A STABILITY RESULT FOR NONLINEAR NEUMANN PROBLEMS IN REIFENBERG FLAT DOMAINS IN $\mathbb{R}^N$

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Abstract

In this paper we prove that if Ω_k is a sequence of Reifenberg-flat domains in $\mathbb{R}^N$ that converges to Ω for the complementary Hausdorff distance and if in addition the sequence Ω_k has a "uniform size of holes", then the solutions u_k of a Neumann problem of the form

$$(0.1) \qquad \begin{cases} -\operatorname{div} a(x, \nabla u_k) + b(x, u_k) = 0 & \text{in } \Omega_k \\ a(x, \nabla u_k) \cdot \nu = 0 & \text{on } \partial\Omega_k \end{cases}$$

converge to the solution u of the same Neumann problem in Ω. The result is obtained by proving the Mosco convergence of some Sobolev spaces, that follows from the extension property of Reifenberg-flat domains.

Introduction

In this paper we study the stability of solutions for the following nonlinear Neumann problem

$$(0.2) \qquad \begin{cases} -\operatorname{div} a(x, \nabla u) + b(x, u) = 0 & \text{in } \Omega \\ a(x, \nabla u) \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded subset of $\mathbb{R}^N$, $a\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $b\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions satisfying suitable monotonicity, coerciveness and growth conditions (see (1.1)–(1.3) below). More precisely, we are interested in the following question. Let Ω_k be a sequence of open sets in $\mathbb{R}^N$ that converges to Ω for the complementary Hausdorff distance. Let u_k be the sequence of solutions for the problem (0.2) in Ω_k and let u be the solution associated to Ω. Is it true that u_k converges to u? If the answer is positive we say that the problem (0.2) is stable along the sequence Ω_k (see Definition 3).

This question was studied by many authors in the last decade, principally in dimension 2, for smooth and non smooth domains. The case of non smooth domain, and more specifically domains with cracks, appears typically in applications from fracture mechanics. In addition, the problem of stability is linked to the notions of Mosco convergence or Gamma convergence, which are powerful tools to study the semicontinuity of functionals in shape optimization problems or for instance the Mumford-Shah functional in image processing.

Recently, G. Dal Maso, F. Ebobisse and M. Ponsiglione [**3**] proved that in dimension 2, if Ω_k tends to Ω for the complementary Hausdorff distance in $\mathbb{R}^2$, $|\Omega_k|$ tends to $|\Omega|$ and if the number of the connected components of the complements Ω_k^c are uniformly bounded, then the problem (0.2) is stable. This stability property extends the corresponding results of A. Chambolle and F. Doveri [**2**] and also D. Bucur and N. Varchon [**1**]. In particular, the authors showed that the stability is equivalent to the convergence of some Banach spaces in the sense of Mosco.

According to the authors knowledge, only few results have been proven in higher dimensions. In [**8**], A. Giacomini proves a stability result in $\mathbb{R}^N$ for fractured locally Lipschitz domains where an approach involving the Mosco convergence is also used.

A famous example called the "Neumann Sieve" ([**7**], [**15**], [**17**]), shows that in many cases the stability is not true. The general idea, for instance in dimension 2, is that if one considers a sequence of 1-dimensional sets E_k in $B(0,1)$ with more and more holes of size $\frac{1}{k}$ but with always same global length, it could happen that $\{E_k\}$ converges to a segment while the problem (0.2) is not stable along the sequence $\{B(0,1)\backslash E_k\}$.

In this paper, we prove that if $\{\Omega_k\}$ converges to Ω for the complementary Hausdorff distance in $\mathbb{R}^N$ and if $\{\Omega_k\}$ is a sequence of Reifenberg flat domains with "uniform size of holes", then problem (0.2) is stable along the sequence $\{\Omega_k\}$ (see Theorem 7). The main point is to prove that the limit in the sense of Mosco, of the space of restrictions of functions in $W^{1,p}(\Omega_k)$ is the space of restrictions of functions in $W^{1,p}(\Omega)$.

The notion of "uniform size of holes" (Definition 8) is here to avoid the "Neumann Sieve" problem as described above. Indeed, we want to allow crack domains but according to our definition of uniform size of holes, the "tips" of the cracks can never become arbitrary closer to each other at the limit.

Our theorem partially extends the result of [**3**] in any dimension, and it also can be considered as an extension of [**8**] since the regularity of our sets is weaker than Lipschitz. Indeed, Reifenberg flat sets are ones that

are locally well-approximated by hyperplanes, at every scale. This allows for instance some "Hölderian spirals" or fractal boundaries (see [6]).

Reifenberg flat sets are rough domains that have naturally been used in the study of boundary regularity and the regularity of free boundaries coming from a minimization problem (see for instance [4], [5], [10], [12], [14] and references therein). As we shall see in the sequel, Reifenberg flat domains are moreover very well-adapted in the construction of good extensions for functions in the Sobolev space, which is the key ingredient for proving the Mosco convergence.

This sort of technics were also used in [11], and probably appeared for the first time in [9] while Peter Jones was seeking some geometrical conditions to define a class of extension domains. Notice however that our extension is slightly different than the classical extension of Jones, since we allow ourselves to modify the function inside the domain in a very small neighborhood of the boundary. On the other hand there is a little topological difficulty to overcome, coming from the fact that our domains admit some cracks (i.e. holes in the boundary).

Once the Extension Lemma is established, the Mosco convergence comes fairly easily and this completes the proof of our main result. Let us list here some notations that will be used along the present paper.

Notation.

$\partial\Omega := \overline{\Omega}\backslash\Omega$, the topological boundary of Ω.

$W^{1,p}(\Omega) :=$ the Sobolev space.

$C_c^1(\Omega) :=$ space of $C^1(\Omega)$ functions with compact support.

$\nu :=$ the outward normal vector.

$|A| :=$ the Lebesgue measure of the Borel set A.

$d_H :=$ the Hausdorff distance (defined in (1.7)).

$A^c :=$ the complement of the set A.

$A\triangle B := A\backslash B \cup B\backslash A$.

$B(x,r) :=$ the ball with center at x and radius r.

$C :=$ a positive constant, that could vary from line to line, and that only could depend on dimension.

