma(z,w) + t_1 + \cdots + t_{m-1})^m}$$

and thus

$$\left| \int_0^{5\tilde{r}} \cdots \int_0^{5\tilde{r}} D_z K_B \left(z, w', w_n - \sum_{i=1}^{m-1} t_i \right) dt_1 \dots dt_{m-1} \right|$$
$$\lesssim_m \frac{\prod_{i=1}^n \mathscr{F}_i(z, |\rho(z)| + |\rho(w)| + \gamma(z,w))}{|\rho(z)| + |\rho(w)| + \gamma(z,w)}.$$

Now, to finish, it suffices to consider the partition of U_k defined in the proof of Lemma 3.5. □

End of the proof of Theorem 3.2: By Lemma 3.5

$$|\nabla^m P_B b_k(z)| \leq C\,|\rho(z)|^{-m}\,2^{-k\alpha} \lesssim_f C\,|\rho(z)|^{-m+\alpha}\,,$$

and by Lemma 3.6

$$|\nabla P_B g_k(z)| \lesssim |\rho(z)|^{-1}\left[\|g_k\|_\infty + \cdots + \|X_1 \ldots X_{m-1} g_k\|_\infty\right].$$

But

$$\|g_k\|_\infty + \cdots + \|X_1 \ldots X_{m-1} g_k\|_\infty \lesssim \|g_k\|_\infty + \left\|\nabla^{m-1} g_k\right\|_\infty\,,$$

and, because $m-1 > \alpha$,

$$\left\|\nabla^{m-1} g_k\right\|_\infty \lesssim_{f,m} 2^{(m-1)k} 2^{-k\alpha} \lesssim_{f,m} |\rho(z)|^{-m+1+\alpha}\,.$$

This completes the proof. □

As an immediate corollary of Lemma 3.5 and the fact that a $\mathscr{C}^1$ function satisfying $|\nabla f(z)| \lesssim |\rho(z)|$ is in $BMO(\Omega)$ (Lemma 7 of [**MS94**]), we have:

Corollary. *Under the conditions stated in Theorem 3.1, the Bergman projector maps continuously $L^\infty(\Omega)$ into $BMO(\Omega)$.*

3.3. Nonisotropic Hölder estimates. Let $V_1, \ldots, V_N$ be a covering of the boundary as defined in Section 3.1 (for $m = 1$).

For each k, if p is a point of V_k, let Φ_p be the change of variables associated to p and $\delta(p)$; we denote by (z_i) the corresponding coordinates. Recall that V_k is chosen so that, if $\tilde\rho = \rho \circ \Phi_p^{-1}$, and $w, z \in \Phi_p(V_k)$ such that $w_j = z_j$ for $j < n$ and $w_n = z_n + t$, $t \in \mathbb{R}$, then $\tilde\rho(w) - \tilde\rho(z) \simeq t$ (Proposition 2.3).

In Section 2.2, we defined the function $\gamma_k(p,q)$ for points p and q in the euclidean ball $B(p_k, r_k)$. Let us define a global version γ of these functions putting, for arbitrary points p and q in Ω,

$$\gamma(p,q) = \inf\left\{\gamma_k(p,q) \text{ such that } p, q \in V_k,\ 1 \leq k \leq N\right\},$$

if there exists k such that $p, q \in V_k$ and $\gamma(p,q) = 1$ if not. Then we denote

$$\rho(p,q) = \min\left\{\gamma(p,q) + |\rho(p)| + |\rho(q)|\,, |p-q|\right\}.$$

With these notations, we define now the space $\Gamma_\alpha(\Omega)$:

Definition 3.2. For $\alpha < 1/M$, $\Gamma_\alpha(\Omega)$ is the space of continuous functions f on $\overline{\Omega}$ such that

$$|f(p) - f(q)| \lesssim \rho(p,q)^\alpha.$$

Remark. Note that with this definition, $\Gamma_\alpha(\Omega)$ is independent of the choice of the covering V_k (Remark 2.1).

In the previous section, we proved that the Bergman projection maps continuously Λ_α into itself. We will now see, that, the properties of holomorphic functions in Ω related to the geometry give a better:

Theorem 3.3. *Let Ω be a bounded pseudo-convex domain of finite type in $\mathbb{C}^n$ with locally diagonalizable Levi form. For $\alpha < 1/M$, the Bergman projection maps continuously Λ_α into Γ_α.*

By Theorem 3.2, it suffices to show that a holomorphic function in Λ_α belongs automatically to Γ_α. We prove this in two steps. First, we show that the derivatives of a holomorphic function in $\Lambda_\alpha$satisfy some non-isotropic estimates in terms of the functions $\mathscr{F}_i$. The proof of the theorem is then done in Section 3.3.2. In Section 3.3.3 we indicate briefly how the theorem can be extended for $\alpha \geq 1/M$.

3.3.1. Nonisotropic estimates of the derivatives of a holomorphic function in $\boldsymbol{\Lambda_\alpha}$. For $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, D^a denotes the derivative, in the (z_i) coordinate system, $D^a = \frac{\partial^{|a|}}{\partial z_1^{a_1} \ldots \partial z_n^{a_n}}$. Recall that we denote $\mathscr{F}^{a/2} = \prod_{i=1}^n \mathscr{F}_i^{a_i/2}$.

