Publ. Mat. **60** (2016), 113–170
DOI: 10.5565/PUBLMAT_60116_05

FINE GRADINGS ON $\mathfrak{e}_6$

Cristina Draper* and Antonio Viruel†

Abstract: There are fourteen fine gradings on the exceptional Lie algebra $\mathfrak{e}_6$ over an algebraically closed field of zero characteristic. We provide their descriptions and a proof that any fine grading is equivalent to one of them.

2010 Mathematics Subject Classification: 17B25, 17B70.

Key words: Graded algebra, exceptional Lie algebra, maximal abelian diagonalizable group.

1. Introduction

The concept of grading on algebraic structures has a long tradition in Mathematics, appearing in the classics as [**11**, Chapter III, §3]. Gradings are the natural ambient for algebraic structures arising from geometrical situations, so all the algebraic structures considered in Algebraic Topology are graded. Gradings of Lie algebras have its source in the original work of Jordan [**33**] towards a mathematical description of Quantum Mechanics. It is remarkable the role of the gradings of Lie algebras in mathematical physics [**35**] and particularly in particle physics [**13**]. Moreover, gradings are in the background of any ordered choice of basis, as recalled in [**39**].

Motivated by these manifold applications of gradings on Lie algebras, Patera et al. initiated in [**39**] a systematic classification of all the possible group gradings on finite-dimensional Lie algebras. For algebraically closed fields of characteristic zero, the gradings on the classical Lie algebras have been studied in [**6, 7, 9, 20, 23**] and the gradings in some exceptional ones, namely, $\mathfrak{f}_4$ and $\mathfrak{g}_2$, in [**18, 19, 15, 8, 24**]. Lately, some authors have already studied the case of prime characteristic, [**6**] in the classical case (with the exception of $\mathfrak{d}_4$) and [**24**] in $\mathfrak{f}_4$ and $\mathfrak{g}_2$. Over arbitrary fields, almost nothing is known: for example, over the real field the gradings on simple Lie algebras have been studied only for the exceptional algebras $\mathfrak{f}_4$ and $\mathfrak{g}_2$ [**12**] and in classical ones for some examples in low-dimensions [**30**].

*Partially supported by MINECO grant MTM2010-15223, MTM2013-41208-P, and by the Junta de Andalucía PAI projects FQM-336, FQM-2467, and FQM-3737.
†Partially supported by the MINECO grants MTM2010-18089, MTM2013-41768-P, and by the Junta de Andalucía PAI projects FQM-2863 and FQM-213.

Summarizing, the classification of fine group gradings on finite-dimensional simple Lie algebras over an algebraically closed field is almost complete, remaining just the E-family to be fully studied, although some examples of gradings on $\mathfrak{e}_6$ have been described (for instance, in [**22**]). This work intends to be a first step towards filling that gap by giving a complete description of the fine group gradings of the smallest representative of Lie algebras of the "E-series":

Theorem 1. *There are* 14 *fine group gradings on* $\mathfrak{e}_6$ *up to equivalence. The following table describes them in terms of the associated MAD-group* Q *in* $\operatorname{Aut}\mathfrak{e}_6$*, the type of the fine grading, and the dimension of the fixed subalgebra* $\mathfrak{fix}(Q) = L_e$*.*

Quasitorus	*Isomorphic to*	*Type*	$\dim L_e$
$\mathcal{Q}_1$	$\mathbb{Z}_3^4$	$(72,0,2)$	0
$\mathcal{Q}_2$	$(\mathbb{F}^*)^2 \times \mathbb{Z}_3^2$	$(60,9)$	2
$\mathcal{Q}_3$	$\mathbb{Z}_3^2 \times \mathbb{Z}_2^3$	$(64,7)$	0
$\mathcal{Q}_4$	$(\mathbb{F}^*)^2 \times \mathbb{Z}_2^3$	$(48,1,0,7)$	2
$\mathcal{Q}_5$	$(\mathbb{F}^*)^6$	$(72,0,0,0,0,1)$	6
$\mathcal{Q}_6$	$(\mathbb{F}^*)^4 \times \mathbb{Z}_2$	$(72,1,0,1)$	4
$\mathcal{Q}_7$	$\mathbb{Z}_2^6$	$(48,1,0,7)$	0
$\mathcal{Q}_8$	$\mathbb{F}^* \times \mathbb{Z}_2^4$	$(57,0,7)$	1
$\mathcal{Q}_9$	$\mathbb{Z}_3^3 \times \mathbb{Z}_2$	$(26,26)$	0
$\mathcal{Q}_{10}$	$(\mathbb{F}^*)^2 \times \mathbb{Z}_2^3$	$(60,7,0,1)$	2
$\mathcal{Q}_{11}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^4$	$(48,13,0,1)$	0
$\mathcal{Q}_{12}$	$\mathbb{F}^* \times \mathbb{Z}_2^5$	$(73,0,0,0,1)$	1
$\mathcal{Q}_{13}$	$\mathbb{Z}_2^7$	$(72,0,0,0,0,1)$	0
$\mathcal{Q}_{14}$	$\mathbb{Z}_4^3$	$(48,15)$	0

The first five MAD-groups contain no outer automorphisms. Note that the type of the fine grading (how many pieces are of each size) jointly with the dimension of the fixed subalgebra are enough to distinguish the conjugacy class of the quasitorus inducing such grading.

Due to the dimension of $\mathfrak{e}_6$, the techniques previously used for the other Lie algebras seem to be inefficient in this case. The classical Lie algebras were studied by taking into consideration the gradings on the associative matrix algebras in which they live. The gradings on $\mathfrak{g}_2$ are induced from the gradings on the octonions, described in [**21**]. The gradings on $\mathfrak{f}_4$ have been submitted to several attacks. The first of them uses this tool: the (fine) grading is produced by a maximal diagonalizable subgroup of the automorphism group of the algebra, and this subgroup lives in the normalizer of a maximal torus. In particular, if the group is a maximal torus, we obtain the root decomposition. In [**19**], all the elements in this normalizer are constructed (as matrices of size 52) and then a detailed analysis of the possible cases is realized. It turns out very difficult to apply this technique in this case, which would require more than one hundred thousand square matrices of size 78. What has been done is to work, not in the normalizer, but in its quotient by the torus, isomorphic to the group of automorphisms of the root system (the extended Weyl group). This allows to work with matrices of size 6, which can be implemented in any computer. Even more, we use the computer only in two proofs throughout the paper. The remaining computations, although long, are made by hand, and the representatives or the orbits of the Weyl group are extracted from [**2**].

Another technique used in this work is to take advantage of the knowledge of the elementary p-groups and their centralizers in the complex case. Thus, there is an effort to use these results for algebraically closed fields of characteristic zero. This saves some computational work. Some basic facts about the structure of the MAD-groups are studied in order to make use of such elementary groups, although most of the useful information is extracted and generalized from [**19, 20, 15**].

About the results, we would like to mention the great amount of fine gradings on $\mathfrak{e}_6$. If we take into account that on $\mathfrak{g}_2$ there are 2 fine gradings, and on $\mathfrak{f}_4$ there are 4, we find that the number in this case is much bigger, proportionally speaking. Of course the reason is a greater symmetry in $\mathfrak{e}_6$ than in the two previously mentioned cases: the presence of outer automorphisms is a signal of such symmetry. Besides we find some remarkable gradings: It was thought that every outer fine grading on a simple Lie algebra of type A was induced by a quasitorus of automorphisms containing an outer order two automorphism. Elduque proved that this is not true in [**23**], in which he gave a revised version of the classification of fine gradings on the simple classical Lie algebras with new arguments (and some new fine grading). The same fact occurs in the case of the gradings on $\mathfrak{e}_6$. There is an outer fine grading such that the MAD-group producing the grading does not contain outer automorphisms of order 2. This is an unexpected $\mathbb{Z}_4^3$-grading and it turns out a nice symmetry based on the number 4. Over other finite groups, a $\mathbb{Z}_2^7$-grading and a $\mathbb{Z}_3^4$-grading appear.

The structure of the work is as follows. In Section 2, we recall some generalities about gradings and we study the quasitori of the automorphism group producing the gradings. We delve into the structure of a

MAD-group, that is, the maximal abelian diagonalizable group which is producing a fine grading. Besides, in this section about generalities, we recall the elementary p-groups of $\operatorname{Int}\mathfrak{e}_6^{\mathbb{C}}$ and try to translate the result to a more general field. All this will be used in Section 3, after describing some examples of fine gradings on $\mathfrak{e}_6$, to prove that they are all the possible cases of fine gradings produced by inner automorphisms. This only needs a technical result, the fact that every MAD-group contains a minimal elementary non-toral p-group (even more, of minimum rank). This is proved in Section 4, in which the Weyl group is recalled, its conjugacy classes exhibited, and the result is obtained by means of a technical play, developed with the help of a computer in some points, in which the representatives of the orbits in this Weyl group and the elements in a maximal torus fixed by them have an important role. Afterwards, in Section 5 we describe all the possible fine gradings, up to equivalence, produced by a group of automorphisms that are not all inner. We consider the gradings obtained by extending gradings on $\mathfrak{f}_4$ and the ones obtained by extending gradings on $\mathfrak{c}_4$, and, afterwards, we show an example of (outer) grading which is not in any of the two previous situations. Finally Section 6 is devoted to proving that there are no more fine gradings than seen before. Again we make use of a technical result, proved with a computer. It also contains the descriptions of the MAD-groups in computational terms in Equation (10).

At the end of the paper we have added an appendix with natural automorphisms of $\mathfrak{e}_6$ whose projections on the extended Weyl group are representatives of the conjugacy classes used through the work.

2. Generalities

2.1. Notions about gradings. If L is a finite-dimensional Lie algebra and G is an abelian group, we will say that a decomposition $\Gamma : L = \oplus_{g\in G} L_g$ is a *G-grading* whenever for all $g, h \in G$, $L_g L_h \subset L_{g+h}$ and G is generated by the set $\operatorname{Supp}(\Gamma) := \{g \in G : L_g \neq 0\}$, called the *support* of the grading. The subspaces L_g are referred to as the *homogeneous components* of the grading.

Given two gradings $\Gamma : L = \oplus_{g\in G} L_g$ and $\Gamma' : L = \oplus_{g'\in G'} L_{g'}$ over two abelian groups G and G', Γ is said to be a *refinement* of Γ' (or Γ' a *coarsening* of Γ) if for any $g \in G$ there is $g' \in G'$ such that $L_g \subseteq L_{g'}$. That is, any homogeneous component of Γ' is the direct sum of several homogeneous components of Γ. The refinement is *proper* if there are $g \in G$ and $g' \in G'$ such that $L_g \subsetneq L_{g'}$. A grading is said to be *fine* if it admits no proper refinements.

The gradings Γ and Γ' are said to be *equivalent* if the sets of homogeneous subspaces are the same up to isomorphism, that is, if there are an automorphism $f \in \operatorname{Aut} L$ and a bijection between the supports $\alpha\colon \operatorname{Supp}(\Gamma) \to \operatorname{Supp}(\Gamma')$ such that $f(L_g) = L_{\alpha(g)}$ for any $g \in \operatorname{Supp}(\Gamma)$. The *type* of a grading Γ (following [**31**]) is the sequence of numbers $(h_1, \dots, h_r)$ where h_i is the number of homogeneous components of dimension i, $i = 1, \dots, r$, with $h_r \neq 0$. Thus $\dim L = \sum_{i=1}^{r} ih_i$. This sequence is of course an invariant up to equivalence. Our objective is to classify fine gradings up to equivalence, because any grading is obtained as a coarsening of some fine grading.