1. Stability for the nonlinear Neumann problem

Let Ω be an open bounded subset of $\mathbb{R}^N$ and let the real numbers p and q satisfy $1 < p \leq 2 \leq q < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. In this section

we prove the stability for problem (0.2) where $a\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $b\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions satisfying the following assumptions: there exist $0 < c_1 \leq c_2$, $\alpha \in L^q(\mathbb{R}^N)$, and $\beta \in L^1(\mathbb{R}^N)$ such that, for almost every $x \in \mathbb{R}^N$ and for every $\xi, \xi_1, \xi_2 \in \mathbb{R}^N$ with $\xi_1 \neq \xi_2$

$$(1.1) \qquad \langle a(x, \xi_1) - a(x, \xi_2), (\xi_1 - \xi_2) \rangle > 0;$$

$$(1.2) \qquad |a(x, \xi)| \leq \alpha(x) + c_2 |\xi|^{p-1};$$

$$(1.3) \qquad a(x, \xi) \cdot \xi \geq -\beta(x) + c_1 |\xi|^p.$$

We assume that b satisfies the same inequalities, with possible different α, β, c_1 and c_2. For simplicity we use the same notation.

$$(1.4) \qquad \langle b(x, \xi_1) - b(x, \xi_2), (\xi_1 - \xi_2) \rangle > 0;$$

$$(1.5) \qquad |b(x, \xi)| \leq \alpha(x) + c_2 |\xi|^{p-1};$$

$$(1.6) \qquad b(x, \xi) \cdot \xi \geq -\beta(x) + c_1 |\xi|^p.$$

Throughout the paper we assume that (0.2) is satisfied in the usual weak sense of Sobolev spaces, that is $u \in W^{1,p}(\Omega)$ and

$$\int_\Omega \langle a(x, \nabla u), \nabla \varphi \rangle + \langle b(x, u), \varphi \rangle \, \mathrm{d}x = 0$$

for all $\varphi \in W^{1,p}(\Omega)$. It is well known that problem (0.2) has a unique solution in $W^{1,p}(\Omega)$ (see [13]).

If A and B are two nonempty closed subsets of $\mathbb{R}^N$, we define the Hausdorff distance between A and B by

$$(1.7) \qquad d_H(A, B) := \max\{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\}$$

where $d(x, A) = \operatorname{dist}(x, A)$.

Definition 1. Let $\{\Omega_k\}_{k \in \mathbb{N}}$ and Ω be some nonempty open subsets of $\mathbb{R}^N$. We say that Ω_k converges to Ω for the complementary Hausdorff distance if $d_H(\Omega_k^c, \Omega^c)$ tends to 0.

Remark 2. If Ω_k tends to Ω for the complementary Hausdorff distance, then $d_H(\partial \Omega_k, \partial \Omega)$ tends to zero, but observe that the converse is false in general (for instance take $\Omega_k := (\frac{1}{k}, +\infty)$ and $\Omega := (-\infty, 0)$).

Suppose that Ω_k converges to Ω for the complementary Hausdorff distance, and let u_k be a weak solution of the problem

(1.8)
$$\begin{cases} -\operatorname{div} a(x, \nabla u_k) + b(x, u_k) = 0 & \text{in } \Omega_k \\ a(x, \nabla u_k) \cdot \nu = 0 & \text{on } \partial \Omega_k. \end{cases}$$

Definition 3. We say that problem (0.2) is stable along the sequence Ω_k, if the following holds: Let $u_k \in W^{1,p}(\Omega_k)$ be a sequence of solutions of the problem (1.8) in Ω_k. Then $(1_{\Omega_k} u_k, 1_{\Omega_k} \nabla u_k)$ converges strongly to $(1_\Omega v, 1_\Omega \nabla v)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N, \mathbb{R}^N)$ and v is a solution of problem (0.2).

We seek conditions on Ω_k to make the problem (0.2) stable.

Definition 4. A (δ, r_0)-Reifenberg-flat domain $\Omega \subset \mathbb{R}^N$ is an open bounded set such that for each $x \in \partial \Omega$ and for any $r \leq r_0$ there exist a hyperplane $P(x, r)$ containing x such that

(1.9)
$$\frac{1}{r} \sup_{y \in \partial \Omega \cap B(x,r)} \operatorname{dist}(y, P(x,r)) \leq \delta.$$

Our definition of Reifenberg-flat domains is not exactly the same that could be found in the literature, essentially from the fact that we did not take a bilateral definition of the distance in (1.9) in order to allow cracks (when the domain lies in each side of its boundary). Using the terminology of [11], our Reifenberg-flat domains should be called "weak Reifenberg-flat domains". Observe that our definition allows the fact that $\partial \Omega$ could have an infinite number of connected components. Moreover in our definition, Ω is not supposed to be connected.

The topological disc theorem of Reifenberg [16] says that, under some additional separation conditions and if δ is small enough, then the boundary of a (δ, r_0)-Reifenberg-flat domain is locally the bi-hölderian image of a $N-1$ dimensional unit disc. In addition, this is optimal since a Reifenberg-flat domain can admit some Hölder spiral. It is worth mentioning that a Reifenberg-flat domain could have a fractal "snowflake-like" boundary (see [6]).

Now we consider a topological assumption in order to avoid the problem of "Neumann Sieves" (see [15]). For a Reifenberg-flat domain Ω and for any ball $B(x, r)$ centered at $\partial \Omega$ and with radius $r \leq r_0$, let us define the sets $D^+(x, r)$ and $D^-(x, r)$ by the following way. Let $P(x, r)$ be the hyperplane given by the definition of Reifenberg flatness of Ω. Denote by $z^\pm(x, r)$ two points of $B(x, r)$ that lie at distance $3r/4$ from $P(x, r)$ and whose orthogonal projections on $P(x, r)$ are equal to x. Then we set $D^\pm(x, r) := B(z^\pm, r/4)$ as in the following picture.
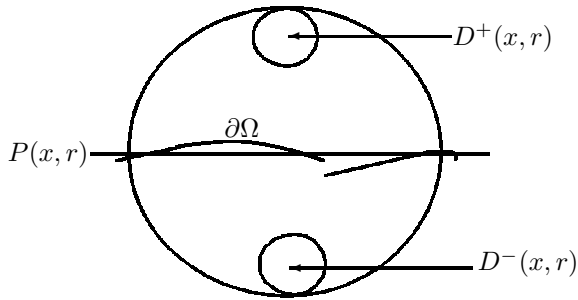
Definition 5. Let Ω be a (δ, r_0)-Reifenberg flat domain. We say that Ω has a uniform size of holes with constant $C_0 \geq 1/2$, if for every ball $B(x, r)$ centered at $\partial\Omega$ with radius $r \leq \frac{1}{2}r_0$ and such that $D^+(x, r)$ and $D^-(x, r)$ lie in the same connected component of $B(x, r) \cap \Omega$, there exists a ball $B(y, s)$ centered on $P(x, 2r) \cap B(x, 2r)$ with radius $s > \frac{1}{C_0}r$ such that $B(y, s) \cap \partial\Omega = \emptyset$. By convention if Ω is such that $D^\pm(x, r)$ always lie in different connected components we say that Ω has a uniform size of holes with $C_0 = 1/3$.