Proposition 3.3. *Let $f \in \Lambda_\alpha(\Omega) \cap \mathscr{H}(\Omega)$. Let $p \in V_k$, Φ_p the change of coordinates corresponding to p and $\delta = |\rho(p)|$, and D^a a derivative. Then there exits a constant C, depending only on Ω, f and a, such that*

$$\left|D^a(f \circ \Phi_p^{-1})(0)\right| \leq C\left[\mathscr{F}(p, |\rho(p)|)^{a/2} \left|\rho(p)\right|^\alpha + 1\right].$$

The starting point of the proof is the following lemma which is valid for any domain Ω (c.f. [**MS94**, Lemma 8, p. 197]):

Lemma 3.7. *Let $f \in \Lambda_\alpha(\Omega) \cap \mathscr{H}(\Omega)$. Let ∂^s be a derivative of length $|s| > \alpha$ (in the canonical coordinate system). There exists a constant C depending only on f, $|s|$ and Ω such that, for all $q \in \Omega$,*

$$|\partial^s f(q)| \leq C \left|\rho(q)\right|^{-|s|+\alpha}.$$

Let $p \in V_k$. This lemma and the regularity properties of the change of variables $z = \Phi_p(Z)$ (Proposition 2.1) imply that, if D^a denotes a derivative with respect to the z variable of length $|a| > \alpha$, then there exists a constant C depending only on f, $|a|$ and Ω such that, for all $z \in \Phi_p(\Omega)$,

$$\left|D^a\left(f \circ \Phi_p^{-1}\right)(z)\right| \leq C\left|\rho(\Phi_p^{-1}(z))\right|^{-|a|+\alpha}. \tag{3.2}$$

We want to use this inequality through Cauchy's formula, so, we need to know that certain polydisks are contained in $\Phi_p(\Omega)$:

Lemma 3.8. *There exists two constants $c < 1$ and $\nu > 0$, depending only on Ω, such that, for $0 < t < \delta_0$, if z belongs to the polydisk*

$$\left\{|z_i| \leq c\mathscr{F}_i^{-1/2}(p,|\rho(p)|)\left(\frac{|\rho(p)|+t}{|\rho(p)|}\right)^{\frac{1}{2M^2}},\ \text{for } i<n,\right.$$

$$\left.|z_n+t| \leq c\,|\rho(p)|\left(\frac{|\rho(p)|+t}{|\rho(p)|}\right)^{\frac{1}{2M^2}}\right\},$$

then

$$\rho\circ\Phi_p^{-1}(z) = \tilde{\rho}(z) \leq \frac{1}{4}\left[\rho(p)-\nu t\right].$$

Proof: Let z be a point in the polydisk described in the lemma and $z' = z + (0,\dots,0,t)$. Taylor's formula gives

$$|\tilde{\rho}(z')-\tilde{\rho}(0)| \leq \sum_{|s|\leq M} * c^{|s|} D^s\tilde{\rho}(0)\mathscr{F}^{-s/2}(p,|\rho(p)|)\left(\frac{|\rho(p)|+t}{|\rho(p)|}\right)^{\frac{|s|}{2M^2}} + A,$$

where $*$ are absolute constants and, by Proposition 2.2 (1), and (2.1), A satisfies

$$A \leq K_1 c^{M+1}\,|\rho(p)|^{\frac{M+1}{M}-\frac{M+1}{2M^2}} \leq K_2\,|\rho(p)|\,c.$$

Moreover, Proposition 2.2 (2) implies

$$|D^s\tilde{\rho}(0)| \leq K\,|\rho(p)|\,\mathscr{F}^{s/2}(p,|\rho(p)|)$$

and

$$|\tilde{\rho}(z')-\tilde{\rho}(0)| \leq cK_3(|\rho(p)|+t).$$

On the other hand, by Proposition 2.3, there exists a constant ν such that $\tilde{\rho}(z)-\tilde{\rho}(z') < -\nu t$, which implies the lemma, for c small enough. □

Proof of Proposition 3.3: Let $l > \alpha$ be an integer. Using (3.2) and the previous lemma, Cauchy's formula applied at the point $(0,\dots,0,-t)$

shows that there exists a constant K depending only on f, l, Ω and $|s|$ such that

$$\text{(3.3)}\quad \left|D^a \frac{\partial^l}{\partial z_n^l} f\circ\Phi_p^{-1}(0,\dots,0,-t)\right| \\ \leq K\mathscr{F}^{a/2}(p,|\rho(p)|)\left(\frac{|\rho(p)|}{|\rho(p)|+t}\right)^{\frac{1}{2M^2}}\left[\frac{1}{|\rho(p)|+t}\right]^{-\alpha+l}.$$

We choose $l=[\alpha]+1$. The function $f\circ\Phi_p^{-1}$ being holomorphic, we have $\frac{\partial^l}{\partial z_n^l}\left(f\circ\Phi_p^{-1}\right)=\frac{\partial}{\partial x_n}\frac{\partial^{l-1}}{\partial z_n^{l-1}}\left(f\circ\Phi_p^{-1}\right)$, and

$$D^a\frac{\partial^{l-1}}{\partial z_n^{l-1}}\left(f\circ\Phi_p^{-1}\right)(0,\dots,0,-t)=D^a\frac{\partial^{l-1}}{\partial z_n^{l-1}}\left(f\circ\Phi_p^{-1}\right)(0,\dots,0,-\delta_0) \\ -\int_t^{\delta_0} D^a\frac{\partial^l}{\partial z_n^l}\left(f\circ\Phi_p^{-1}\right)(0,\dots,0,-u)\,du=A+B.$$

Note first that $|A|$ is bounded by a constant depending only on $|a|$ and f, because $\Phi_p^{-1}(0,\dots,0,-\delta_0)$ belongs to a fixed compact of Ω.