Given a grading $\Gamma : L = \oplus_{g\in G} L_g$, consider the free abelian group $\widetilde{G}$ generated by $\operatorname{Supp}(\Gamma)$ and subject to the relations $g_1 + g_2 = g_3$ if $0 \neq [L_{g_1}, L_{g_2}] \subset L_{g_3}$. The group $\widetilde{G}$ is called the *universal grading group* of Γ. Note that we have a $\widetilde{G}$-grading $\widetilde{\Gamma} : L = \oplus_{\widetilde{g}\in\widetilde{G}} L_{\widetilde{g}}$ equivalent to Γ, where $L_{\widetilde{g}}$ is the sum of the homogeneous spaces L_g of Γ such that the class of g in $\widetilde{G}$ is $\widetilde{g}$. Besides $\widetilde{G}$ has the following universal property: given any coarsening $L = \oplus_{h\in H} L_h$ of $\widetilde{\Gamma}$, there exists a unique group epimorphism $\alpha\colon \widetilde{G} \to H$ such that $L_h = \oplus_{\widetilde{g}\in\alpha^{-1}(h)} L_{\widetilde{g}}$.

The ground field $\mathbb{F}$ will be supposed to be algebraically closed and of characteristic zero throughout this work. In this context, the group of automorphisms of the algebra L is an algebraic linear group. There is a deep connection between gradings on L and quasitori of the group of automorphisms $\operatorname{Aut} L$, according to [**38**, §3, p. 104]. If $L = \oplus_{g\in G} L_g$ is a G-grading, the map $\psi\colon \mathfrak{X}(G) = \operatorname{Hom}(G, \mathbb{F}^\times) \to \operatorname{Aut} L$ mapping each $\alpha \in \mathfrak{X}(G)$ to the automorphism $\psi_\alpha\colon L \to L$ given by $L_g \ni x \mapsto \psi_\alpha(x) := \alpha(g)x$ is a group homomorphism. Since G is finitely generated, then $\psi(\mathfrak{X}(G))$ is an algebraic quasitorus. And conversely, if Q is a quasitorus and $\psi\colon Q \to \operatorname{Aut} L$ is a homomorphism, $\psi(Q)$ is formed by semisimple automorphisms and we have a $\mathfrak{X}(Q)$-grading $L = \oplus_{g\in\mathfrak{X}(Q)} L_g$ given by $L_g = \{x \in L\colon \psi(q)(x) = g(q)x \;\forall\, q \in Q\}$, with $\mathfrak{X}(Q)$ a finitely generated abelian group.

If $\Gamma : L = \oplus_{g\in G} L_g$ is a G-grading, then the set of automorphisms of L such that every L_g is contained in some eigenspace is an abelian group formed by semisimple automorphisms, called the *diagonal group* of the grading, and denoted by $\operatorname{Diag}(\Gamma)$. If $\psi\colon \mathfrak{X}(G) \to \operatorname{Aut} L$ is the related homomorphism, then the group $\operatorname{Diag}(\Gamma)$ contains $\psi(\mathfrak{X}(G))$, and both groups coincide when G is the universal grading group of Γ.

The grading is fine if and only if $\operatorname{Diag}(\Gamma)$ is a maximal abelian subgroup of semisimple elements, usually called a *MAD-group* ("maximal abelian diagonalizable" group). It is convenient to observe that the number of conjugacy classes of MAD-groups in $\operatorname{Aut} L$ equals the number of equivalence classes of fine gradings on L, and that if Q is a MAD-group, then $\mathfrak{X}(Q)$ is the universal group of the induced fine grading.

A grading is *toral* if it is a coarsening of the root decomposition of a semisimple Lie algebra L relative to some Cartan subalgebra. In other words, if the grading is produced by a quasitorus contained in a torus of the automorphism group of L. If L_e denotes the identity component of a grading on L, such grading is toral if and only if L_e contains a Cartan subalgebra of L [**18**, Subsection 2.4]. As L_e in any case is a reductive subalgebra [**31**, Remark 3.5], the grading is toral if and only if the rank of L_e coincides with the rank of L.

If Q is a MAD-group of $\operatorname{Aut} L$, then the homogeneous component L_e, that is, the subalgebra fixed by Q, is an abelian subalgebra whose dimension coincides with the dimension of Q (by definition, the dimension of the maximal torus contained in Q) according to [**19**, Corollary 5]. Moreover, for any quasitorus Q of $\operatorname{Aut} L$, the dimension of L_e is, at least, the dimension of Q. Indeed, take Q' a MAD-group containing Q, then L_e contains $\mathfrak{fix}(Q')$ and $\mathfrak{fix}(Q')$ is a subalgebra of dimension equal to $\dim Q'$, as above. Hence $\dim L_e \geq \dim \mathfrak{fix}(Q') \geq \dim Q$. In particular, if $L_e = 0$ (if the grading is *special*, by using terminology of [**31**]), the grading is produced by a finite quasitorus. The converse is true, in the case of a fine grading, again by [**19**, Corollary 5]: if the MAD-group producing the grading (the diagonal group) is finite, then the grading is special. This kind of gradings has an extra-property:

Lemma 1. *Every homogeneous element in a fine grading on a simple Lie algebra produced by a finite quasitorus is semisimple.*

Proof: If $L = \oplus_g L_g$ is a grading, for any nonzero element $x \in L_g$, there is $x_s \in L_g$ semisimple and $x_n \in L_g$ nilpotent such that $x = x_s + x_n$ and $[x_s, x_n] = 0$ according to [**38**, Theorem 3.3] (its semisimple and nilpotent parts also belong to L_g). Hence either every homogeneous element is semisimple or there exists a homogeneous nilpotent element. The latter case does not happen if the grading is in the conditions of the lemma, because if $x \in L_g$ is nilpotent, there are a semisimple element $h \in L_e$ and a nilpotent one $y \in L_{-g}$ such that $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$ by [**38**, Theorem 3.4], so that $L_e \neq 0$. □

The general fact is that every homogeneous element in a fine grading on a simple Lie algebra is either semisimple or nilpotent [**19**, Proposition 10], which implies that it is very easy to find bases formed by semisimple or nilpotent elements.

It is worth pointing out that in this case of a grading $L = \oplus_{g\in G} L_g$ on a simple Lie algebra L, if k is the Killing form of L (non-degenerate), then $k(L_g, L_h) = 0$ if $g, h \in G$ with $g+h \neq e$. Thus L_{-g} can be identified with L_g^*.

2.2. Some techniques for group gradings. We collect in this and the next subsections some key results from the structure of a MAD-group, extracted mainly from [**20, 19, 15**].

As in [**19**, Section 5], the Platonov's analogue of Borel–Serre Theorem tells us that every quasitorus of $\mathfrak{G} := \operatorname{Aut}\mathfrak{e}_6$ normalizes some of the maximal tori of $\operatorname{Aut}\mathfrak{e}_6$. More precisely, the quasitorus is the product of a torus T with a finite group, and we can take a maximal torus containing T such that the quasitorus is contained in its normalizer:

Lemma 2 ([**15**, Lemma 3])**.** *If H_1 is a toral subgroup of $\mathfrak{G}$ and H_2 is a diagonalizable subgroup of $\mathfrak{G}$ which commutes with H_1, then there is a maximal torus T of $\mathfrak{G}$ such that $H_1 \subset T$ and H_2 is contained in the normalizer $\mathfrak{N}(T)$.*

Assume we have fixed $\mathcal{T}$ a maximal torus of $\operatorname{Aut}\mathfrak{e}_6$. If $f \in \mathfrak{N}(\mathcal{T})$, then $f\mathcal{T}f^{-1} = \mathcal{T}$ and we define

$$(1)\qquad \begin{aligned} &\mathcal{T}^{\langle f\rangle} := \mathfrak{C}_{\mathfrak{G}}(f) \cap \mathcal{T} = \{t \in \mathcal{T} \mid ftf^{-1} = t\},\\ &Q(f) := \langle f, \mathcal{T}^{\langle f\rangle}\rangle, \end{aligned}$$

where we use the notation $\langle S_1, \dots, S_l\rangle$ for the quasitorus of $\operatorname{Aut}\mathfrak{e}_6$ generated by $S_1 \cup \dots \cup S_l \subset \operatorname{Aut}\mathfrak{e}_6$ (the closure, with the Zarisky topology, of the group generated by them). The quasitori of the form $Q(f)$ have proved to be relevant for the study of the MAD-groups in [**19**], since every MAD-group of $\operatorname{Aut}\mathfrak{f}_4$ is $Q(f)$ for certain $f \in \operatorname{Aut}\mathfrak{f}_4$. This also happens for the MAD-groups of $\operatorname{Aut}\mathfrak{d}_4$ containing outer automorphisms of order 3, although not for the MAD-groups containing outer automorphisms of order 2 (for more details about these MAD-groups, see [**20**]). Through this paper we will also find several MAD-groups in the set $\{Q(f) \mid f \in \operatorname{Aut}\mathfrak{e}_6\}$, namely, all the five MAD-groups contained in $\operatorname{Int}\mathfrak{e}_6$ (Section 4) and six of the MAD-groups not contained in $\operatorname{Int}\mathfrak{e}_6$ (Section 6).

Lemma 3. *If $f \in \mathfrak{N}(\mathcal{T})$ has order $r \in \mathbb{N}$, then the set $\mathcal{T}^{\langle f\rangle}$ is equal to $\mathcal{S}^{\langle f\rangle} \cdot \mathcal{H}^{\langle f\rangle}$ for the subtorus $\mathcal{S}^{\langle f\rangle} = \{(tf)^r \mid t \in \mathcal{T}\}$ of $\mathcal{T}$ and a (finite) subgroup $\mathcal{H}^{\langle f\rangle} \subset \{t \in \mathcal{T} \mid t^r = 1_{\mathfrak{G}}\}$ such that $\mathcal{S}^{\langle f\rangle} \cap \mathcal{H}^{\langle f\rangle} = \{1_{\mathfrak{G}}\}$.*

Proof: All is proved in [**15**, Lemma 6], except for the fact that the subtorus $S = \{(tf)^r \mid t \in \mathcal{T}\} \subset \mathcal{S}^{\langle f\rangle}$ fills the whole $\mathcal{S}^{\langle f\rangle}$. If $s \in \mathcal{S}^{\langle f\rangle}$, take $t \in \mathcal{S}^{\langle f\rangle}$ such that $t^r = s$ (a torus contains roots of all its elements). As $\mathcal{S}^{\langle f\rangle} \subset \mathcal{T}^{\langle f\rangle}$, then $tf = ft$, so that $s = t^r = (tf)^r \in S$. $\square$

Although $\mathcal{H}^{\langle f\rangle}$ is not determined by f and $\mathcal{T}$, we will use such notation for any group satisfying that $\mathcal{T}^{\langle f\rangle} = \mathcal{S}^{\langle f\rangle} \cdot \mathcal{H}^{\langle f\rangle}$ and $\mathcal{S}^{\langle f\rangle} \cap \mathcal{H}^{\langle f\rangle} = \{1_{\mathfrak{G}}\}$.