Remark 6. If Ω has a uniform size of holes with constant $C_0 \geq 1/2$ then Ω has a uniform size of holes with any constant $C > C_0$. Therefore, we will usually take C_0 as being the minimal constant satisfying the property of Definition 5 which can never be less than $1/2$ for obvious geometrical reasons. Moreover when we mean that a sequence Ω_k has a uniform size of holes with same constant C_0, we say that all the minimal constants associated to Ω_k are bounded by a same constant C_0 (see also the examples given below).

The rest of the section is devoted to the proof of the following result.

Theorem 7. *Let $C_0 \in [2, +\infty) \cup \{1/3\}$ be a constant and assume that $\{\Omega_k\}_{k\in\mathbb{N}}$, Ω are (δ, r_0)-Reifenberg flat domains with $\delta < 10^{-3}C_0^{-1}$, having a uniform size of holes with same constant C_0, and such that Ω_k converges to Ω for the complementary Hausdorff distance. Then the Neumann problem (0.2) is stable along the sequence Ω_k.*

Before passing to the details of the proof of Theorem 7, we give two examples for domains described in the discussion above.

1.1. Examples.

We would like to emphasize the fact that according to Definition 5, in Theorem 7 we do not consider "sieve domains" with holes in the boundary that are becoming smaller when Ω_k tends to Ω, but only "crack domains" which contain some holes but with a fixed size bigger than $C_0^{-1} r_0$ and controlled shape. To illustrate this, let us show two basic examples in dimension 2.

1.1.1. A counterexample.

Let us consider the 2 dimensional Sieve Domain defined as follow. For a fixed ε set $r_n := \varepsilon 2^{-n}$,

$$\Gamma_n := [0,1] \setminus \bigcup_{k=1}^{2^n} [k2^{-n} - r_n, k2^{-n}]$$

and $\Omega_n := B((1/2,0),1) \setminus \Gamma_n$.
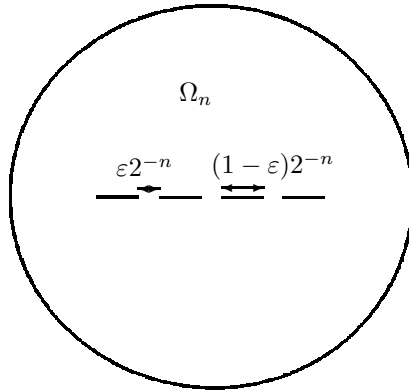
Since the boundary of Ω_n is "smooth", for every $\delta \in (0,1)$, one can easily obtain a radius $r_\delta \in (0, \frac{1}{10})$ such that all the Ω_n are (δ, r_δ)-Reifenberg flat domains. Now for n big enough, we claim that the minimal constant C_0^n of uniform size of holes in Ω_n is bounded from below by $C2^n$. Indeed, we denote $x_0 := (1/2,0) \in \mathbb{R}^2$ and consider the ball $B(x_0, r_\delta/2)$. The corresponding domains $D^\pm(x_0, r_\delta/2)$ lie in the same connected components of Ω and $P(x, r_\delta/2)$ is the first axis. Now since $r_\delta < \frac{1}{10}$, any ball $B(y,s)$ centered on $P(x_0, r_\delta) \cap B(x_0, r_\delta)$ and such that $B(y,s) \cap \partial\Omega_n = \emptyset$ must have a radius $s < \varepsilon 2^{-(n+1)}$. This means that

$$C_0^n \geq r_\delta \frac{2^{n+1}}{\varepsilon}$$

thus the sequence Ω_n cannot have a uniform size of hole with same constant C_0.

1.1.2. An example with fractured domains.

Further, we give another example where the domains Ω_n have now a uniform size of holes with same constant. Of course we could also take some "non-fractured" domains for which the constant is fixed to $\frac{1}{3}$ by convention, but let us consider the following more instructive crack situation. Let S be the segment $[-1,1] \times \{0\} \subset \mathbb{R}^2$ and $\Omega := B(0,2) \backslash S$. Now let $\Gamma_n := \{(t, f_n(t)); t \in [-a_n, a_n] \subset \mathbb{R}\}$ be a sequence of Lipschitz graphs with same Lipschitz constant L and such that $d_H(\Gamma_n, S) \leq 2^{-n-3}$.
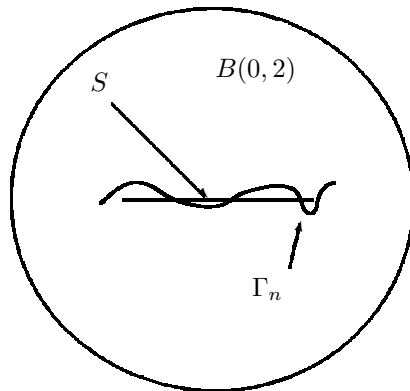
It is not difficult to see that the sequence $\Omega_n := B(0,2) \backslash \Gamma_n$ is a sequence of (δ, r_0)-Reifenberg-flat domains converging to $\Omega \backslash S$ for the complementary Hausdorff distance. Indeed, they are Reifenberg-flat with a good choice of δ and r_0 depending only on the Lipschitz constant L. Moreover we have that $d_H(\Omega_n^c, \Omega^c) = d_H(\Gamma_n, S) \leq 2^{-n-3}$. Finaly, Ω_n and Ω have all a uniform size of holes with constant $C_0 \leq 10$. This is easy to prove for Ω because the only way for a ball $B(x,r)$ to be centered on $\partial\Omega$ and having the property that $B(x,r) \cap \Omega$ is connected is to be centered on S with a radius $r > \text{dist}(x, E(S))$ where $E(S)$ are the two endpoints of S. In this case it is clear that $B(x, 2r)$ contains a ball of radius $s > 10^{-1}r$ centered on the first axis that does not meet $\partial\Omega$. Now one can see this also for Ω_n by exactly the same argument, replacing the endpoints of S by the endpoints of Γ_n, using also that Γ_n is a graph.

Of course in our example, Lipschitz graphs was assumed for convenience and one could try to weaken the regularity assumption on Γ_n taking for instance a sequence of connected Reifenberg-flat sets with admitting some tangents at the endpoints, but the proof becomes a bit more technical in this case.

1.2. Extension Lemma.

The main ingredient for proving the Mosco convergence will be the Extension Lemma contained in this section. For every function $u \in L^1(B(x,r))$ we denote by $m^{\pm}(u)$ the average of u on $D^{\pm}(x,r)$

$$(1.10) \qquad m^{\pm}(u) := \frac{1}{|D^{\pm}(x,r)|} \int_{D^{\pm}(x,r)} u(x)\,\mathrm{d}x.$$

We begin with the following fact.