If α is not an integer, (3.3) implies

$$|B|\leq K\mathscr{F}^{a/2}(p,\rho(p))\int_t^{\delta_0}\frac{du}{(|\rho(p)|+u)^{l-\alpha}} \\ \leq K_1\mathscr{F}^{a/2}(p,|\rho(p)|)\frac{1}{(|\rho(p)|+t)^{l-\alpha-1}},$$

and a simple iteration gives the estimate of the proposition.

If α is an integer (then $l=\alpha+1$), the same inequality implies

$$|B|\leq K\mathscr{F}^{a/2}(p,\rho(p))\,|\rho(p)|^{1/2M^2}\int_t^{\delta_0}\frac{du}{(|\rho(p)|+u)^{1+\frac{1}{2M^2}}} \\ \leq K\mathscr{F}^{a/2}(p,|\rho(p)|)\left(\frac{|\rho(p)|}{(|\rho(p)|+t)}\right)^{1/2M^2},$$

and we may follow the same proof as before. □

3.3.2. Proof of Theorem 3.3. The result is a trivial consequence of Theorem 3.2 and the following lemma:

Lemma 3.9. *For $\alpha<1/M$, a holomorphic function in $\Lambda_\alpha(\Omega)$ verifies $|f(p)-f(q)|\lesssim\rho(p,q)^\alpha$, $p,q\in\Omega$.*

Proof: Let us denote by $Z = (Z_i)$ the coordinate system associated to each V_k as in the beginning of Section 3.1. We showed, at the beginning of the proof of Lemma 3.3, that if $Z \in V_{k_0} \cap \Omega$, the point $Z_t = (Z_1, \dots, Z_{n-1}, Z_n - t)$ belongs to $U_{k_0} \cap \Omega$ for $0 \leq t \leq \tilde{r}$.

To prove the lemma, f being in Λ_α, it suffices to establish the estimate when $\rho(p,q) = \gamma(p,q) + \rho(p) + \rho(q)$, and, f being bounded, when p and q are in some V_l for $l \geq 1$ and so that $\rho(p,q)$ is small (i.e. $\ll \tilde{r}$).

To simplify the notations, let $\tau = \rho(p,q)$, and consider the associated points p_τ and q_τ (in the Z coordinates). By Proposition 2.8, $q_\tau \in B_{\exp}(p_\tau, K\tau)$, and there exist coefficients $(u_i)_{1\leq i \leq 2n}$ such that $q_\tau = \exp_{p_\tau}(u_1, \dots, u_{2n})$ and $|u_i| \lesssim \mathscr{R}'_i(p_\tau, \tau)$.

Thus, if we denote $f_\tau(p) = f(p) - f(p_\tau)$, and the similarly for q, we have

$$\begin{aligned} f(p) - f(q) &= f_\tau(p) - f_\tau(q) + f(p_\tau) - f(q_\tau) \\ &= f_\tau(p) - f_\tau(q) + \int_0^1 \sum u_i \mathcal{Y}_i f(\exp_{p_\tau}(tu_1, \dots, tu_{2n}))\, dt. \end{aligned}$$

The function f being holomorphic the last integral can be written

$$\int_0^1 \sum v_i L_i f(\exp_{p_\tau}(tu_1, \dots, tu_{2n}))\, dt,$$

where $|v_i| \lesssim \mathscr{R}_i(p_\tau, \tau)$. Now, in the coordinate system defined by $\Phi_{w(t)}$, $w(t) = \exp_{p_\tau}(tu_1, \dots, tu_{2n})$, $L_i(w(t)) = \frac{\partial}{\partial z_i}$, and Proposition 3.3 gives (recall $\alpha < 1/M$)

$$|L_i f(w(t))| \lesssim \mathscr{F}_i^{1/2}(w(t), |\rho(w(t))|)\, |\rho(w(t))|^\alpha .$$

But, $\rho(w(t))$ belongs to $[\rho(p_\tau), \rho(q_\tau)]$, and thus, by the properties of coordinate system (Z_i), $|\rho(w(t))| \sim \tau$. Then, Proposition 2.9 gives $\mathscr{F}_i(w_t, |\rho(w(t))|) \sim \mathscr{F}_i(p_\tau, \tau)$ and

$$|f(p_\tau) - f(q_\tau)| \lesssim \tau^\alpha.$$

f being holomorphic, $\frac{d}{dt}(f(p_t)) = \frac{\partial f}{\partial Z_n}(p_t)$, then, if we write $f_\tau(p)$ as an integral of $\frac{d}{dt}(f(p_t)$ between 0 and τ, $\alpha < 1$ and Lemma 3.7 imply (recall $|\rho(p_t)| \gtrsim t$)

$$|f_\tau(p)| \leq C(f) \int_0^\tau t^{\alpha-1}\, dt = C(f)\tau^\alpha.$$

The similar inequality is clearly true replacing p by q. Then the lemma is proved. □

3.3.3. The case $\alpha \geq 1/M$. The spaces Λ_α are well adapted to our purpose for all $\alpha > 0$, but those given by Definition 3.2 cannot be used, for $\alpha \geq 1/M$, to obtain results on the Bergman projection (for example, if $\alpha > 1/M$ it forces the function to be constant in certain directions). To get through the general case we have to define the spaces Γ_α in another way.