Remark 1. A way for identifying $Q(f)$ with $Q(g)$ for $f, g \in \mathfrak{N}(\mathcal{T})$ having the same projection on the Weyl group ($f\mathcal{T} = g\mathcal{T}$) was developed in [**19**]. Consider the group $\mathfrak{S}^{\langle f\rangle} := \{f^{-1}tft^{-1} \mid t \in \mathcal{T}\} \subset \mathcal{T}$. Denote by $\operatorname{Ad} F\colon \mathfrak{G} \to \mathfrak{G}$ the conjugation $\operatorname{Ad} F(h) = FhF^{-1}$ if $F \in \mathfrak{G}$. If $s = f^{-1}tft^{-1} \in \mathfrak{S}^{\langle f\rangle}$ ($t \in \mathcal{T}$), then $\operatorname{Ad} t(f) = fs$, so that $Q(f)$ is conjugate to $Q(fs)$, since $\operatorname{Ad} t$ does not move the torus $\mathcal{T}$. As obviously $Q(f) = Q(ft)$ if $t \in \mathcal{T}^{\langle f\rangle}$, then $Q(f)$ is conjugate to $Q(ft)$ for all $t \in \mathcal{T}^{\langle f\rangle}\mathfrak{S}^{\langle f\rangle}$. A sufficient condition (also necessary) for having $\mathcal{T}^{\langle f\rangle}\mathfrak{S}^{\langle f\rangle} = \mathcal{T}$ is that $\mathcal{T}^{\langle f\rangle} \cap \mathfrak{S}^{\langle f\rangle}$ is finite, following the arguments in the proof of [**19**, Proposition 6]. This condition is easy to check in practice.

Remark 2. If $f \in \mathfrak{N}(\mathcal{T})$, then $\mathcal{T}^{\langle f\rangle}$ is finite if and only if all the elements ft have the same order, independently of the element $t \in \mathcal{T}$. This result and its proof are almost the same than those ones in [**19**, Lemma 1], where we were working with the projections on the Weyl group.

2.3. Structure of a MAD-group of Aut $\mathfrak{e}_6$.

Lemma 4 ([**15**, Lemma 4]). *If a prime p does not divide the order of the (extended) Weyl group of L for L a simple Lie algebra, then every abelian p-group $H \leq \operatorname{Aut} L$ is toral.*

Lemma 5 ([**15**, Corollary 1]). *Any non-toral quasitorus of* $\operatorname{Aut} L$ *for L a simple Lie algebra contains a non-toral p-group for some prime p.*

Now, take into account that

- the order of the Weyl group of $\mathfrak{e}_6$ is $3^4 2^6 5$;
- there are no non-toral 5-groups of $\operatorname{Aut}\mathfrak{e}_6$ (see Lemma 14 afterwards or [**27**, Lemma 10.3] for the complex case);

to conclude that any non-toral quasitorus of $\operatorname{Aut}\mathfrak{e}_6$ contains either a non-toral 2-group or a non-toral 3-group.

Our purpose is to go further:

Proposition 1. *If Q is a MAD-group of* $\operatorname{Aut}\mathfrak{e}_6$ *contained in* $\operatorname{Int}\mathfrak{e}_6$, *different from a maximal torus, then Q contains a non-toral group isomorphic to either $\mathbb{Z}_2^3$ or $\mathbb{Z}_3^2$.*

Hence any MAD-group contained in $\operatorname{Int}\mathfrak{e}_6$ contains a non-toral elementary p-group. The point is that it is not true that, for any simple Lie algebra L, any non-toral quasitorus of $\operatorname{Aut} L$ contains a non-toral elementary p-group for some prime p. The condition of being maximal is necessary, as the following example shows:

Example. By using the notations in [**19**],

$$Q = \langle \{t_{-1,1,-1,1}, t_{1,-1,-1,1}, \widetilde{\sigma}_{105} t_{1,1,1,i}\}\rangle \leq \operatorname{Aut}\mathfrak{f}_4$$

is a non-toral quasitorus isomorphic to $\mathbb{Z}_2^2 \times \mathbb{Z}_8$ which does not contain any 2-elementary non-toral subquasitorus.

Examples of this situation in $\operatorname{Int}\mathfrak{e}_6$ will appear in Section 4 for $p = 3$ and in Section 6 for $p = 2$.

This makes necessary an ad-hoc proof of Proposition 1 (for $L = \mathfrak{e}_6$ and Q a MAD-group), which we have to postpone until Section 4. The purpose of the first part of this paper is to find the MAD-groups of $\mathfrak{G}_0 := \operatorname{Int}\mathfrak{e}_6$ by using that they must live in the centralizers of such subgroups of type $\mathbb{Z}_2^3$ or $\mathbb{Z}_3^2$. But all the non-toral elementary p-subgroups in $\mathfrak{G}_0$ are well-known in the literature for the case $\mathbb{F} = \mathbb{C}$. In order to distinguish the complex case from the abstract one, we will denote $\mathfrak{e}_6^{\mathbb{C}}$ the complex Lie algebra of type E_6 and $\mathfrak{G}^{\mathbb{C}} = \operatorname{Aut}\mathfrak{e}_6^{\mathbb{C}}$ and $\mathfrak{G}_0^{\mathbb{C}} = \operatorname{Int}\mathfrak{e}_6^{\mathbb{C}}$ the corresponding algebraic groups (in fact, Lie groups).

Theorem 2. *Let p be a prime and $Q \leq \mathfrak{G}_0^{\mathbb{C}} = \operatorname{Int}\mathfrak{e}_6^{\mathbb{C}}$ be a non-toral elementary abelian p-subgroup. Then either $p = 2$ or $p = 3$, and Q is (up to conjugacy) one of the following subgroups:*

(1) *If $p = 2$:*
- $V_2^3 \cong \mathbb{Z}_2^3$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_2^3) = V_2^3 \times \mathrm{PSL}(3)$,
- $V_2^4 \cong \mathbb{Z}_2^4$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_2^4) = V_2^3 \times \mathrm{GL}(2)$,
- $V_2^5 \cong \mathbb{Z}_2^5$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_2^5) = V_2^3 \times (\mathbb{C}^*)^2$.

(2) *If $p = 3$:*
- $V_3^{2a} \cong \mathbb{Z}_3^2$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_3^{2a}) = V_3^{2a} \times \mathrm{PSL}(3)$,
- $V_3^{2b} \cong \mathbb{Z}_3^2$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_3^{2b}) = V_3^{2b} \times G_2$,
- $V_3^{3a} \cong \mathbb{Z}_3^3$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_3^{3a}) = V_3^{2a} \times ((\mathbb{C}^*)^2 \rtimes \mathbb{Z}_3)$,
- $V_3^{3b} \cong \mathbb{Z}_3^3$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_3^{3b}) = V_3^{3b}\,\mathbb{Z}_3^3$,
- $V_3^{3c} \cong \mathbb{Z}_3^3$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_3^{3c}) = V_3^{2b} \times \mathrm{SL}(2)$,
- $V_3^{3d} \cong \mathbb{Z}_3^3$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_3^{3d}) = V_3^{2b} \times \mathrm{GL}(2)$,
- $V_3^{4a} \cong \mathbb{Z}_3^4$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_3^{4a}) = V_3^{4a}$,
- $V_3^{4b} \cong \mathbb{Z}_3^4$ *such that* $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(V_3^{4b}) = V_3^{2b} \times (\mathbb{C}^*)^2$.

We are using the notation $\mathfrak{C}_A(B) = \{x \in A \mid xb = bx \ \forall\, b \in B\}$ for the centralizers ($B \leq A$), and $A = B \times C$ if $A = BC$ for B and C normal subgroups of A such that $B \cap C = \{1_A\}$. As usual, if only B is a normal subgroup, the used notation is that one for semidirect product, $A = B \rtimes C$.

Proof: Either $p = 2$ or $p = 3$, as above. Now, the data involving non-toral elementary abelian 2-subgroups in $\operatorname{Int} \mathfrak{e}_6^{\mathbb{C}}$ can be read in [**36**, p. 78], where V_2^3, V_2^4, and V_2^5 are denoted by $\langle V_2, c\rangle$, $\langle V_3, c\rangle$, and $\langle V_4, c\rangle$ respectively. The reader may also consult [**27**, Theorem 8.2 and its proof, p. 279] where non-toral elementary abelian 2-subgroups in $\operatorname{Aut} \mathfrak{e}_6^{\mathbb{C}}$ are identified with non-toral elementary abelian 2-subgroups in $\operatorname{Aut} \mathfrak{f}_4^{\mathbb{C}}$. The non-toral abelian 2-subgroups in $\operatorname{Aut} \mathfrak{f}_4$ are described in [**19**].

The data related to non-toral elementary abelian 3-subgroups in $\operatorname{Int} \mathfrak{e}_6^{\mathbb{C}}$ can be read in [**4**, Theorem 8.10, p. 148]. $\square$

The important point is that the p-subgroups of $\mathfrak{G}_0^{\mathbb{C}}$ (respectively, of $\mathfrak{G}^{\mathbb{C}}$) are in bijective correspondence with the p-subgroups of $\mathfrak{G}_0$ (respectively, of $\mathfrak{G}$) if the transcendence degree of the field extension $\mathbb{F}|\mathbb{Q}$ is infinite. But, even if this is not the case, we will find only one non-toral subgroup of type $\mathbb{Z}_2^3$ and just two non-toral groups of type $\mathbb{Z}_3^2$ in $\mathfrak{G}_0$. This will be consequence of the following result, communicated to the authors by A. Elduque.

Proposition 2. *Let $\mathbb{F}$ be an algebraically closed field of zero characteristic, and $\mathfrak{l}$ be a simple Lie algebra over $\mathbb{F}$. Let $\mathfrak{l}^{\mathbb{C}}$ denote a simple complex Lie algebra of same type as $\mathfrak{l}$, and consider $\mathcal{G} = \operatorname{Aut} \mathfrak{l}$ and $\mathcal{G}^{\mathbb{C}} = \operatorname{Aut} \mathfrak{l}^{\mathbb{C}}$. Then, there is an injective map*

$$\Omega\colon \{\textit{finite subgroups of } \mathcal{G}\} \to \{\textit{finite subgroups of } \mathcal{G}^{\mathbb{C}}\}.$$

Furthermore, if P_1 and P_2 are two finite subgroups of $\mathcal{G}$, P_1 and P_2 are conjugate in $\mathcal{G}$ if and only if $\Omega(P_1)$ and $\Omega(P_2)$ are conjugate subgroups of $\mathcal{G}^{\mathbb{C}}$.