Proposition 8. *Let $p \geq 1$, $C_0 \geq 1/2$ and let Ω be a (δ, r_0)-Reifenberg flat domain with $\delta < 10^{-3}C_0^{-1}$ and having uniform size of holes with constant C_0. Then for every ball $B(x,r)$ centered at $\partial\Omega$ with $r \leq \frac{1}{4}r_0$ such that $D^+(x,r)$ and $D^-(x,r)$ lie in the same connected component of $B(x,r) \cap \Omega$, we have that*

$$(1.11) \qquad |m^+(u) - m^-(u)| \leq C \frac{1}{r^{N-1}} \int_{B(x,3r)} |\nabla u|\,\mathrm{d}x$$

for any function $u \in W^{1,1}(B(x,3r) \cap \Omega)$, and where C is depending on dimension N and constant C_0.

Proof: The proof is an easy consequence of the classical Poincaré inequality. Indeed, the definition of Uniform size of holes implies that there exist a ball $B(x,s)$ centered on $P(x,2r) \cap B(x,2r)$ such that $B(x,s) \cap \partial\Omega = \emptyset$, and with $s > \frac{1}{C_0}r$. Now the proposition follows from the following fact: since $\partial\Omega$ is at distance less than $2\delta r < 2.10^{-3}C_0^{-1}r$ from $P(x,2r)$ in $B(x,2r)$, one can define a Lipschitz domain A contained in $B(x,3r)$, that contains $D^+(x,r) \cup D^-(x,r) \cup B(x,\frac{s}{2})$, and such that the Poincaré constant in A is less than $C(C_0, N)r$, where $C(C_0, N)$ is only depending on C_0 and N and is independent from all the possible positions of $B(x,s)$. $\square$

Next we give the Extension lemma.

Lemma 9. *Let r_0 be a positive radius, $\tau \in (0, r_0)$ and $C_0 \in [1/2, +\infty) \cup \{1/3\}$. Let Ω_1 and Ω_2 be two (δ, r_0)-Reifenberg-flat domains with $\delta < 10^{-3}C_0^{-1}$, having a uniform size of holes with same constant C_0 and such that*

$$d_H(\Omega_1^c, \Omega_2^c) \leq 10^{-3}C_0^{-1}\tau.$$

Then setting

$$W(\tau) := \{y \in \mathbb{R}^N; d(y, \partial\Omega_1) \leq \tau\},$$

for any $v \in W^{1,p}(\Omega_1)$ there exists a function $\tilde{v} \in W^{1,p}(\Omega_2)$ such that $v = \tilde{v}$ in $\Omega_1 \backslash W(10\tau)$ and

$$(1.12) \qquad \|\tilde{v}\|_{L^p(\Omega_2 \cap W(10\tau))} \leq C\|v\|_{L^p(\Omega_1 \cap W(10\tau))}$$

$$(1.13) \qquad \|\nabla\tilde{v}\|_{L^p(\Omega_2 \cap W(10\tau))} \leq C_1\|\nabla v\|_{L^p(\Omega_1 \cap W(100\tau))}$$

with C depending only on the dimension N, while C_1 depends on constant C_0 and on dimension N.

Proof: Let $\tau \in (0, r_0)$ be fixed and let $B_i := B(x_i, \tau)$ be a family of balls of radius τ, centered at $x_i \in \partial\Omega_1$, and maximal for the property that $\frac{1}{10}B_i \cap \frac{1}{10}B_j = \emptyset$ for all $i, j \in I$ with $i \neq j$. Notice that by this way,

$$W(9\tau) \subset \bigcup_{i \in I} 10B_i \subset W(10\tau).$$

We want to construct a partition of unity associated to $\{B_i\}_{i \in I}$. For all i, define a function $\varphi_i \in C_c^1(10B_i)$, such that $\varphi = 1$ in $8B_i$, $|\nabla\varphi| \leq \tau^{-1}$ and let φ_0 be a function that is equal to 1 in $\Omega_1 \backslash \bigcup_{i \in I} 10B_i$, $\varphi_0 = 0$ in $\bigcup_{i \in I} 8B_i$ and $\varphi_0 + \sum_{i \in I} \varphi_i \geq 1$ in $\Omega_1 \cup \bigcup_{i \in I} 10B_i$. Moreover, we can assume that there is a constant C such that for all $x \in 10B_i \backslash 8B_i$, $|\nabla\varphi_0(x)| \leq C\tau^{-1}$. Indeed, such a function φ_0 can be obtained by setting

$$\varphi_0(x) := \prod_{i \in I} l(d(x, x_i)/\tau)$$

where l is a Lipschitz function equal to 0 in $[0, 8]$, equal to 1 in $[10, +\infty)$ and $l'(x) \leq 10$. Finally, define

$$\theta_i := \frac{\varphi_i}{\varphi_0 + \sum_{i \in I} \varphi_i} \quad \text{for } i \in I \cup \{0\}$$

thus we now have a partition of unity in $\Omega_1 \cup \bigcup_{i \in I} B_i$. To define a function $\tilde{v} \in W^{1,p}(\Omega_2)$, it suffices to define an extension of v in $\Omega_2 \cap \bigcup_{i \in I} 10B_i$ since $\Omega_1 \triangle \Omega_2$ is contained in $\bigcup_{i \in I} 10B_i$.

Let $P_i := P(x_i, 10\tau)$ be the hyperplane associated to Ω_1 given by Definition 4. Recall that since $d_H(\Omega_1^c, \Omega_2^c) \leq 10^{-3}C_0^{-1}\tau$, we have that $10B_i \cap (\partial\Omega_2) \subset \{y \in 10B_i; d(y, P_i) \leq 2.10^{-2}C_0^{-1}\tau\}$.

For each ball B_i, we denote by $D^\pm(x_i, 10\tau)$ the two balls defined just before Definition 8 associated to Ω_1, and we assume this notation to be coherent with the orientation of $\partial\Omega_1$. More precisely we assume the notation of positive and negative signs of $D^\pm(x_i, 10\tau)$, to be chosen in such a way that for any i, j satisfying $B(x_i, 10\tau) \cap B(x_j, 10\tau) \neq \emptyset$, we have that $D^+(x_i, 10\tau)$ and $D^+(x_j, 10\tau)$ both lie in the same connected component of $B(x_i, 100\tau) \setminus P(x_i, 100\tau)$. This of course can be done without loss of generality, starting from a first choice of $D^+(x_j, 10\tau)$ and then defining the others successively. Then we need to consider three cases.