The functions in Λ_α can be characterized by the existence of a decomposition in a sum of functions whose derivatives are well controlled in an isotropic way (Proposition 3.2). Similarly, for $\alpha < 1/M$, the space Γ_α can be characterized (see Lemma 3.9 for the fact that if f satisfy Definition 3.3 below then $f \in \Gamma_\alpha(\Omega)$, and the methods of [**MS94**] and Proposition 2.5 for the converse) using such a decomposition but with non-isotropic estimates for the derivatives. It is then natural to define the space Γ_α, for all $\alpha > 0$, as follows (recall that we denote by $Z = (Z_i)$ the coordinate system associated to each V_l as in the beginning of Section 3.1).

Definition 3.3. A function f belongs to Γ_α, $0 < \alpha < +\infty$, if it is in $\Lambda_\alpha(V_0)$ and, for every $l \geq 1$ and every integer k there exists functions f_k and g_k, defined in $V_l \cap \Omega$, such that $f = f_k + g_k$ in $V_l \cap \Omega$ and:

(1) $\|f_k\|_{L^\infty} \lesssim_f 2^{-k\alpha}$.
(2) If $Z \in V_l \cap \Omega$ and $|\rho(Z)| \geq 2^{-k}$, for all integer $m \geq M\alpha$,
$$|\nabla^m g_k(Z)| \lesssim_{f,m} 2^{mk} 2^{-k\alpha}.$$
(3) If $Z \in V_l \cap \Omega$ and $|\rho(Z)| < 2^{-k}$, D^a being a derivative of length $|a| \geq M\alpha$ with respect to the coordinate system $\Phi = \Phi_Z$ associated to Z and $\delta = |\rho(Z)|$,
$$\left|D^a(g_k \circ \Phi^{-1})(0)\right| \lesssim_{f,a} \mathscr{F}^{a/2}(Z, 2^{-k}) 2^{-k\alpha}.$$

Theorem 3.4. *Let Ω be a bounded pseudo-convex domain of finite type in $\mathbb{C}^n$ with locally diagonalizable Levi form. For $0 < \alpha < +\infty$, the Bergman projection maps continuously Λ_α into Γ_α.*

Once again, the result follows the next lemma:

Lemma 3.10. *For all $\alpha > 0$, each function in $\Lambda_\alpha(\Omega)$ belongs to $\Gamma_\alpha(\Omega)$.*

Proof: Recall that we showed, at the beginning of the proof of Lemma 3.3, that if $Z \in V_l \cap \Omega$, the point $Z_t = (Z_1, \dots, Z_{n-1}, Z_n - t)$ belongs to $U_l \cap \Omega$ for $0 \leq t \leq \tilde{r}$.

Let $s > \alpha$ be an integer. By integration by parts, we have, for $Z \in V_l \cap \Omega$ and f holomorphic in $U_l \cap \Omega$,

$$f(Z) = \frac{(-1)^{s-1}}{(s-1)!} \int_0^{\tilde{r}} t^{s-1} \frac{d^s}{dt^s} f(Z_t)\, dt + E(Z),$$

where E is $\mathscr{C}^\infty$ on $\overline{V_l \cap \Omega}$.

For k such that $2^{-k} \leq \tilde{r}$, we define f_k and g_k by

$$f_k(Z) = \frac{(-1)^{s-1}}{(s-1)!} \int_0^{2^{-k}} t^{s-1} \frac{d^s}{dt^s} f(Z_t)\, dt,$$

$$g_k(Z) = \frac{(-1)^{s-1}}{(s-1)!} \int_{2^{-k}}^{\tilde{r}} t^{s-1} \frac{d^s}{dt^s} f(Z_t)\, dt + E(Z)$$

so that $f = f_k + g_k$.

Then

Lemma. *With the previous notations, if $f \in \Lambda_\alpha(\Omega)$ is holomorphic, we have*

$$|f_k(Z)| \leq C(f) 2^{-k\alpha}$$

and, if Φ_p is the change of coordinates associated to $p = Z_{2^{-k}}$ and $\delta = |\rho(p)|$ and $z = \Phi_p(Z)$, for all derivative D^a,

$$\left| D^a (g_k \circ \Phi_p^{-1})(z) \right| \leq C(f,a) \left[\mathscr{F}^{a/2}(Z_{2^{-k}}, |\rho(Z_{2^{-k}})|)\, |\rho(Z_{2^{-k}})|^\alpha + 1 \right].$$

Proof: By the choice of the coordinate system, $|\rho(Z_t)| \gtrsim t$, and, f being holomorphic, $\frac{d^s}{dt^s}(f(Z_t)) = \frac{\partial^s f}{\partial Z_n^s}(Z_t)$, then $s > \alpha$ and Lemma 3.7 imply

$$|f_k(Z)| \leq C(f) \int_0^{2^{-k}} t^{-s+\alpha+s-1}\, dt = C(f) 2^{-k\alpha}.$$

To estimate the derivatives of g_k, let us first integrate by parts:

$$g_k(z) = \tilde{E}(Z) + \sum_{l=0}^{s-1} *2^{-kl} \frac{d^l}{dt^l} f(Z_t)_{|t=2^{-k}} = \tilde{E}(Z) + \sum_{l=0}^{s-1} *2^{-kl} \frac{d^l f}{dZ_n^l}(Z_{2^{-k}}),$$

where the $*$ are absolute constants depending only on s and $\tilde{E}$ is $\mathscr{C}^\infty$ on $\overline{V_l \cap \Omega}$.