Proof: Take $P = \{f_1, \dots, f_s\}$ a finite subgroup of $\mathcal{G}$. Let $d = \dim \mathfrak{l}$, and fix $B = \{b_1, \dots, b_d\}$ a basis of $\mathfrak{l}$ such that $[b_i, b_j] \in \sum_{b \in B} \mathbb{Q} b$ for all i, j (for instance, take a Chevalley basis). Thus $\mathfrak{l}^{\mathbb{C}} \cong \mathfrak{l}^{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ and $\mathfrak{l} \cong \mathfrak{l}^{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{F}$, for $\mathfrak{l}^{\mathbb{Q}} = \sum_{b \in B} \mathbb{Q} b$ the $\mathbb{Q}$-Lie algebra of same type as $\mathfrak{l}$ spanned by B. Note that, for each $f_i \in P$ and each $j \leq d$, $f_i(b_j) = \sum_{k=1}^{d} c_{ijk} b_k$ for some $c_{ijk} \in \mathbb{F}$. Take $S = \{c_{ijk} \mid i \leq s;\, j, k \leq d\}$, which is a finite subset of $\mathbb{F}$. Take $\mathbb{K} = \mathbb{Q}(S)$, a subfield of $\mathbb{F}$. Let n be the transcendence degree of the field extension $\mathbb{K}|\mathbb{Q}$, which is obviously finite (as S). That means that there is an algebraically independent subset $\{X_1, \dots, X_n\} \subset \mathbb{K}$ such that the extension $\mathbb{K}|\mathbb{Q}(X_1, \dots, X_n)$ is a finite (algebraic) extension. Every element of $\mathbb{K}$ is a root of some nonzero polynomial with coefficients in $\mathbb{Q}(X_1, \dots, X_n)$, concretely we can take $\{a_1, \dots, a_m\} \subset \mathbb{K}$ such that $\mathbb{K} = (\dots(\mathbb{Q}(X_1, \dots, X_n)(a_1))\dots)(a_m)$ and p_i is the minimal polynomial of a_i with coefficients in the field $(\dots(\mathbb{Q}(X_1, \dots, X_n)(a_1))\dots)(a_{i-1})$. Now recall that $\mathbb{C}|\mathbb{Q}$ has infinite transcendence degree, therefore there exists $\{x_1, \dots, x_n\}$ an algebraically independent subset of $\mathbb{C}$. It is clear that the fields $\mathbb{Q}(X_1, \dots, X_n)$ and $\mathbb{Q}(x_1, \dots, x_n)$ are isomorphic. Now we construct K a subfield of $\mathbb{C}$ isomorphic to $\mathbb{K} = \mathbb{Q}(S)$, simply by adjoining to $\mathbb{Q}(x_1, \dots, x_n)$ elements $\{y_1, \dots, y_m\} \subset \mathbb{C}$ such that p_i is just the minimal polynomial of y_i with coefficients in the field $(\dots(\mathbb{Q}(x_1, \dots, x_n)(y_1))\dots)(y_{i-1}) \leq \mathbb{C}$. Let $\Psi\colon \mathbb{K} \to K$ be a field isomorphism between $\mathbb{K}$ the subfield of $\mathbb{F}$ and K the subfield of $\mathbb{C}$. Now define $\widetilde{f_i}\colon \mathfrak{l}^{\mathbb{C}} \to \mathfrak{l}^{\mathbb{C}}$ by linearity, with $\widetilde{f_i}(b_j) = \sum_{k=1}^d \Psi(c_{ijk})b_k \in \mathfrak{l}^{\mathbb{C}}$. It is clear that $\widetilde{f_i} \in \mathcal{G}^{\mathbb{C}}$ (it is an automorphism). Hence $\Omega(P) := \{\widetilde{f_1}, \dots, \widetilde{f_s}\}$ is the desired subgroup of $\mathcal{G}^{\mathbb{C}}$.

Now assume we have $P_1 = \{f_1, \dots, f_z\}$ and $P_2 = \{g_1, \dots, g_z\}$ two finite subgroups of $\mathcal{G}$. Thus $f_i(b_j) = \sum_{k=1}^d c_{ijk}b_k$ and $g_i(b_j) = \sum_{k=1}^d d_{ijk}b_k$ for some $c_{ijk}, d_{ijk} \in \mathbb{F}$, and more concretely, these scalars c_{ijk} and d_{ijk} live in an extension $\mathbb{K}$ of $\mathbb{Q}$ such that there is an isomorphism $\Psi\colon \mathbb{K} \to K$ onto a subfield of $\mathbb{C}$. The fact of being P_1 and P_2 conjugate in $\mathcal{G}$ is equivalent to the existence of $\{a_{jk} \mid j, k \leq d\} \subset \mathbb{F}$ verifying certain polynomial equations. Namely, if $\varphi\colon \mathfrak{l} \to \mathfrak{l}$ is the automorphism such that $\varphi f_i \varphi^{-1} = g_i$ and $A = (a_{jk}) \in \mathrm{Mat}_{d\times d}(\mathbb{F})$ is its matrix relative to the basis B, the conditions for φ are equivalent to the existence of solutions of the following polynomial equations, for each i, j, l:

$$\sum_k c_{ijk}a_{kl} = \sum_k d_{ikl}a_{jk},$$
$$\sum_k \alpha_{ijk}a_{kl} = \sum_{s,t} a_{is}a_{jt}\alpha_{stl},$$
$$(\det(A)+1)(\det(A)-1) = 0,$$

if $\alpha_{ijk} \in \mathbb{Q}$ are such that $[b_i, b_j] = \sum_k \alpha_{ijk}b_k$. So we have $I \triangleleft \mathbb{K}[Y_{ij} \mid i, j \leq d]$ an ideal of polynomials such that the conjugacy of P_1 and P_2 is equivalent to the existence in $\mathbb{F}^r$ ($r = d^2$) of some common zero of all the polynomials in I. But, as $\mathbb{F}$ is algebraically closed, by weak Nullstellensatz, this is equivalent to the fact that $1 \in I$. By similar arguments the conjugacy of $\Omega(P_1)$ and $\Omega(P_2)$ is equivalent to the existence in $\mathbb{C}^r$ of some common zero of all the polynomials in I (passing through Ψ) and again this is equivalent to the fact that $1 \in I$. $\square$

In the following section we will provide descriptions of three quasitori of types $\mathbb{Z}_2^3$, $\mathbb{Z}_3^2$ and $\mathbb{Z}_3^2$ (non-conjugate), which will be unique by Theorem 2 jointly with the above proposition applied to $\mathfrak{l} = \mathfrak{e}_6$.

To finish this section, we would like to state some results that, though simple, will be highly useful through this paper:

Lemma 6 ([**4**, Theorem 8.2(3)])**.** *Let $\mathcal{G}$ be a linear algebraic group over an algebraically closed field. Assume that $\mathcal{G}$ is a connected reductive group such that its commutator subgroup is simply connected. If Q is a subquasitorus of $\mathcal{G}$ generated by at most two elements, then Q is toral.*

This cannot be applied to our context, since $\operatorname{Int}\mathfrak{e}_6$ has fundamental group $\mathbb{Z}_3$, so we will generalize it a little bit.

Lemma 7. *Let $\mathcal{G}$ be a linear algebraic group over an algebraically closed field. Assume that $\mathcal{G}$ is a connected reductive group such that its commutator group has fundamental group $\mathbb{Z}_n$. If Q is a quasitorus of $\mathcal{G}$ generated by at most two elements and the order of Q is prime to n, then Q is toral.*

Proof: Let $\mathcal{G}$ be a linear algebraic group over an algebraically closed field, and let $\mathcal{G}'$ denote its commutator subgroup $[\mathcal{G},\mathcal{G}]$. Then, the short exact sequence of groups $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}/\mathcal{G}'$ is indeed a fibration of connected topological spaces that induces a short exact sequence of fundamental groups $\pi_1(\mathcal{G}') \to \pi_1(\mathcal{G}) \to \pi_1(\mathcal{G}/\mathcal{G}')$. Therefore $\pi_1(\mathcal{G}') \cong \mathbb{Z}_n$ can be identified with a subgroup of $\pi_1(\mathcal{G})$, and we denote by $p\colon \widetilde{\mathcal{G}} \to \mathcal{G}$ the n-sheeted cover of $\mathcal{G}$ associated to that subgroup. Then $\widetilde{\mathcal{G}}$ is again a connected reductive linear algebraic group whose commutator subgroup is denoted by $\widetilde{\mathcal{G}}'$, and p is an algebraic epimorphism that identifies $\widetilde{\mathcal{G}}$ with a central extension of $\mathcal{G}$ by $\mathbb{Z}_n$. We now claim that $\widetilde{\mathcal{G}}'$ is simply connected. Indeed, since p is a group epimorphism, its restriction $p|_{\widetilde{\mathcal{G}}'}\colon \widetilde{\mathcal{G}}' \to \mathcal{G}'$ is so, and the snake lemma shows that $\mathbb{Z}_n \cong Z(\widetilde{\mathcal{G}}) \subset \widetilde{\mathcal{G}}'$ is in the kernel of $p|_{\widetilde{\mathcal{G}}'}$, what shows that $\widetilde{\mathcal{G}}'$ is the universal cover of $\mathcal{G}'$.

Now, let Q be a quasitorus of $\mathcal{G}$ generated by two elements and define $\widetilde{Q} = p^{-1}(Q)$. Then $\widetilde{Q}$ is a quasitorus of $\widetilde{\mathcal{G}}$, that fits in a central short exact sequence of discrete groups $\mathbb{Z}_n \to \widetilde{Q} \to Q$. Since $|Q|$ is coprime to n, then $H^*(Q;\mathbb{Z}_n) = 0$ and the previous exact sequence splits, that is $\widetilde{Q} \cong \mathbb{Z}_n \times Q$ and $\widetilde{Q}$ can be generated by two elements (as Q is so). Then the result follows from Lemma 6 applied to $\widetilde{Q}$ and $\widetilde{\mathcal{G}}$. □

Consequently there are no non-toral 2-groups of $\operatorname{Int}\mathfrak{e}_6$ with less than 3 factors.

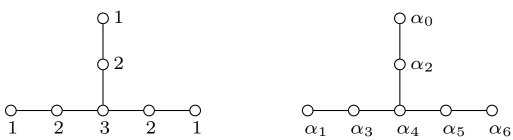
Lemma 8 ([**20**, Lemma 2]). *If L is a simple Lie algebra, T is a torus of* $\operatorname{Aut} L$ *and H is a toral subgroup of* $\operatorname{Aut} L$ *commuting with T, then HT is toral.*

Remark 3. An immediate consequence of Lemma 8 is that, if $\mathcal{T}$ is a maximal torus of $\operatorname{Aut}\mathfrak{e}_6$, and $f \in \mathfrak{N}(\mathcal{T})$ is an inner automorphism such that $\mathcal{T}^{\langle f\rangle}$ is a torus, then $Q(f)$ is toral. Moreover, if $\mathcal{T}^{\langle f\rangle} \cong (\mathbb{F}^*)^l \times \mathbb{Z}_2$, then $Q(f)$ is also toral, by applying Lemma 7 and Lemma 8.

3. Description of the inner gradings

A grading produced by a quasitorus contained in $\operatorname{Int}\mathfrak{e}_6$, the identity component of $\operatorname{Aut}\mathfrak{e}_6$, will be called an *inner grading*. And we call an *outer grading* any grading that is not inner. Of course the fine inner gradings are produced by MAD-groups of $\operatorname{Aut}\mathfrak{e}_6$ contained in $\operatorname{Int}\mathfrak{e}_6$. For describing them, first we fix some notation about the finite order inner automorphisms.