- **Cracktip case:** $D^+(x_i, 10\tau)$ and $D^-(x_i, 10\tau)$ lie both in the same connected component of $B(x, 10\tau) \cap \Omega_2$. Let I_c be the set of indices corresponding to the balls in this situation and for every $i \in I_c$ define $m_i := m^+(v)$ (as in (1.10)).

- **Boundary case 1:** $D^+(x_i, 10\tau)$ and $D^-(x_i, 10\tau)$ lie in different connected components of $B(x, 10\tau) \cap \Omega_2$. Let I_{b_1} be the set of indices corresponding to the balls in this situation. For every $i \in I_{b_1}$, let A_i^+ and A_i^- be the two connected components of $10B_i \setminus \partial\Omega_2$ that contain respectively $D^+(x_i, 10\tau)$ and $D^-(x_i, 10\tau)$ and define $m_i^\pm := m^\pm(v)$.

- **Boundary case 2:** One of $D^\pm(x_i, 10\tau)$ lies in Ω_2 while the other one lies in Ω_2^c. Let I_{b_2} be the set of indices corresponding to the balls in this situation. For every $i \in I_{b_2}$, let m_i be equal to the one of $m^\pm(v)$ corresponding to ball $D^\pm(x_i, 10\tau)$ that lies in Ω_2. Finally as for I_{b_1}, let A_i^+ and A_i^- be the two connected components of $10B_i \setminus \partial\Omega_2$ that contain respectively $D^+(x_i, 10\tau)$ and $D^-(x_i, 10\tau)$.

Notice that since τ is chosen sufficiently small, the preceding 3 cases would be the same if Ω_1 was taken instead of Ω_2 in their definition.

We are now ready to define $\tilde{v}$. It could be that Ω_2 has some tiny connected components hidden in some $10B_i \setminus (A_i^+ \cup A_i^-)$ for $i \in I_{b_1}$. In those components, let us define $\tilde{v}$ to be equal to 0. Then, anywhere else, i.e. for all $x \in A := (\Omega_2 \setminus \bigcup_{i \in I_{b_1}} 10B_i) \cup (\bigcup_{i \in I_{b_1}} A_i^+ \cup A_i^-)$, define

$$(1.14) \quad \tilde{v}(x) := \theta_0 v(x) + \sum_{i \in I_c \cup I_{b_2}} \theta_i(x) m_i + \sum_{i \in I_{b_1}} \theta_i(x)(m_i^+ 1_{A_i^+}(x) + m_i^- 1_{A_i^-}(x)).$$

The function $\tilde{v}$ is now well defined for every $x \in \Omega_2$. We claim that $\tilde{v} \in W^{1,p}(\Omega_2)$ and that (1.12) and (1.13) are satisfied. Let us first show (1.12). Using Hölder inequality and extending v by 0 outside of Ω_1,

we have for all $i \in I_{b_2} \cup I_c$,

$$(1.15) \qquad |m_i|^p \le C \left(\frac{1}{|B_i|} \int_{B_i} |v| \, \mathrm{d}x \right)^p \le C|B_i|^{-1} \int_{B_i} |v|^p$$

and by the same way for $i \in I_{b_1}$ we also have

$$(1.16) \qquad |m_i^{\pm}|^p \le C|B_i|^{-1} \int_{B_i} |v|^p.$$

In addition, since B_i are in a bounded cover (with a universal constant C), the sums in (1.14) are locally finite. Thus, since $\theta_i(x) \le 1_{10B_i}(x)$ and using (1.15) and (1.16),

$$\left\| \sum_{i \in I_c \cup I_{b_2}} \theta_i(x)m_i \right\|_{L^p(\Omega_2)}^p \le C \sum_{i \in I_c \cup I_{b_2}} \int_{10B_i} |m_i|^p$$

$$\le C \sum_{i \in I} \int_{10B_i} |v|^p \, \mathrm{d}x \le C\|v\|_{L^p(W(10\tau))}^p$$

and by the same way

$$\left\| \sum_{i \in I_{b_1}} \theta_i(x)(m_i^+ 1_{A_i^+}(x) + m_i^- 1_{A_i^-}(x)) \right\|_{L^p(\Omega_2)}^p \le C \sum_{i \in I} \int_{10B_i} |v|^p \, \mathrm{d}x$$

$$\le C\|v\|_{L^p(W(10\tau))}^p$$

thus taking the L^p norm in (1.14) we deduce that (1.12) holds. Let us now prove (1.13), which will also imply that $\tilde{v} \in W^{1,p}(\Omega_2)$. By definition, $\tilde{v} = v$ in $\Omega_2 \setminus \bigcup_{i \in I} 10B_i$, thus all we have to prove is that

$$(1.17) \qquad \int_{A \cap \bigcup_{i \in I} 10B_i} |\nabla \tilde{v}|^p \le \int_{\Omega_1 \cap W(60\tau)} |\nabla v|^p.$$

Let i_0 be a fixed index such that $x \in A \cap 10B_{i_0}$. Let I_x be the finite set of indices $i \in I$ such that $x \in B_i$. All balls B_i for $i \in I_x$ are contained in $B(x_{i_0}, 20\tau)$. Let us first make the following assumption on the orientation:

$$(1.18) \qquad \begin{array}{l} \text{For any } i \in (I_{b_1} \cup I_{b_2}) \cap I_x, \, x \text{ lie in the same connected} \\ \text{component as } D(x_{i_0}, 10\tau)^+ \text{ of } B(x_i, 10\tau) \cap \Omega_2. \end{array}$$

Next we define a convex domains D_x^+, as being the convex hull of all the $D^+(x_i, 10\tau)$ for $i \in I_x$. Since $(\partial\Omega_1 \cup \partial\Omega_2) \cap B(x_{i_0}, 20\tau)$ is contained in $\{y; \, d(y, P_0) \le \frac{2\tau}{10}\}$ for some hyperplane P_0, we have that D_x^+ does not meet $(\partial\Omega_1 \cup \partial\Omega_2) \cap B(x_{i_0}, 20\tau)$ and is contained in $B(x_{i_0}, 100\tau)$.