Then, Proposition 3.3, (2.1) and the fact that $|\rho(Z_{2^{-k}})| \gtrsim 2^{-k}$ imply

$$\left| D^a (g_k \circ \Phi_p^{-1})(z) \right| \leq C(f) \left[\mathscr{F}^{a/2}(Z_{2^{-k}}, |\rho(Z_{2^{-k}})|)\, |\rho(Z_{2^{-k}})|^\alpha + 1 \right]. \qquad \square$$

To prove Lemma 3.10, it suffices to show that the functions $g_k$satisfies properties (2) and (3) of Definition 3.3. Denote by $\tilde{z}$ and z the coordinate system associated to Φ_p and Φ_q where $q = Z^q = Z$ and $p = Z_{2^{-k}}$, and write $L_i = \sum b_i^j \frac{\partial}{\partial z_j} = \sum \tilde{b}_i^j \frac{\partial}{\partial \tilde{z}_j}$, $\frac{\partial}{\partial z_i} = \sum a_i^j L_j$, $\frac{\partial}{\partial \tilde{z}_i} = \sum \tilde{a}_i^j L_j$. Because $|\rho(Z)| < |\rho(Z_{2^{-k}})|$, Propositions 2.5 and 2.9 imply that all the derivatives $\left|D_z^a(b_i^j)\right|$, $\left|D_z^a(a_i^j)\right|$, $\left|D_{\tilde{z}}^a(\tilde{b}_i^j)\right|$ and $\left|D_{\tilde{z}}^a(\tilde{a}_i^j)\right|$ are

$$\lesssim_a \mathscr{F}^{a/2}(Z, |\rho(Z_{2^{-k}})|)\mathscr{F}_i^{1/2}(Z, |\rho(Z_{2^{-k}})|)\mathscr{F}_j^{-1/2}(Z, |\rho(Z_{2^{-k}})|).$$

It follows immediately that

$$\left|D_{\tilde{z}}^a(g_k \circ \Phi^{-1})(0)\right| \lesssim_a \left|D_z^a(g_k \circ \Phi_p^{-1})(z)\right|.$$

Now, $|\rho(Z)| < 2^{-k}$ implies $|\rho(Z_{2^{-k}})| \sim 2^{-k}$, and, because $|a| \geq M\alpha$, (2.1) and (2.2) give the required inequality.

If $|\rho(Z)| \geq 2^{-k}$, the result is trivial because the function $\mathscr{F}^{a/2}(p, \delta)\delta^\alpha$ is decreasing in the case $|a| \geq M\alpha$. □

4. The Szegö projection

The theory of "non isotropic smoothing" operator (NIS operator) was introduced in [**NRSW89**] to study the Szegö projection for domains of finite type in $\mathbb{C}^2$, and then extended to decoupled domains of $\mathbb{C}^n$ in [**CG94**] and to convex domains of finite type in [**MS97**].

These operators are defined in relation with a (non-isotropic) good pseudo-distance which confers to the domain a structure of homogeneous space. Then the method consists to prove first that the class of NIS operators is stable under basic operations (composition, derivation, Lie brackets. . .), second that a NIS operator of order 0 maps certain spaces into themselve (L^p spaces, non-isotropic Hölder spaces. . .), and, finally, that the Szegö projection is a NIS operator of order 0. Relations between the Szegö projection and the Bergman kernel and pointwise estimates of this kernel are used to prove the last point.

In the case of convex domains of finite type J. D. McNeal and E. M. Stein [**MS97**] obtained also classical Hölder estimates proving pointwise estimates for the Szegö kernel inside the domain.

Note that the NIS operators has also been used in [**CNS92**] to study the $\overline{\partial}$-Neumann problem.

The goal of this section is to show that the NIS operator theory and its applications to the Szegö projection can be developed in the case of domains of finite type with locally diagonalizable Levi form. The general ideas of the NIS operators theory being now well known, we will only show how to adapt this theory to our context (giving slighty modified definitions for bump functions and NIS operators) and give the essential steps without detailled proofs. The proofs can easily be deduced from [**MS97**].

4.1. NIS operators and bump functions. As we did in Section 3, we cover $\overline{\Omega}$ by open sets V_k, $0 \leq k \leq N_0$, the sets V_k, $1 \leq k \leq N_0$, being a covering of $\partial\Omega$. For each $k \geq 1$, let $\{L_1^{(k)}, \ldots, L_{n-1}^{(k)}\}$ be a basis of the complex tangent space diagonalizing the Levi form in $V_k \cap \partial\Omega$.