3.1. Inner automorphisms of finite order. Recall that the finite order automorphisms of the simple Lie algebras are completely described in [**34**, Chapter 8]. The $\mathbb{Z}_m$-inner gradings on $\mathfrak{e}_6$ can be obtained by assigning weights $\bar{p} = (p_0, \dots, p_6)$ ($p_i \in \mathbb{Z}_{\geq 0}$) to the nodes of the extended affine diagram of $\mathfrak{e}_6$, $E_6^{(1)}$, such that $\sum_{i=0}^{6} p_i n_i = m$, for n_i the label of the corresponding node, that is, $n_0 = 1$ and $\alpha_0 = -\sum n_i\alpha_i$ denotes minus the maximal root, for $\{\alpha_i\}_{i=1}^{6}$ a set of simple roots.



The subalgebra fixed by this automorphism is reductive of rank 6, and the Dynkin diagram of its semisimple part is just the one obtained when removing the nodes with non-zero weights from $E_6^{(1)}$.

In particular, there are 5 conjugacy classes of order three automorphisms. We say that an order three automorphism f is of type $3B$, or that $f \in 3B$, if f is obtained with the choice of weights $(0,0,0,0,0,1,1)$ (for some Cartan subalgebra and some set of simple roots). Thus the fixed subalgebra is of type $\mathfrak{a}_5$ direct sum with a one-dimensional center (denoted by Z), whose dimension is 36. The same notations will be used for the remaining order three automorphisms, according to the following table:

Type	Fixed subalgebra	dim	$\bar{p}$
$3B$	$\mathfrak{a}_5 \oplus Z$	36	$(0,0,0,0,0,1,1)$
$3C$	$3\mathfrak{a}_2$	24	$(0,0,0,0,1,0,0)$
$3D$	$\mathfrak{d}_4 \oplus 2Z$	30	$(1,1,0,0,0,0,1)$
$3E$	$\mathfrak{a}_4 \oplus \mathfrak{a}_1 \oplus Z$	28	$(1,0,1,0,0,0,0)$
$3F$	$\mathfrak{d}_5 \oplus Z$	46	$(2,1,0,0,0,0,0)$

We have chosen the names of the types of the automorphisms according to [**27**, Table VI]. Besides, throughout this paper we use the notations $\mathfrak{a}_l$, $\mathfrak{b}_l$, $\mathfrak{c}_l$, $\mathfrak{d}_l$, $\mathfrak{g}_2$, $\mathfrak{f}_4$, and $\mathfrak{e}_l$ for the simple finite-dimensional Lie algebras, instead of capital letters, to avoid confusions.

In the same way, there are two conjugacy classes of order two inner automorphisms:

Type	Fixed subalgebra	dim	$\bar{p}$
$2A$	$\mathfrak{a}_5 \oplus \mathfrak{a}_1$	38	$(0,0,1,0,0,0,0)$
$2B$	$\mathfrak{d}_5 \oplus Z$	46	$(1,1,0,0,0,0,0)$

We observe that the type of any order three or two inner automorphism of $\mathfrak{e}_6$ is determined by the dimension of its fixed subalgebra.

3.2. A $\mathbb{Z}_3^4$-grading. Let $V_1 = V_2 = V_3 = V$ be a three-dimensional vector space. Let $B_V = \{u_0, u_1, u_2\}$ be a basis of V. Take

$$\mathcal{L} = \mathrm{sl}(V_1) \oplus \mathrm{sl}(V_2) \oplus \mathrm{sl}(V_3) \oplus V_1 \otimes V_2 \otimes V_3 \oplus V_1^* \otimes V_2^* \otimes V_3^*$$

with the product given as in [**1**, Chapter 13], which is a simple Lie algebra of type $\mathfrak{e}_6$. Note that we have a $\mathbb{Z}_3$-grading on $\mathfrak{e}_6 \equiv \mathcal{L}$ by doing

$$\begin{aligned} \mathcal{L}_{\bar{0}} &= \mathrm{sl}(V_1) \oplus \mathrm{sl}(V_2) \oplus \mathrm{sl}(V_3), \\ \mathcal{L}_{\bar{1}} &= V_1 \otimes V_2 \otimes V_3, \\ \mathcal{L}_{\bar{2}} &= V_1^* \otimes V_2^* \otimes V_3^*. \end{aligned} \tag{2}$$

Consider $F_1 \in \mathrm{Aut}\,\mathfrak{e}_6$ the order three grading automorphism, that is, $F_1|_{\mathcal{L}_{\bar{i}}} = \omega^i\,\mathrm{id}$ for $\omega \in \mathbb{F}$ a primitive cubic root of the unit and $i = 0, 1, 2$. Take F_2 the automorphism which *permutes the V_i-components*, that is, the only one verifying $F_2(u \otimes v \otimes w) = v \otimes w \otimes u$ for all $u, v, w \in V$. As $F_2(\mathcal{L}_{\bar{i}}) \subset \mathcal{L}_{\bar{i}}$, the automorphisms F_1 and F_2 commute.

Now, if $A \in \mathrm{SL}(3)$ and $f_A \in \mathrm{SL}(V)$ is the endomorphism whose associated matrix relative to B_V is A, we call $\Psi(A)$ the only automorphism of $\mathfrak{e}_6$ whose action in $\mathcal{L}_{\bar{1}}$ is $u \otimes v \otimes w \mapsto f_A(u) \otimes f_A(v) \otimes f_A(w)$. Note that the uniqueness is a consequence of the fact that $\mathcal{L}_{\bar{2}} = [\mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{1}}]$ ($[\mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{1}}]$ is an $\mathcal{L}_{\bar{0}}$-submodule of $\mathcal{L}_{\bar{2}}$, which is irreducible) and $\mathcal{L}_{\bar{0}} = [\mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{2}}]$. It is straightforward to check that the only possible extension preserves the bracket. Thus we have a well-defined map

$$\Psi\colon \mathrm{SL}(3) \to \mathfrak{C}_{\mathrm{Aut}\,\mathfrak{e}_6}(F_1, F_2) \tag{3}$$

with kernel $\{I_3, \omega I_3, \omega^2 I_3\}$, where I_n will denote the identity matrix of size n throughout the text. Take the following invertible matrices

$$b := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad c := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{4}$$

and $F_3 := \Psi(b)$ and $F_4 := \Psi(c)$ the related order three automorphisms. As $bc = \omega cb$, thus F_3 and F_4 commute. Consider the subgroup of automorphisms

$$\mathcal{Q}_1 := \langle F_1, F_2, F_3, F_4 \rangle \leq \mathrm{Int}\,\mathfrak{e}_6,$$

which is isomorphic, as abstract group, to $\mathbb{Z}_3^4$. If we denote a homogeneous component of the $\mathbb{Z}_3^4$-grading induced by $\mathcal{Q}_1$ on $\mathcal{L}$ by

$$L_{(\bar{i},\bar{j},\bar{k},\bar{l})} = \{x \in \mathcal{L} \mid F_1(x) = \omega^i x,\ F_2(x) = \omega^j x,\ F_3(x) = \omega^k x,\ F_4(x) = \omega^l x\} \tag{5}$$

for $i, j, k, l \in \{0, 1, 2\}$, and denote by u_{ijk} the element $u_i \otimes u_j \otimes u_k$ in $\mathcal{L}_{\bar{1}}$, then it is easy to check that:

- $\sum_j L_{(\bar{1},\bar{j},\bar{0},\bar{0})} = \langle u_{000} + u_{111} + u_{222}, u_{012} + u_{120} + u_{201}, u_{021} + u_{210} + u_{102} \rangle$ is fixed by F_2, so that it coincides with $L_{(\bar{1},\bar{0},\bar{0},\bar{0})}$, while $L_{(\bar{1},\bar{j},\bar{0},\bar{0})} = 0$ if $j = 1, 2$.
- $\sum_j L_{(\bar{1},\bar{j},\bar{0},\bar{1})} = \langle u_{000} + \omega^2 u_{111} + \omega u_{222}, u_{012} + \omega^2 u_{120} + \omega u_{201}, u_{021} + \omega^2 u_{210} + \omega u_{102} \rangle$ is not fixed by F_2, which acts in the three elements with eigenvalues 1, ω, and ω^2 respectively. The same happens with $\sum_j L_{(\bar{1},\bar{j},\bar{0},\bar{2})}$, by duality.
- $\sum_j L_{(\bar{1},\bar{j},\bar{1},\bar{0})} = \langle x_0 := u_{001} + \omega^2 u_{112} + \omega u_{220}, x_1 := u_{010} + \omega^2 u_{121} + \omega u_{202}, x_2 := u_{100} + \omega^2 u_{211} + \omega u_{022} \rangle$ is not fixed by F_2, but this automorphism sends each x_i to x_{i+1}, so that it acts with eigenvalues 1, ω, ω^2 in $x_0 + x_1 + x_2$, $x_0 + \omega^2 x_1 + \omega x_2$, and $x_0 + \omega x_1 + \omega^2 x_2$ respectively.

On the other hand, $\langle F_3, F_4 \rangle$ breaks each copy $\mathrm{sl}(V_i)$ of $\mathcal{L}_{\bar{0}}$ in 8 pieces of dimension one (the non-toral $\mathbb{Z}_3^2$-grading on $\mathfrak{a}_2$ usually called *Pauli grading*, see for instance [**9**]), so, $\sum_j L_{(\bar{0},\bar{j},\bar{k},\bar{l})}$ has dimension three if $(\bar{k}, \bar{l}) \neq (\bar{0}, \bar{0})$ and breaks into three one-dimensional pieces when F_2 is applied. Therefore all the homogeneous components have dimension one, except for the following cases:

$$\begin{aligned}
\dim L_{(\bar 0,\bar j,\bar 0,\bar 0)} &= 0 \quad \text{for any } j\in\{0,1,2\},\\
\dim L_{(\bar i,\bar j,\bar 0,\bar 0)} &= 0 \quad \text{for any } i,j\in\{1,2\},\\
\dim L_{(\bar i,\bar 0,\bar 0,\bar 0)} &= 3 \quad \text{for any } i\in\{1,2\}.
\end{aligned}$$

Consequently the type of this grading is $(72, 0, 2)$.

In order to prove that we have found our first fine grading, note the following result:

Lemma 9. *The non-toral quasitorus* $\mathcal{P}_1 := \langle F_1, F_2\rangle$ *has centralizer* $\mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_1) \cong \mathcal{P}_1 \times \mathrm{PSL}(3)$*. So it has type* V_3^{2a} *with the notations in Theorem 2.*

Proof: First we are going to prove that $\mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_1) \cong \mathcal{P}_1 \operatorname{Im}\Psi$, for the map Ψ defined in Equation (3). It is clear that $\mathfrak{fix}(\mathcal{P}_1) = \{x^1 + x^2 + x^3 \mid x \in \mathrm{sl}(V)\} \cong \mathrm{sl}(V)$ is an algebra of type $\mathfrak{a}_2$, where if $x \in \mathrm{sl}(V)$, we are denoting by x^i the element x in $\mathrm{sl}(V_i)$, for $i = 1, 2, 3$. Take F any automorphism belonging to $\mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_1)$. This F preserves all the homogeneous components of the $\mathbb{Z}_3^2$-grading $\mathcal{L} = \oplus L_{(\bar i,\bar j)}$ produced by $\mathcal{P}_1$. Note that $\dim L_{(\bar 1,\bar 0)} = \dim L_{(\bar 2,\bar 0)} = 11$ ($F_2 \in 3D$ and fixes a subalgebra of dimension 30 isomorphic to $\mathfrak{d}_4 \oplus 2Z$) but $\dim L_{(\bar i,\bar j)} = 8$ for the six remaining homogeneous components, which are $L_{(\bar 0,\bar 0)} = L_e$-irreducible modules of adjoint type.