Define

$$m_x^+ := \frac{1}{|D_x^+|} \int_{D_x^+} v(x)\, dx, \qquad \text{and}$$

$$m_{i_0} := \frac{1}{|D(x_{i_0}, 20\tau)^+|} \int_{D(x_{i_0}, 20\tau)^+} v(x)\, dx.$$

Since $\nabla\theta_0 + \sum_{i\in I} \nabla\theta_i = 0$ we have, using our assumption (1.18),

$$\nabla\tilde{v}(x) = \theta_0(x)\nabla v(x) + \nabla\theta_0(x)v(x) + \sum_{i\in I_c\cup I_{b_2}} \nabla\theta_i m_i + \sum_{i\in I_{b_1}} \nabla\theta_i m_i^+ 1_{A_i^+}(x)$$

$$= \theta_0(x)\nabla v(x) + \underbrace{\nabla\theta_0(x)(v(x) - m_{i_0})}_{f_1}$$

$$+ \underbrace{\sum_{i\in I_c\cup I_{b_2}} \nabla\theta_i(m_i - m_{i_0}) + \sum_{i\in I_{b_1}} \nabla\theta_i(m_i^+ - m_{i_0})}_{f_2}.$$

To estimate f_1, since $\mathrm{supp}(\nabla\theta_0) \cap B(x_{i_0}, 60\tau)$ is contained in a Lipschitz domain with universal constant, we infer that

$$(1.19) \quad \int_{B(x_{i_0}, 10\tau)} |f_1(x)|^p \leq \frac{1}{\tau^p} \int_{B(x_{i_0}, 60\tau)} 1_{supp(\nabla\theta_0(x))}|v(x) - m_{i_0}|^p$$

$$\leq C \int_{B(x_{i_0}, 60\tau)} |\nabla v|^p.$$

Now we estimate f_2 still using the Poincaré inequality. Firstly for any $i \in I_x \cap I_{b_1}$ we have that

$$(1.20) \quad |m_i^+ - m_{i_0}| \leq |m_i^+ - m_x^+| + |m_x^+ - m_{i_0}| \leq \frac{C}{\tau^{N-1}} \int_{D_x^+} |\nabla v|.$$

If $i \in I_c$, we have the same inequality with m_i instead of m_i^+, because by definition $m_i = m^+(v)$ in this case.

Therefore we have proved that under assumption (1.18),

$$(1.21) \quad |f_2(x)| \leq C\frac{1}{\tau^N} \int_{B(x_{i_0}, 100\tau)} |\nabla v|\, dx.$$

Now assume that instead of (1.18) we had:

(1.22)   For any $i \in (I_{b_1} \cup I_{b_2}) \cap I_x$, x lie in the same connected component as $D(x_{i_0}, 10\tau)^-$ of $B(x_i, 10\tau) \cap \Omega_2$.

Then we can proceed exactly as above with negative signs in the notations instead of positive signs, except for inequality (1.20) with $i \in I_c$ since by convention in this case we have set $m_i := m^+(v)$. However, since the boundary of $\partial\Omega_1$ has a uniform size of holes, we can estimate the difference $|m^+(v) - m^-(v)|$ by (1.11). Indeed, we know that $D_{\Omega_1}^{\pm}(x_i, 10\tau)$ lie in the same connected component of $B(x_i, 10\tau) \cap \Omega_2$. This means that there is a ball $B(y_i, s)$ centered on $P(x_i, 10\tau) \cap B(x_i, 20\tau)$ with radius $s > C_0^{-1} 20\tau$, such that $B(y_i, s) \cap \partial\Omega_1 = \emptyset$. But since $d_H(\Omega_1^c, \Omega_2^c) < 10^{-3} C_0^{-1} \tau$, we deduce that $B(y_i, \frac{s}{2}) \cap \partial\Omega_2 = \emptyset$ thus $D_{\Omega_2}^{\pm}(x_i', 20\tau)$ (the one associated to Ω_2 for a $x_i' \in \partial\Omega_2$ satisfying $\mathrm{dist}(x_i, x_i') \leq 10^{-3} C_0^{-1} \tau$), lie also in the same connected components of $B(x_i', 20\tau) \cap \Omega_2$. Then Proposition 8, together with an other application of the Poincaré inequality to estimate the difference between the average on $D_{\Omega_2}^{\pm}(x_i', 20\tau)$ and the average on $D_{\Omega_1}^{\pm}(x_i, 10\tau)$, gives

$$|m_i - m_x^-| \leq |m_i - m^-(v)| + |m^-(v) - m_x^-| \leq \frac{C}{\tau^{N-1}} \int_{B(x_i, 60\tau)} |\nabla u|$$

where C is now depending on C_0.

Therefore, in any cases we have proved (1.23). Subsequently,

$$(1.23) \qquad \|f_2(x)\|_{L^p(B(x_{i_0}, 10\tau))} \leq \|\nabla v\|_{L^p(B(x_{i_0}, 100\tau))}.$$

In conclusion we have obtained,

$$\|\nabla \tilde{v}(x)\|_{L^p(\Omega_2 \cap 10 B_{i_0})} \leq C \|\nabla v\|_{L^p(\Omega_1 \cup 100 B_{i_0})},$$

and (1.13) follows using the fact that $\{B_i\}$ forms a locally finite covering of $W(10\tau)$. $\qquad\square$

1.3. Mosco-convergence.

For every open set $\Omega \subset \mathbb{R}^N$ we define the closed linear subspace X_Ω of $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N, \mathbb{R}^N)$ by

$$(1.24) \qquad X_\Omega := \{(u 1_\Omega, \nabla u 1_\Omega); \ u \in W^{1,p}(\Omega)\}.$$

Definition 10 (Mosco-convergence)**.** Let Ω_k and Ω be open subsets of $\mathbb{R}^N$ and let X_{Ω_k} and X_Ω be the corresponding subspaces of $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N, \mathbb{R}^N)$ defined by (1.24). We say that X_{Ω_k} converges to X_Ω in the sense of Mosco if the following two properties hold:

(M1) for every $u \in W^{1,p}(\Omega)$, there exists a sequence $u_k \in W^{1,p}(\Omega_k)$ such that $u_k 1_{\Omega_k}$ converges to $u 1_\Omega$ strongly in $L^p(\mathbb{R}^N)$ and $\nabla u_k 1_{\Omega_k}$ converges to $\nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^N, \mathbb{R}^N)$;

$(M2)$ if h_k is a sequence of indices converging to ∞, u_k is a sequence such that $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, and $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^N)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^N, \mathbb{R}^N)$ to a function ψ, then there exists $u \in W^{1,p}(\Omega)$ such that $\phi = u 1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^N$.

The Mosco convergence is a great tool to study stability for Neumann problems. In particular, we have the following result coming from [**3**, Theorem 2.3], which is stated in $\mathbb{R}^2$ in [**3**] but can be extended in $\mathbb{R}^N$ with the same proof.

Theorem 11 ([**3**])**.** *Let Ω_k and Ω be open subsets of $\mathbb{R}^N$. Then Ω is stable for the problems (0.2) along the sequence Ω_k, if and only if X_{Ω_k} converges to X_Ω in the sense of Mosco.*

According to Theorem 11, Theorem 7 will be a consequence of the following result.