First we define the "bump functions" as follows:

Let $x_0 \in \partial\Omega \cap V_k$ and $\delta > 0$ such that the "polydisk" $P_{x_0}(\delta)$ (see Section 2.2) is contained in V_k. A function $\Phi \in \mathscr{C}^N(P_{x_0}(\delta) \cap \partial\Omega)$ with compact support is called a normalized "bump function" of order N if, for every list $\mathscr{L}$ (associated to the vector fields $L_i^{(k)}$ or $\overline{L}_i^{(k)}$, $i < n$) of length less than N,

$$|\mathscr{L}\Phi| \leq \mathscr{F}(x_0, \delta)^{-\mathscr{L}/2}.$$

Note that the properties of the change of coordinates (Section 2.2) and Proposition 2.7 show that, for $\alpha \in]0, 1[$, there exists a constant $C_{N,\alpha}$ such that, for $k \geq 1$, if $P_{x_0}(\delta) \subset V_k$, there exists a function Φ identically equal to 1 in $P_{x_0}(\alpha\delta) \cap \partial\Omega$ such that $\Phi/C_{N,\alpha}$ is a normalized bump function of order N.

These bump functions are used to define partitions of unity to localize the problems in the open sets V_k so that we can work with the vector fields $L_i^{(k)}$.

Now we give the definition, adapted to our situation, of a NIS operator of order a:

Definition 4.1. An operator $\mathscr{A}$ mapping $\mathscr{C}^\infty(\partial\Omega)$ into itself and defined by

$$\mathscr{A} f(x) = \int_{\partial\Omega} A(x, y) f(y)\, d\sigma(y),$$

where A is $\mathscr{C}^\infty$ outside the diagonal is called an operator of order $a = (a_1, \ldots, a_n)$ if there exists a family of operators $\mathscr{A}_\varepsilon$ defined by kernels A_ε satisfying the five following properties:

(1) For $f \in \mathscr{C}^\infty(\partial\Omega)$, $\mathscr{A}_\varepsilon f$ converges to $\mathscr{A} f$ in $\mathscr{C}^\infty(\partial\Omega)$.
(2) $A_\varepsilon \in \mathscr{C}^\infty(\partial\Omega \times \partial\Omega)$.
(3) For $k = 1, \ldots, N_0$, and all lists $\mathscr{L}$ and $\mathscr{L}'$ associated to vector fields $L_i^{(k)}$ and $\overline{L}_i^{(k)}$ of length α and β,
   (a) for all $(x, y) \in V_k \times V_k$,

$$\left|\mathscr{L}_x \mathscr{L}'_y A_\varepsilon(x, y)\right| \lesssim_{\alpha,\beta} \mathscr{F}(x, \gamma_k(x, y))^{\frac{-a+\mathscr{L}+\mathscr{L}'}{2}},$$

(b) for all $(x,y) \notin \bigcup_{k\geq 1}(V_k \times V_k)$,

$$\left|\mathscr{L}_x\mathscr{L}'_y A_\varepsilon(x,y)\right| \lesssim_{\alpha,\beta} 1,$$

the constants in the inequalities being independent of ε.

(4) For $r \in \mathbb{N}^*$, there exists $N_r \in \mathbb{N}^*$ such that, for all normalized bump function Φ on $P_{x_0}(\delta) \cap \partial\Omega$ (contained in V_k) of order $\geq N_r$ and all list $\mathscr{L}$ (as before),

$$\|\mathscr{L}(A_\varepsilon\Phi)\|_\infty \lesssim_r \mathscr{F}^{\frac{-a+\mathscr{L}}{2}}(x_0,\delta),$$

the estimate being uniform in ε.

(5) The properties (1)–(4) are satisfied by the adjoint $\mathscr{A}^*$ of $\mathscr{A}$.

Following the methods of [**NRSW89**] and [**MS97**], with the previous definition it is easy to prove the next stability properties:

Proposition 4.1. (1) *If $\mathscr{A}_1$ and $\mathscr{A}_2$ are NIS operators of respective order a and b, and if $\sum_{a_k+b_k\geq 0}\frac{a_k+b_k}{2} + \sum_{a_k+b_k<0}\frac{a_k+b_k}{M} < \frac{2n}{M}$, then $\mathscr{A}_1 \circ \mathscr{A}_2$ is a NIS operator of order $a+b$.*

(2) *If $\mathscr{A}$ is a NIS operator of order a and if D is a differential operator of order 1, then the family of kernels $D_x A_\varepsilon(x,y)$ defines a NIS operator of order $a-(1,0,\dots,0)$.*

(3) *If $\mathscr{A}$ is a NIS operator of order a and $h \in \mathscr{C}^\infty(\partial\Omega)$, the operator $[\mathscr{A},h]$ is a NIS operator of order $a+(\frac{1}{M},0,\dots,0)$.*

4.2. The Szegö projection as an operator of order 0. We use the relation, shown in [**NRSW89**], between the Szegö projection and the Bergman kernel. Let us recall it briefly.

For $x \in \partial\Omega$, let $N(x,t)$ denote the integral curve of N normalized so that $N(x,0)=x$ and $\rho(N(x,t))=t$. Choose b sufficiently small so that $N(x,t) \in \Omega \cap \left(\bigcup_{k\geq 1} V_k\right)$, $0 \leq t \leq b$. For $g \in \mathscr{C}^\infty(\overline{\Omega})$, define the kernel A_{g_ε} by

$$A_{g_\varepsilon}(x,y) = \int_\varepsilon^b K_B(N(x,t),y)g(N(x,t))\,dt.$$

We assume $|\nabla\rho| \equiv 1$ on $\partial\Omega$. Thus, if $f \in \mathscr{C}^\infty(\partial\Omega)$ and if $\tilde{f}$ denotes a $\mathscr{C}^\infty(\overline{\Omega})$ extension of f, Stokes formula shows that

$$\int_{\partial\Omega} A_{g_\varepsilon}(x,w)f(w)\,d\sigma(w) = \int_\varepsilon^b g(N(x,t))P_B\left(\left\langle\partial\tilde{f},\partial\rho\right\rangle\right)(N(x,t))\,dt + \int_\varepsilon^b g(N(x,t))P_B(\tilde{f}\Delta\rho)(N(x,t))\,dt,$$

where P_B is the Bergman projection.