In particular the map F leaves $L_e = \mathfrak{fix}(\mathcal{P}_1) \cong \mathrm{sl}(V)$ invariant. Thus $F|_{L_e} \in \operatorname{Aut}\mathrm{sl}(V) \cong \mathrm{PSL}(V) \rtimes \mathbb{Z}_2$, concretely $\operatorname{Aut}\mathrm{sl}(V) = \{\operatorname{Ad} f, \theta \operatorname{Ad} f \mid f \in \mathrm{SL}(V)\}$, for $\operatorname{Ad} f(x) = fxf^{-1}$ and for the outer order 2 automorphism of $\mathrm{sl}(V)$ given by $\theta(x) = -x^t$. If $F|_{L_e} \in \{\operatorname{Ad} A, \theta \operatorname{Ad} A\}$ for $A \in \mathrm{SL}(3)$ (identified with $\mathrm{SL}(V)$ by means of B_V), by replacing F with $F\Psi(A^{-1})$ we can assume that $F|_{L_e} \in \{\mathrm{id}, \theta\}$.

Assume that $F|_{L_e} = \mathrm{id}$ and we are going to check that then $F \in \langle F_1, F_2\rangle$. As $L_{(\bar 0,\bar 1)}$ is L_e-irreducible, the restriction $F|_{L_{(\bar 0,\bar 1)}} \in \operatorname{Hom}_{L_e}(L_{(\bar 0,\bar 1)}, L_{(\bar 0,\bar 1)}) = \mathbb{F}\,\mathrm{id}$, according to Schur's Lemma. So there is $\alpha \in \mathbb{F}^*$ such that $F|_{L_{(\bar 0,\bar 1)}} = \alpha\,\mathrm{id}$. But $[[L_{(\bar 0,\bar 1)}, L_{(\bar 0,\bar 1)}], L_{(\bar 0,\bar 1)}] = L_e$ (F_2 produces a $\mathbb{Z}_3$-grading on $\mathcal{L}_{\bar 0}$ with fixed subalgebra of type $\mathfrak{a}_2$), so $\alpha^3 = 1$, and changing, if necessary, F with either FF_2 or FF_2^2, we can assume that $F|_{L_{(\bar 0,\bar 1)}} = \mathrm{id}$. Hence $F|_{\mathcal{L}_{\bar 0}} = \mathrm{id}$ (recall that $\mathcal{L}_{\bar 0} = \oplus_j L_{(\bar 0,\bar j)}$). Again we can apply Schur's Lemma, because $\mathcal{L}_{\bar 1}$ is an $\mathcal{L}_{\bar 0}$-irreducible module, thus $F|_{\mathcal{L}_{\bar 1}} = \beta\,\mathrm{id}$ for some nonzero scalar $\beta \in \mathbb{F}$. By similar arguments, $\beta^3 = 1$, and we change, if necessary, F with FF_1 or FF_1^2 to obtain that $F|_{\mathcal{L}_{\bar 1}} = \mathrm{id}$. From that we conclude (after multiplying by elements in $\langle F_1, F_2\rangle$) that $F = \mathrm{id}_{\mathcal{L}}$.

Now assume that we have $F \in \mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_1)$ such that $F|_{L_e} = \theta$. Thus, $F^2|_{L_e} = \theta^2 = \mathrm{id}$. As above, this implies that $F^2 \in \mathcal{P}_1$, so that $\langle F_1, F_2, F\rangle$ is isomorphic as abstract group to $\mathbb{Z}_3 \times \mathbb{Z}_6$. Take $G \in \mathfrak{G}$ of order 6 such that $\langle G, F_2\rangle = \langle F_1, F_2, F\rangle$. Now $\mathfrak{fix}(G|_{\mathfrak{fix}F_2 \cong \mathfrak{d}_4+2Z}) = \mathfrak{fix}(\langle F_1, F_2, F\rangle) = \mathfrak{fix}F|_{\mathfrak{fix}(\mathcal{P}_1)\cong \mathrm{sl}(V)}$ is identified with the algebra $\mathfrak{fix}\theta = \{x \in \mathrm{sl}(3) \mid x = -x^t\} = \mathfrak{so}\,(3) \cong \mathfrak{a}_1$. But $G|_{\mathfrak{d}_4}$ is an automorphism of $\mathfrak{d}_4$ of order r a divisor of 6, so that the possibilities for fixed subalgebras of rank different from 4 are just $2\mathfrak{a}_1$ and $\mathfrak{a}_1 \oplus Z$ if $r = 6$, $\mathfrak{g}_2$ and $\mathfrak{a}_2$ if $r = 3$, and $\mathfrak{b}_3 \oplus Z$ if $r = 2$, of course not contained in an algebra of type $\mathfrak{a}_1$. We have got a contradiction in this case.

Note that we have really proved $\mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_1) = \mathcal{P}_1\Psi(\mathrm{SL}(3))$. But this is a direct product. Indeed, take $A \in \mathrm{SL}(3)$ and $n_1, n_2 \in \{0, 1, 2\}$ such that $\Psi(A) = F_1^{n_1}F_2^{n_2}$. On one hand, the fact $\Psi(A)|_{\mathcal{L}_{\bar{0}}} = \mathrm{id}$ implies that A commutes with $\mathrm{sl}(3)$ and hence is in the center of $\mathrm{gl}(3)$, this center equal to $\mathbb{F}I_3$, so that there is $\gamma \in \mathbb{F}$ such that $A = \gamma I_3$. As $\det A = 1$, then $\gamma^3 = 1$. On the other hand, the elements $u_{ijk} \in \mathcal{L}_{\bar{1}} = V^{\otimes 3}$ must be eigenvectors of $F_1^{n_1}F_2^{n_2}$, thus $n_2 = 0$. But $\Psi(A)(u_{ijk}) = \gamma^3 u_{ijk} = u_{ijk} = F_1^{n_1}(u_{ijk})$, so $\omega^{n_1} = 1$ and $n_1 = 0$. In conclusion, $\mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_1) = \mathcal{P}_1 \times \Psi(\mathrm{SL}(3)) \cong \mathbb{Z}_3^2 \times \mathrm{PSL}(3)$, since the kernel of Ψ is $\{I_3, \omega I_3, \omega^2 I_3\}$. □

Hence $\mathcal{Q}_1$ is a MAD-group of $\mathrm{Aut}\,\mathfrak{e}_6$, isomorphic to $\mathbb{Z}_3^4$, since $\langle b, c\rangle$ is a MAD-group of $\mathrm{PSL}(3)$. Moreover, this grading is fine not only as a group-grading, but it is possible to check that it is also fine as a general grading (as partition into subspaces such that the product of two of them is contained in some other).

Now we would like to pay attention to some subquasitori of $\mathcal{Q}_1$ for further use.

Remark 4. The subquasitorus $\langle F_2, F_3, F_4\rangle \cong \mathbb{Z}_3^3$ of $\mathcal{Q}_1$ is toral.

To check it, recall that if L_g is a homogeneous component of the grading induced by $\mathcal{Q}_1$, then L_g and L_{-g} are composed by semisimple elements by Lemma 1. As $[L_g, L_{-g}] \subset L_e = 0$, then the elements in $L_g \oplus L_{-g}$ are semisimple too. Hence, the subalgebra fixed by $\langle F_2, F_3, F_4\rangle$, that is $\mathfrak{h} = L_{(\bar{1},\bar{0},\bar{0},\bar{0})} \oplus L_{(\bar{2},\bar{0},\bar{0},\bar{0})}$, is a toral subalgebra. As $\mathfrak{h}$ has dimension 6, it is a Cartan subalgebra.

Remark 5. Observe that:

- $\langle F_1, F_3, F_4\rangle \cong \mathbb{Z}_3^3$ is a non-toral quasitorus of $\mathcal{Q}_1$, the Jordan subgroup in [**3**] (appearing also in [**38**, Chapter 3, §3.13]).
- It is a minimal non-toral elementary 3-group of $\mathfrak{G}_0$. In particular it does not contain any non-toral group isomorphic to $\mathbb{Z}_3^2$.

As $\sum_j \dim L_{(\bar 0,\bar j,\bar 0,\bar 0)} = 0$, $\sum_j \dim L_{(\bar\imath,\bar j,\bar 0,\bar 0)} = 3+0+0 = 3$ if $i=1,2$, and $\sum_j \dim L_{(\bar\imath,\bar j,\bar k,\bar l)} = 1+1+1 = 3$ if $(k,l)\neq(0,0)$, then this quasitorus induces a non-toral grading of type $(0,0,26)$ with $L_e=0$, which is just the Jordan grading in [**22**, Main Theorem (vi)]. The second item is a consequence of the fact that $L_g\oplus L_{-g}$ is a Cartan subalgebra for all $0\neq g\in\mathbb{Z}_3^3$, by reasoning as in Remark 4.

We will use these notations through the whole section.

3.3. A $\mathbb{Z}_3^2\times\mathbb{Z}^2$-grading. Lemma 9 suggests us another interesting grading. Take for any scalars $\alpha,\beta\in\mathbb{F}^*$, the automorphism $T_{\alpha,\beta}=\Psi(p_{\alpha,\beta})$, for

$$p_{\alpha,\beta}=\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \frac{1}{\alpha\beta}\end{pmatrix}. \tag{6}$$

The quasitorus

$$\mathcal{Q}_2 := \langle\{F_1,F_2,T_{\alpha,\beta}\mid \alpha,\beta\in\mathbb{F}^*\}\rangle\leq \operatorname{Int}\mathfrak{e}_6$$

is isomorphic to $\mathbb{Z}_3^2\times(\mathbb{F}^*)^2$.

Lemma 10. *The quasitorus $R_1=\langle F_1,F_2,T_{\omega,1},T_{\xi,\xi}\rangle\cong\mathbb{Z}_3^4$ for ξ a ninth root of the unit such that $\xi^3=\omega^2$ is of type V_3^{4b}, with the notations in Theorem 2. Its centralizer in $\mathfrak{G}$ is just $\mathcal{Q}_2$.*

Proof: As $\mathcal{P}_1\subset R_1$, then $\mathfrak{C}_{\mathfrak{G}}(R_1)\subset\mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_1)=\mathcal{P}_1\times\Psi(\mathrm{SL}(3))\cong\mathbb{Z}_3^2\times\mathrm{PSL}(3)$. More concretely, $\mathfrak{C}_{\mathfrak{G}}(R_1)=\mathcal{P}_1\times\mathfrak{C}_{\Psi(\mathrm{SL}(3))}\langle T_{\omega,1},T_{\xi,\xi}\rangle$. But if $A\in\mathrm{SL}(3)$ commutes with $p_{\omega,1}$, then A is diagonal and hence it belongs to $\{p_{\alpha,\beta}\mid\alpha,\beta\in\mathbb{F}^*\}$. □

Consequently $\mathcal{Q}_2$ is again a MAD-group of $\operatorname{Aut}\mathfrak{e}_6$, because $\mathfrak{C}_{\mathfrak{G}}(\mathcal{Q}_2)\subset\mathfrak{C}_{\mathfrak{G}}(R_1)=\mathcal{Q}_2$, that is, it is self-centralizing.