Theorem 12. *Let $r_0 > 0$, $C_0 \in [1/2, +\infty) \cup \{1/3\}$ and let $\{\Omega_k\}_{k \in \mathbb{N}}$ and Ω be (δ, r_0)-Reifenberg flat domains with $\delta < 10^{-3} C_0^{-1}$ and having a uniform size of holes with same constant C_0. Assume that Ω_k converges to Ω for the complementary Hausdorff distance. Then X_{Ω_k} converges to X_Ω in the sense of Mosco.*

Proof: Let $u \in W^{1,p}(\Omega)$ and assume without loss of generality that $\tau_k := d_H(\Omega_k^c, \Omega^c) < C_0^{-1} 10^{-4} r_0$. Set $W(t) := \{x; d(x, \partial\Omega) \leq t\}$ and let $\tilde{u}_k$ be the extension function given in Lemma 9 for $\Omega_1 := \Omega$, $\Omega_2 := \Omega_k$, $v := u$, $\tau := C_0 10^3 \tau_k$. Since $\tilde{u}_k \in W^{1,p}(\Omega_k)$, all we have to prove is that $(\tilde{u}_k 1_{\Omega_k}, \nabla \tilde{u}_k 1_{\Omega_k})$ converges strongly to $(u 1_\Omega, \nabla u 1_\Omega)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N, \mathbb{R}^N)$. Since $u = \tilde{u}_k$ in $\Omega \setminus W(10\tau_k)$ we have that

$$\left(\int_{\mathbb{R}^N} |\tilde{u} 1_{\Omega_k} - u 1_\Omega|^p \, dx \right)^{\frac{1}{p}} = \left(\int_{W(10\tau_k)} |\tilde{u} 1_{\Omega_k} - u 1_\Omega|^p \, dx \right)^{\frac{1}{p}}$$

$$\leq \left(\int_{W(10\tau_k)} |\tilde{u} 1_{\Omega_k}|^p \, dx \right)^{\frac{1}{p}} + \left(\int_{W(10\tau_k)} |u 1_\Omega|^p \, dx \right)^{\frac{1}{p}}$$

$$\leq C \|u\|_{L^p(W(10\tau_k))}$$

which tends to zero when k tends to $+\infty$ because $\partial\Omega$ is closed and $|\partial\Omega| = 0$. For the gradients, a similar argument can be done, using (1.13). That

is,

$$\left(\int_{\mathbb{R}^N} |\nabla \tilde{u} 1_{\Omega_k} - \nabla u 1_\Omega|^p \, \mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_{W(10\tau_k)} |\nabla \tilde{u} 1_{\Omega_k} - \nabla u 1_\Omega|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{W(10\tau_k)} |\nabla \tilde{u} 1_{\Omega_k}|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$$

$$+ \left(\int_{W(10\tau_k)} |\nabla u 1_\Omega|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$$

$$\leq C \|\nabla u\|_{L^p(W(60\tau_k))}$$

which tends to 0, thus $(M1)$ is proved.

Let us now prove $(M2)$. Let $\varphi \in C^\infty(\mathbb{R}^N)$ be compactly supported in Ω. Then we know by the weak convergence that

(1.25)
$$\begin{cases} \int_\Omega u_k 1_{\Omega_{h_k}} \varphi \, \mathrm{d}x \xrightarrow{k \to +\infty} \int_\Omega \phi \varphi \, \mathrm{d}x \\ \int_\Omega \langle \nabla u_k 1_{\Omega_{h_k}}, \varphi \rangle \, \mathrm{d}x \xrightarrow{k \to +\infty} \int_\Omega \psi \varphi \, \mathrm{d}x. \end{cases}$$

On the other hand, since Ω_{h_k} converges to Ω for the complementary Hausdorff distance, for k large enough $1_{\Omega_{h_k}}$ is equal to 1 everywhere on the support of ϕ. Thus (1.25) shows that u_k converges to ϕ in $\mathcal{D}'(\Omega)$ and ∇u_k converges to ψ in $\mathcal{D}'(\Omega)$. By uniqueness of the limit in $\mathcal{D}'(\Omega)$ we conclude that

(1.26)
$$\psi = \nabla \phi \quad \text{in} \quad \mathcal{D}'(\Omega).$$

Moreover since $\psi \in L^p(\mathbb{R}^N, \mathbb{R}^N)$, we deduce that (1.26) in true in $L^p(\Omega, \mathbb{R}^N)$ thus $\psi = \nabla \phi$ a.e. in Ω and therefore $\phi|_\Omega \in W^{1,p}(\Omega)$. To conclude, all we have to show is that $\varphi = \psi = 0$ in Ω^c. To see this, we use a similar argument as above by defining a function φ compactly supported in Ω^c. By the weak convergence, and because Ω_k converges to Ω for the complementary Hausdorff distance, we deduce that $\int_\Omega \phi \varphi \, \mathrm{d}x = 0$. This holds for any function φ compactly supported in Ω^c. Since $\phi \in L^p(\mathbb{R}^N)$ we conclude that $\phi = 0$ in Ω^c. In a similar way we obtain that $\psi = 0$ in Ω^c and since the Lebesgue measure of $\partial\Omega$ is zero, the proof is complete. $\qquad \square$

2. Stability for problems with mixed boundary conditions

In this last section we state some similar stability result for the problem with mixed boundary conditions, directly coming from the proof of Theorem 6.3 in [**3**]. The main motivation is that those statements could be useful to obtain estimates of solutions for elliptic partial differential equations in Reifenberg-flat domains using a perturbation argument.

Let $A \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary ∂A, and let $\partial_D A$ be a relatively open subset of ∂A. For every compact set $K \subset \bar{A}$, for every $g \in W^{1,p}(A)$, and for every pair of function a and b satisfying the properties (1.1)–(1.3) we consider the solutions u of the mixed problem

$$(2.1) \quad \begin{cases} -\operatorname{div} a(x, \nabla u) + b(x, u) = 0 & \text{in } A \backslash K, \\ u = g & \text{on } \partial_D A \backslash K, \\ a(x, \nabla u) \cdot \nu = 0 & \text{on } \partial(A \backslash K) \backslash (\partial_D A \backslash K). \end{cases}$$

Definition 13. We say that the problem (2.1) is stable along the sequence (K_k, g_k), if the following holds: Let $u_k \in W^{1,p}(A \backslash K)$ be a sequence of solutions of the problem (2.1) corresponding to K_k and g_k. Then $(u_k 1_{A \backslash K_k}, \nabla u_k 1_{A \backslash K_k})$ converges strongly to $(v 1_{A \backslash K}, \nabla v 1_{A \backslash K})$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N, \mathbb{R}^N)$ and v is a solution of problem (2.1).