As the Bergman projection maps $\mathscr{C}^\infty(\overline{\Omega})$ into itself, the preceding formula shows that the operator $\mathscr{A}_{g_\varepsilon}$, defined with the kernel A_{g_ε}, converges, in $\mathscr{C}^\infty(\partial\Omega)$, to the operator $\mathscr{A}_g$ defined by the kernel

$$A_g(x,y) = \int_0^b g(N(x,t))K_B(N(x,t),y)\,dt.$$

Following the calculus made in [**MS97**], it is easy to show that $\mathscr{A}_g$ is an operator of order 0.

Our domain being of finite type, the Szegö projection P_S maps $\mathscr{C}^\infty(\partial\Omega)$ into itself. Thus, writing, for $f \in \mathscr{C}^\infty(\partial\Omega)$,

$$\frac{d}{dt}P_S f(N(x,t)) = \sum_{i=1}^n g_i(N(x,t))\frac{\partial P_S f}{\partial z_i}(N(x,t)),$$

by integration by parts we immediately obtain that

$$P_S f = \lim_{\varepsilon\to 0}\sum_{i=1}^n \mathscr{A}_{g_{i\varepsilon}}\left(\frac{\partial\rho}{\partial z_i}P_S f\right),$$

the convergence being in $\mathscr{C}^\infty(\partial\Omega)$. Using the fact that $f - P_S f$ is orthogonal to holomorphic functions and that the kernel of $\mathscr{A}_{g_{i\varepsilon}}$ is anti-holomorphic in the second variable, we easily obtain

$$P_S f = \lim_{\varepsilon\to 0}\left(P_\varepsilon f + Q_\varepsilon P_S f\right), \tag{4.1}$$

in $\mathscr{C}^\infty(\partial\Omega)$, where $P_\varepsilon = \sum_{i=1}^n \frac{\partial\rho}{\partial z_i}\mathscr{A}_{g_{i\varepsilon}}$ and $Q_\varepsilon = \tilde{Q} + \sum_{i=1}^n \left[\mathscr{A}_{g_{i\varepsilon}}, \frac{\partial\rho}{\partial z_i}\right]$, with $\tilde{Q}$ the operator of order $\left(\frac{1}{M}, 0\ldots,0\right)$ associated to the Poisson kernel.

In other words, this method allows us to write P_S as a limit of well controlled operators. Then the fact that P_S is of order 0 follows Proposition 4.1.

4.3. Estimates for the Szegö projection.

4.4. L^p estimates. The restriction to $\partial\Omega\times\partial\Omega$ of the function γ defined in Section 3.3 is a pseudo-distance which defines a structure of homogeneous space on $\partial\Omega$. Thus, we can use the T(1) Theorem of [**DJS85**] to show that a NIS operator of order 0 is bounded on $L^2(\partial\Omega)$, an then also on $L^p(\partial\Omega)$, $1 < p < +\infty$. Using commutations properties (similar to Corollary 4.2 of [**MS97**]), we obtain

Theorem 4.1. *For $1 < p < +\infty$ and $s \in \mathbb{N}$, the Szegö projection maps continuously $L^p_s(\partial\Omega)$ into itself.*

4.5. Hölder estimates. Let us first consider the calssical Hölder space $\Lambda_\alpha(\partial\Omega)$ (i.e. defined with the euclidean distance). Following (4.1), we can write $P_S = P + QP_S$, where P is an operator of order 0 and Q an operator of order $\left(\frac{1}{M}, \dots, 0\right)$, and, iterating, $P_S = \sum_{i=0}^{N-1} Q^i P + Q^N P_S$. The Szegö projection being autoadjoint, we also have $P_S = \sum_{i=1}^{N-1} P^* Q^{*i} + P_S Q^{*N}$. Then, writing $P_S = P_S^* P_S$, we approximate P_S by operators whose kernels are in $\mathscr{C}^\infty(\partial\Omega \times \partial\Omega)$ holomorphic in the first variable and antiholomorphic in the second, and we can prove that the Szegö projection satisfies interior estimates in the sense of Definition 4 of [**MS97**], and reproduce without any difficulty the proof of this paper to obtain:

Proposition 4.2. *The Szegö projection maps continuously* $\Lambda_\alpha(\Omega)$ *into itself for all* $\alpha \in]0, +\infty[$.

Consider now, for $0 < \alpha < 1/M$, $\Gamma_\alpha(\partial\Omega)$, the space of function satisfying a Hölder estimate of exponent α with respect to the pseudo-distance defined with γ. Then $\Gamma_\alpha(\partial\Omega)$ is the restriction to $\partial\Omega$ of the space $\Gamma_\alpha(\Omega)$ introduced in the previous section. We extend this definition to all $\alpha > 0$ by $\Gamma_\alpha(\partial\Omega) = \{F_{|\partial\Omega},\, F \in \Gamma_\alpha(\Omega)\}$.