Our description makes it easy the computation of the simultaneous diagonalization. If we denote by $L_{(\bar\imath,\bar j,\gamma)}$ the set of elements of $\mathcal{L}$ in which F_1 acts with eigenvalue ω^i, F_2 with eigenvalue ω^j and $T_{\alpha,\beta}$ with eigenvalue γ, then we check that all these homogeneous components are zero except for

$$\begin{aligned}&\dim L_{(\bar\imath,\bar j,1)}=2,\\ &\dim L_{(\bar\imath,\bar j,\gamma)}=1 \quad \text{if } \gamma\in\{(\alpha^2\beta)^{\pm1},(\alpha\beta^2)^{\pm1},(\alpha/\beta)^{\pm1}\},\\ &\dim L_{(\bar 1,\bar 0,\gamma)}=1 \quad \text{if } \gamma\in\{\alpha^3,\beta^3,1/(\alpha^3\beta^3)\},\\ &\dim L_{(\bar 2,\bar 0,\gamma)}=1 \quad \text{if } \gamma\in\{1/\alpha^3,1/\beta^3,\alpha^3\beta^3\},\end{aligned} \tag{7}$$

for all $i,j\in\{0,1,2\}$, so that $\mathcal{Q}_2$ produces a fine grading of type $(60,9)$.

Lemma 11. *The quasitorus $\mathcal{P}_2 := \langle F_1 T_{\xi,\xi}, F_2\rangle$ fixes a subalgebra of type $\mathfrak{g}_2$, so it is non-toral of type V_3^{2b}, with the notations in Theorem 2.*

Note that $\mathcal{Q}_2 = \mathcal{P}_1 \times T_2 = \mathcal{P}_2 \times T_2$, for $T_2 = \langle T_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{F}^*\rangle$.

Before proving this lemma, observe some facts about the isomorphy classes of the automorphisms in $\mathcal{P}_2$, as well as in several quasitori.

Remark 6. As $\dim \mathfrak{fix} F_2 = \sum_{i,k,l} \dim L_{(\bar{i},\bar{0},\bar{k},\bar{l})} = 24 + 2\cdot 3 = 30$ and $\dim \mathfrak{fix} F_3 = \sum_{i,j,l} \dim L_{(\bar{i},\bar{j},\bar{0},\bar{l})} = 18 + 2\cdot 3 = 24$, then $F_2 \in 3D$ and $F_3 \in 3C$. Similar arguments tell us that all the non-trivial automorphisms in $\mathcal{P}_2$ are of the class $3D$ and that the only automorphisms in $\mathcal{Q}_1$ which are not of type $3C$ are F_2F and F_2^2F for all $F \in \langle F_3, F_4\rangle$. Therefore:

- $\mathcal{P}_2$ is of type D^8 (this notation means that it contains 8 automorphisms of type $3D$, of course joint with the identity),
- $\mathcal{P}_1$ is of type C^6D^2,
- $\mathcal{Q}_1$ is of type $C^{62}D^{18}$,
- $\langle F_2, F_3, F_4\rangle$ (the Jordan group in Remark 5) is of type C^{26}.

In particular this provides a direct way of knowing when a non-toral subgroup composed by two commuting order three automorphisms is conjugate to either $\mathcal{P}_1$ or $\mathcal{P}_2$.

Proof: Taking into account Equation (7), $\dim \mathfrak{fix}(\mathcal{P}_2) = \dim L_{(\bar{0},\bar{0},1)} + \dim L_{(\bar{1},\bar{0},\omega^2)} + \dim L_{(\bar{2},\bar{0},\omega)} = 4+5+5 = 14$. We try to know more about this subalgebra. As F_2 is of type $3D$, the subalgebra $\mathfrak{fix} F_2$ is of type $\mathfrak{d}_4$ summed with a two-dimensional center. Now $F_1T_{\xi,\xi}$ preserves this subalgebra and its derived subalgebra, that is, $\mathfrak{d}_4$, producing a $\mathbb{Z}_3$-grading on $\mathfrak{d}_4$. This implies that the restriction $F_1T_{\xi,\xi}|_{\mathfrak{d}_4}$ must fix a subalgebra of some of the types $\{\mathfrak{a}_2, \mathfrak{g}_2, 3\mathfrak{a}_1 \oplus Z, \mathfrak{a}_3 \oplus Z\}$, of dimensions $\{8, 14, 10, 16\}$ respectively. As $\mathfrak{fix}(\mathcal{P}_2) = \mathfrak{fix} F_1T_{\xi,\xi}|_{\mathfrak{d}_4+2Z}$ is equal to $\mathfrak{fix} F_1T_{\xi,\xi}|_{\mathfrak{d}_4}$ direct sum with some abelian subalgebra of dimension either 0, 1, or 2, by dimension count the only possibility is that $\mathfrak{fix}(\mathcal{P}_2)$ is a subalgebra of type $\mathfrak{g}_2$. Hence $\mathcal{P}_2$ is non-toral of type V_3^{2b}, since a quasitorus of $\operatorname{Aut}\mathfrak{e}_6$ is non-toral when its fixed subalgebra has rank different from 6. □

In the complex case, Theorem 2 tells us that $\mathfrak{C}_{\mathfrak{G}_0^{\mathbb{C}}}(\mathcal{P}_2) \cong \mathcal{P}_2 \times G_2$ (for G_2 the automorphism group of the octonion algebra), but we cannot translate this result directly to arbitrary fields by applying Proposition 2, since G_2 is of course not finite. Besides we are interested in the centralizers in $\mathfrak{G}$, not in $\mathfrak{G}_0$. In order to understand why it works for an arbitrary $\mathbb{F}$, let us see first the action of $\mathfrak{fix}(\mathcal{P}_2)$ on the remaining homogeneous components.

Remark 7. As $\mathcal{P}_2$ is of type D^8, the $\mathbb{Z}_3^2$-grading Γ_0 induced by $\mathcal{P}_2$ must have all the non-identity components of the same dimension, $(78-14)/8 = 8$ (alternatively see Equation (7)). As L_e is isomorphic to $\operatorname{Der}\mathcal{C}$, the Lie algebra of derivations of an octonion algebra $\mathcal{C}$, and it acts non trivially on any homogeneous component, then each of them is isomorphic as L_e-module to the $\operatorname{Der}\mathcal{C}$-module $\mathcal{C}$ (sum of $\mathcal{C}_0$, the irreducible set of zero trace octonions, with the trivial module $\mathbb{F}1$).

In particular we can compute the type of the grading induced by $\mathcal{Q}_2$ without doing the simultaneous diagonalization: as the root decomposition of $\mathfrak{g}_2$ is of type $(12,1)$ and breaks each module $\mathcal{C}$ in $(6,1)$, hence the grading induced by $\mathcal{Q}_2$ has type $(12,1)+8(6,1) = (60,9)$, as we already knew.

To check that $\mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_2) \cong \mathcal{P}_2 \times G_2$ is equivalent to check that:

a) If $F \in \mathfrak{G}$ preserves the homogeneous components of Γ_0 and $F|_{L_e} = \mathrm{id}$, then $F \in \mathcal{P}_2$.
b) For any $f \in \operatorname{Aut}\mathcal{C} = G_2$, there is an extension $\widetilde{f} \in \operatorname{Aut}\mathcal{L}$ such that $\widetilde{f}(d) = fdf^{-1}$ if $d \in \operatorname{Der}\mathcal{C} = L_e$ and $\widetilde{f}$ preserves the homogeneous components of Γ_0.

If we identify $\rho\colon \mathcal{L} \to \mathcal{L}' := \operatorname{Der}\mathcal{C} \oplus \mathcal{C}^{(g_1)} \oplus \cdots \oplus \mathcal{C}^{(g_8)}$ by means of the L_e-isomorphisms of modules of each component ($\mathcal{C}^{(g_i)}$'s are several copies of $\mathcal{C}$, indexed in a set with eight elements, $\mathbb{Z}_3^2 \setminus \{e\}$), this map ρ is by construction an L_e-isomorphism of modules, which allows to endow $\mathcal{L}'$ with a Lie algebra structure of type $\mathfrak{e}_6$, when asking for ρ to be a Lie homomorphism. Taking into consideration that $\dim \operatorname{Hom}_{\operatorname{Der}\mathcal{C}}(\mathcal{C}_0 \otimes \mathcal{C}_0, \mathcal{C}_0) = \dim \operatorname{Hom}_{\operatorname{Der}\mathcal{C}}(\mathbb{F} \otimes \mathcal{C}_0, \mathcal{C}_0) = 1$, there must exist some fixed nonzero scalars $\alpha_{ij}, \beta_{ij}, \gamma_i \in \mathbb{F}^*$, with $\alpha_{ij} = \alpha_{ji}$, such that

$$
\begin{aligned}
&[x^{(g_i)}, y^{(g_j)}]_{\mathcal{L}'} = \alpha_{ij}(xy - yx)^{(g_i+g_j)} \quad \text{if } g_i \neq 2g_j,\\
&[1^{(g_i)}, y^{(g_j)}]_{\mathcal{L}'} = \beta_{ij} y^{(g_i+g_j)} \quad \text{if } g_i \neq 2g_j,\\
&[x^{(g_i)}, y^{(2g_i)}]_{\mathcal{L}'} = \gamma_i D_{x,y} = \gamma_i([l_x, l_y] + [l_x, r_y] + [r_x, r_y]) \in \operatorname{Der}\mathcal{C},\\
&[1^{(g_i)}, y^{(2g_i)}]_{\mathcal{L}'} = 0,
\end{aligned}
$$

for any $x, y \in \mathcal{C}_0$, where l_x and r_x denote respectively the left and right multiplication operators on $\mathcal{C}$. The scalars can be determined by passing through ρ or simply by using the Jacobi identity (this provides an interesting model of a Lie algebra of type $\mathfrak{e}_6$, not to be developed in this paper). Now, the map $\widetilde{f}$ which sends $d \mapsto fdf^{-1}$ if $d \in \operatorname{Der}\mathcal{C}$ and $x^{(g_i)} \mapsto f(x)^{(g_i)}$ if $x \in \mathcal{C}$, is the required automorphism in item b). Finally, the statement in a) is proved with arguments as used in Lemma 9, with caution, because $\mathcal{C}$ is not $\operatorname{Der}\mathcal{C}$-irreducible.