For all $g \in W^{1,p}(A)$ we denote

$$W_g^{1,p}(A \backslash K, \partial_D A \backslash K) := \{u \in W^{1,p}(A \backslash K) : u = g \text{ on } \partial_D \backslash K\}.$$

As in (1.24) we define the closed linear subspace $X_K^g(A)$ of $L^p(A) \times L^p(A, \mathbb{R}^N)$ by

$$(2.2) \quad X_K^g(A) := \{(u 1_{K^c}, \nabla u 1_{K^c}); \ u \in W_g^{1,p}(A \backslash K, \partial_D A \backslash K)\}.$$

Definition 14 (Mosco-convergence)**.** We say that $X_{K_k}^{g_k}(A)$ converges to $X_K^g(A)$ in the sense of Mosco if the following two properties hold:

($M1'$) for every $u \in W_g^{1,p}(A \backslash K, \partial_D A \backslash K)$, there exists a sequence $u_k \in W_{g_k}^{1,p}(A \backslash K_k, \partial_D A \backslash K_k)$ such that $u_k 1_{K_k^c}$ converges to $u 1_{K^c}$ strongly in $L^p(A)$ and $\nabla u_k 1_{K_k^c}$ converges to $\nabla u 1_{K^c}$ strongly in $L^p(A, \mathbb{R}^N)$;

($M2'$) if h_k is a sequence of indices converging to ∞, u_k is a sequence such that $u_k \in W_{g_k}^{1,p}(A \backslash K_k)$ for every k, and $u_k 1_{K_k^c}$ converges weakly in $L^p(A)$ to a function ϕ, while $\nabla u_k 1_{K_k^c}$ converges weakly in $L^p(A, \mathbb{R}^N)$ to a function ψ, then there exists $u \in W_g^{1,p}(A \backslash K, \partial_D A \backslash K)$ such that $\phi = u 1_{K^c}$ and $\psi = \nabla u 1_{K^c}$ a.e. in $\mathbb{R}^N$.

Remark 15. As it is pointed out in [**3**], by adopting the same proof as in Theorem 11, the Mosco convergence of $X_{K_k}^{g_k}(A)$ to $X_K^g(A)$ is equivalent to the stability of the mixed problem (2.1) along the sequence (K_k, g_k).

Finally we obtain the corresponding result in the case of the mixed boundary conditions.

Theorem 16. *Let A be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary ∂A, let $\partial_D A$ be a relatively open subset of ∂A and let C_0 be a positive constant. Let g_h be a sequence in $W^{1,p}(A)$ converging strongly to a function $g \in W^{1,p}(A)$, and let K_h be a sequence of compact subsets of $\bar{A}$ converging to a set K in the Hausdorff metric. Assume in addition that for every h, $A \backslash K_h$ is a (δ, r_0)-Reifenberg flat domain with $\delta < 10^{-3} C_0^{-1}$ and having a uniform size of holes with constant C_0. Then $X_{K_h}^{g_h}(A)$ converges to $X_K^g(A)$ in the sense of Mosco.*

Proof: The proof is essentially the same as the proof of Theorem 6.3 in [**3**], consisting of an idea due to Chambolle. The only difference is that the sets

$$\Omega_h := \Sigma \setminus (K_h \cup (\partial A \setminus \partial_D A)) \quad \text{and} \quad \Omega := \Sigma \setminus (K \cup (\partial A \setminus \partial_D A))$$

where Σ is an open ball in $\mathbb{R}^N$ such that $\bar{A} \subset \Sigma$, are now satisfying the assumptions of Theorem 12 and thus the Mosco convergence in that setting. We refer the reader to [**3**, Theorem 6.3], for the details in that approach. $\square$

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References

[1] D. Bucur and N. Varchon, Boundary variation for a Neumann problem, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29(4)** (2000), 807–821.

[2] A. CHAMBOLLE AND F. DOVERI, Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets, *Comm. Partial Differential Equations* **22(5–6)** (1997), 811–840.

[3] G. DAL MASO, F. EBOBISSE, AND M. PONSIGLIONE, A stability result for nonlinear Neumann problems under boundary variations, *J. Math. Pures Appl. (9)* **82(5)** (2003), 503–532.

[4] G. DAVID, Hölder regularity of two-dimensional almost-minimal sets in $\mathbb{R}^n$, *Ann. Fac. Sci. Toulouse Math. (6)* **18(1)** (2009), 65–246.

[5] G. DAVID, T. DE PAUW, AND T. TORO, A generalization of Reifenberg's theorem in $\mathbb{R}^3$, *Geom. Funct. Anal.* **18(4)** (2008), 1168–1235.

[6] G. DAVID AND T. TORO, Reifenberg flat metric spaces, snowballs, and embeddings, *Math. Ann.* **315(4)** (1999), 641–710.

[7] T. DEL VECCHIO, The thick Neumann's sieve, *Ann. Mat. Pura Appl. (4)* **147** (1987), 363–402.

[8] A. GIACOMINI, A stability result for Neumann problems in dimension $N \geq 3$, *J. Convex Anal.* **11(1)** (2004), 41–58.

[9] P. W. JONES, Quasiconformal mappings and extendability of functions in Sobolev spaces, *Acta Math.* **147(1–2)** (1981), 71–88.

[10] A. LEMENANT, Sur la régularité des minimiseurs de Mumford-Shah en dimension 3 et supérieure, Thesis University Paris-Sud XI Orsay (2008).

[11] A. LEMENANT, Energy improvement for energy minimizing functions in the complement of generalized Reifenberg-flat sets, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **9(2)** (2010), 351–384.

[12] A. LEMENANT AND E. MILAKIS, Quantitative stability for the first Dirichlet eigenvalue in Reifenberg flat domains in $\mathbb{R}^N$, *J. Math. Anal. Appl.* **364(2)** (2010), 522–533.

[13] J.-L. LIONS, *"Quelques méthodes de résolution des problèmes aux limites non linéaires"*, Dunod, Gauthier-Villars, Paris, 1969.

[14] E. MILAKIS AND T. TORO, Divergence form operators in Reifenberg flat domains, *Math. Z.* **264(1)** (2010), 15–41.

[15] F. MURAT, The Neumann sieve, in: *"Nonlinear variational problems"* (Isola d'Elba, 1983), Res. Notes in Math. **127**, Pitman, Boston, MA, 1985, pp. 24–32.

[16] E. R. REIFENBERG, Solution of the Plateau Problem for m-dimensional surfaces of varying topological type, *Acta Math.* **104** (1960), 1–92.

[17] E. Sánchez-Palencia, Boundary value problems in domains containing perforated walls, in: *"Nonlinear partial differential equations and their applications"*, Collège de France Seminar, Vol. III (Paris, 1980/1981), Res. Notes in Math. **70**, Pitman, Boston, Mass.-London, 1982, pp. 309–325.

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