Note that if a function $f \in \Lambda_\alpha(\partial\Omega)$ is the restriction of a holomorphic function F in Ω (i.e. if it's Poisson integral is holomorphic), then F belongs to $\Lambda_\alpha(\Omega)$. Using then Proposition 4.2 and Lemma 3.10 it is easy to prove:

Theorem 4.2. *For all* $\alpha \in]0, +\infty[$*, the Szegö projection maps continuously* $\Lambda_\alpha(\partial\Omega)$ *into* $\Gamma_\alpha(\partial\Omega)$.

5. Further results

The geometric study of pseudo-convex domains near points of finite type with locally diagonalizable Levi form made in [**CD**] gives two essential properties:

(1) The existence of a change of coordinate Φ_z^δ, attached to a point $z \in \overline{\Omega}$ and a $\delta > 0$, related, uniformly in z and δ, to the functions $\mathscr{F}_i(z, \delta)$ (Section 2.1 and 3.5 of [**CD**]).
(2) The existence of a plurisubharmonic function on Ω whose hessian is controlled, in strings, by the functions $\mathscr{F}_i(z, \delta)$ (Section 4 of [**CD**]).

These local properties where used in [**CD**] to obtain estimates of the Bergman kernel and, in the previous sections, using these estimates and property (1), we proved some estimates for the Bergman and Szegö projections when all the boundary points satisfies our hypothesis.

It is also possible to use the two properties to give precise estimates of the standard invariant metrics of Ω under the local hypothesis of finite type and diagonalizability of the Levi form:

Theorem 5.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\mathscr{C}^\infty$ boundary. Suppose that there is a point of finite type $z_0 \in \partial\Omega$ and a neighborhood U of z_0 where the Levi form is diagonalizable. Let us denote by $B_\Omega(z, L)$ (resp. $C_\Omega(z, L)$, resp. $K_\Omega(z, L)$) the Bergman (resp. Caratheodory, resp. Kobayashi) metric of Ω at the point $z \in \Omega$. Let L_i, $1 \leq i \leq n-1$ be a basis which diagonalizes the Levi form and L_n the complex normal vector field. Then if L is a holomorphic vector at the point $z \in \Omega \cap U$, $L = \sum_{i=1}^n b_i L_i$, we have*

$$B_\Omega(z, L) \simeq C_\Omega(z, L) \simeq K_\Omega(z, L) \simeq \sum_{i=1}^n |b_i| \mathscr{F}_i^{1/2}(z, \delta(z)),$$

where $\delta(z)$ is the distance to the boundary, the constants in the equivalences being independent of z and L.

Note that, using uniform properties of the change of variables $\Phi_{z'}^\delta$, it is enough to prove the equivalence of the metrics on a suitable domain $\Omega'_{z'} = \Phi_{z'}^\delta(\Omega)$. To do that, we follow the method introduced by D. W. Catlin [**Cat89**] in $\mathbb{C}^2$, as S. Cho did in [**Cho02a**] for the case of domains whose Levi form have comparable eigenvalues.

The starting point is an analog of Theorem 3.1 of [**Cat89**] which is easily proved using the proofs of Theorem 4.1 and Proposition 5.1 of [**CD**]:

Theorem 5.2. *There exists a neighborhood U of z_0 such that, for all $\delta > 0$ small enough, there exists a function $\lambda_\delta \in \mathscr{C}^\infty(\mathbb{C}^n)$ with the following properties:*

(1) $|\lambda_\delta(z)| \leq 1$.
(2) *For all $L = \sum b_i L_i$, $\left\langle \partial\overline{\partial}\lambda_\delta(z); L, \overline{L}\right\rangle \gtrsim \sum |b_i|^2 \mathscr{F}_i^{1/2}(z, \delta)$, for $z \in U \cap \{|\rho(z)| < \delta\}$.*
(3) *For $z' \in U \cap \{|\rho(z)| < \delta\}$ and any derivative $D = D^{\alpha_1} \dots D^{\alpha_n}$, in $(\Phi_{z'}^\delta)^{-1}\left(U \cap \{|\rho(z)| < \delta\} \cap P_{z'}(\delta)\right)$,*

$$\left| D\lambda_\delta \circ \Phi_{z'}^\delta \right| \lesssim \mathscr{F}^{\alpha/2}(z', \delta).$$

Then Catlin's proof can be done with anisotropic estimates using cutoff functions associated to the anisotropic polydisk P_z. The function $J_\alpha(\xi')$ introduced by Catlin in Section 4 of [**Cat89**] corresponds to the function $\delta + \gamma(z', \xi')$ (see Lemma 3.5). The existence of the uniform bumping is given by a theorem of S. Cho [**Cho92**] which is valid in any finite type domain.

As shown by J. D. McNeal in [**MN01**], Theorem 5.1 and the estimates on the Bergman kernel recalled in Section 2.3 give an estimate for the $\overline{\partial}$ problem for the norms associated to an invariant metric:

Theorem 5.3. *Let Ω be a domain in $\mathbb{C}^n$ which satisfies the hypothesis of Theorem 3.1. Then there exists a constant $C > 0$ such that, for any $(n,1)$-form α, $\overline{\partial}$-closed, there exists a solution u of the equation $\overline{\partial} u = \alpha$ satisfying*

$$\|u\|_I \leq C \|\alpha\|_I ,$$

where $\|.\|_I$ denotes the norm associated to any of the metrics of Caratheodory, Bergman or Kobayashi.

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