3.4. A $\mathbb{Z}_3^2 \times \mathbb{Z}_2^3$-grading. Let $G_1 \in 2A$ any order two automorphism of $\mathcal{M} = \mathfrak{e}_6$ fixing an algebra of type $\mathfrak{a}_5 \oplus \mathfrak{a}_1$. Let $\mathcal{M} = \mathcal{M}_{\bar{0}} \oplus \mathcal{M}_{\bar{1}}$ be the induced $\mathbb{Z}_2$-grading. Hence there exist U and W vector spaces of dimensions 2 and 6 respectively such that

$$\mathcal{M}_{\bar{0}} = \mathrm{sl}(W) \oplus \mathrm{sl}(U),$$
$$\mathcal{M}_{\bar{1}} = \wedge^3 W \otimes U.$$

We will introduce some notations. If f is an automorphism of the vector space W, we denote by $f^{\wedge 3}$ the automorphism of the vector space $\wedge^3 W$ mapping $w^1 \wedge w^2 \wedge w^3$ ($w^i \in W$) into $f(w^1) \wedge f(w^2) \wedge f(w^3)$. And, if $f \in \operatorname{Aut} A$ and $g \in \operatorname{Aut} B$ (A and B vector spaces), we denote by $f \otimes g$ the automorphism of $A \otimes B$ such that $(f \otimes g)(a_1 \otimes a_2) = f(a_1) \otimes g(a_2)$ for any $a_1 \in A$, $a_2 \in B$. Now fix $B_U = \{u_0, u_1\}$ and $B_W = \{w_i \mid i = 0, \dots, 5\}$ two bases of U and W respectively, and take H_1 and H_2 the only automorphisms of $\mathcal{M}$ whose restrictions to $\mathcal{M}_{\bar{1}}$ are

$$H_1|_{\mathcal{M}_{\bar{1}}} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}_W^{\wedge 3} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_U$$

and

$$H_2|_{\mathcal{M}_{\bar{1}}} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}_W^{\wedge 3} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_U,$$

where we are identifying the automorphisms of W (respectively U) with their matrices relative to B_W (respectively B_U), and b and c are given by Equation (4).

Note that H_1 and H_2 are order six automorphisms commuting with G_1 and between them, so that we can consider the quasitorus

$$\mathcal{Q}_3 := \langle \{H_1, H_2, G_1\} \rangle \leq \operatorname{Int} \mathfrak{e}_6$$

isomorphic, as abstract group, to $\mathbb{Z}_3^2 \times \mathbb{Z}_2^3$. Let us prove that $\mathcal{Q}_3$ is another MAD-group. For that, let us look at their p-subgroups.

Lemma 12. *$\mathcal{P}_3 =: \langle H_1^3, H_2^3, G_1 \rangle$ is a non-toral quasitorus isomorphic to $\mathbb{Z}_2^3$, hence of type V_2^3.*

Proof: An element fixed by G_1 belongs to $\mathrm{sl}(W) \oplus \mathrm{sl}(U)$. If $x_U \in \mathrm{sl}(U)$ is fixed by H_1^3 and H_2^3, then $x_U = 0$. Now, if we write $x_W \in \mathrm{sl}(W)$ in square matrix blocks as $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, the fact that x_W commutes with $\left(\begin{smallmatrix} I_3 & 0 \\ 0 & -I_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b & 0 \\ 0 & -b \end{smallmatrix}\right)^3$ forces B and C to be zero, and the fact that x_W commutes with $\left(\begin{smallmatrix} 0 & I_3 \\ I_3 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & c \\ c & 0 \end{smallmatrix}\right)^3$ forces $A = D$. As $0 = \mathrm{tr}(A) + \mathrm{tr}(D) = 2\,\mathrm{tr}(A)$, hence the fixed subalgebra $\mathfrak{fix}(\mathcal{P}_3) = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)_W \mid A \in \mathrm{sl}(3) \right\}$ is isomorphic to an algebra of type $\mathfrak{a}_2$ and $\mathcal{P}_3$ is non-toral. $\square$

Following similar arguments to Lemma 9, it is not difficult to find the centralizer $\mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_3) \cong \mathcal{P}_3 \times \mathrm{PSL}(3)$. The idea is to consider the well-defined map

$$\Psi' \colon \mathrm{SL}(3) \to \mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_3),$$

where, if $A \in \mathrm{SL}(3)$, $\Psi'(A)$ is the only automorphism of $\mathcal{M}$ whose restriction to $\mathcal{M}_{\bar{1}}$ is

$$\Psi'(A)|_{\mathcal{M}_{\bar{1}}} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}_W^{\wedge 3} \otimes I_U, \tag{8}$$

again with the identifications between automorphisms of W and U and matrices relative to B_W and B_U.

Lemma 13. *$\mathcal{P}_4 := \langle H_1^2, H_2^2 \rangle$ is a non-toral quasitorus isomorphic to $\mathbb{Z}_3^2$ of type V_3^{2b}.*

Proof: We have to find the fixed subalgebra by the automorphisms which are extensions of $\left(\begin{smallmatrix} b^2 & 0 \\ 0 & b^2 \end{smallmatrix}\right)_W^{\wedge 3} \otimes I_U$ and $\left(\begin{smallmatrix} c^2 & 0 \\ 0 & c^2 \end{smallmatrix}\right)_W^{\wedge 3} \otimes I_U$. By reordering some rows and columns in the matrices relative to the endomorphisms of W, we can work with $\left(\begin{smallmatrix} I_2 & 0 & 0 \\ 0 & \omega I_2 & 0 \\ 0 & 0 & \omega^2 I_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 & I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{smallmatrix}\right)$. So, a block matrix $A = (A_{ij})_{i,j=1,2,3}$ ($A_{ij} \in \mathrm{Mat}_{2\times 2}(\mathbb{F})$) commutes with them if and only if $A_{11} = A_{22} = A_{33}$ and $A_{ij} = 0$ if $i \neq j$. As $0 = \mathrm{tr}(A) = 3\,\mathrm{tr}(A_{11})$, then there is a two-dimensional vector subspace W' of W such that we can identify $\mathfrak{fix}(\mathcal{P}_4) \cap \mathrm{sl}(W)$ with $\mathrm{sl}(W')$. Besides all the elements in $\mathrm{sl}(U)$ remain fixed, hence $\mathfrak{fix}(\mathcal{P}_4) \cap \mathcal{L}_{\bar{0}} = \mathrm{sl}(W') \oplus \mathrm{sl}(U) \cong \mathfrak{a}_1 \oplus \mathfrak{a}_1$. It is now easy to check that $\mathfrak{fix}(\mathcal{P}_4) \cap \mathcal{L}_{\bar{1}} = S^3(W') \otimes U$, turning out that $\mathfrak{fix}(\mathcal{P}_4) \cong 2\mathfrak{a}_1 \oplus (V(3) \otimes V(1))$, which is a Lie algebra isomorphic to $\mathfrak{g}_2$ (a well-known fact, see for instance [**10**, Theorem 3.2]). $\square$

As $\mathcal{P}_4$ must be conjugate to $\mathcal{P}_2$ by Proposition 2, we conclude that also the centralizer $\mathfrak{C}_{\mathfrak{G}}(\mathcal{P}_4)$ is a direct product of $\mathcal{P}_4$ with a copy of the group G_2. Consequently, as $\mathcal{Q}_3 = \mathcal{P}_4 \times \mathcal{P}_3$ lives in this centralizer, and the subquasitorus $\mathcal{P}_3 \cong \mathbb{Z}_2^3$ is known to be necessarily a MAD-group of $G_2 = \mathrm{Aut}\,\mathfrak{g}_2$ (see, for instance, [**18**]), these arguments imply that $\mathcal{Q}_3$ is a MAD-group of $\mathrm{Aut}\,\mathfrak{e}_6$.

In this occasion, we compute the type of the grading induced by $\mathcal{Q}_3$ without doing the simultaneous diagonalization (not difficult, but long) but taking into account Remark 7 applied to $\mathcal{P}_4$. It is well-known that the $\mathbb{Z}_2^3$-grading on the octonion algebra $\mathcal{C}$ is a grading of type (8) which induces one of type $(0,7)$ on $\mathrm{Der}\,\mathcal{C}$ (each non-trivial component is a Cartan subalgebra). Hence the fine grading induced by $\mathcal{Q}_3$ has type $(0,7) + 8(8,0) = (64,7)$.

3.5. A $\mathbb{Z}_2^3 \times \mathbb{Z}^2$-grading. We use here the notation in the above subsection. Take $G_2 = H_{\bar{1}}^3$ and $G_3 = H_{\bar{2}}^3$, two order two automorphisms whose restrictions to $\mathcal{M}_{\bar{1}}$ are, of course,

$$G_2|_{\mathcal{M}_{\bar{1}}} = \begin{pmatrix} I_3 & 0 \\ 0 & -I_3 \end{pmatrix}_W^{\wedge 3} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_U,$$

$$G_3|_{\mathcal{M}_{\bar{1}}} = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}_W^{\wedge 3} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_U,$$

and $S_{\alpha,\beta} \in \operatorname{Aut}\mathcal{M}$ the automorphism given by

$$S_{\alpha,\beta}|_{\mathcal{M}_{\bar{1}}} = \begin{pmatrix} p_{\alpha,\beta} & 0 \\ 0 & p_{\alpha,\beta} \end{pmatrix}_W^{\wedge 3} \otimes I_U,$$

that is, $S_{\alpha,\beta} = \Psi'(p_{\alpha,\beta})$ with the notations in Equations (8) and (6). Take, then,

$$\mathcal{Q}_4 := \langle \{G_1, G_2, G_3, S_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{F}^*\} \rangle \cong \mathbb{Z}_2^3 \times (\mathbb{F}^*)^2.$$

It is clear that $\mathcal{Q}_4$ is a MAD-group of $\operatorname{Int}\mathfrak{e}_6$, since $\mathcal{Q}_4$ lives in $\mathfrak{C}_{\operatorname{Aut}\mathfrak{e}_6}(\mathcal{P}_3) = \mathcal{P}_3 \times \Psi'(\mathrm{SL}(3))$, and the torus $\langle p_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{F}^* \rangle \cong (\mathbb{F}^*)^2$ is a maximal torus of $\mathrm{SL}(3)$ (and of $\mathrm{PSL}(3) = \operatorname{Int}\mathfrak{a}_2$).

We can compute the type of the induced fine grading on $\mathcal{M}$ by taking into consideration the $\mathbb{Z}_2^3$-grading induced by $\mathcal{P}_3$. The fixed component $L_e = \mathfrak{fix}(\mathcal{P}_3)$ is a Lie subalgebra of type $\mathfrak{a}_2$, and the other 7 components are, all of them, L_e-modules isomorphic to the adjoint module direct sum with two trivial one-dimensional modules. Thus the grading has one component of dimension 8 and seven of dimension 10. From here it is easy to conclude that all the involved order two automorphisms are of type $2A$, that is, $\mathcal{P}_3$ is of type A^7. If we consider now the $\mathbb{Z}^2$-grading on $\mathcal{M}$ produced by $\langle S_{\alpha,\beta} \rangle$, it produces the root decomposition on the identity component $\mathfrak{a}_2$, which is of type $(6,1)$. And, on each of the homogeneous components it produces the weight decomposition, the part fixed by the two-dimensional torus is the piece of dimension 2 jointly with the two trivial submodules, so of dimension 4 and such component is broken into $(6,0,0,1)$. Thus the grading on $\mathfrak{e}_6$ induced by $\mathcal{Q}_4$ is of type $(6,1,0,0) + 7(6,0,0,1) = (48,1,0,7)$.

3.6. A $\mathbb{Z}^6$-grading. Take as $\mathcal{Q}_5$ a maximal torus of $\operatorname{Aut}\mathfrak{e}_6$, which induces a distinguished fine (group) grading, the Cartan-grading or the root decomposition, which is a $\mathbb{Z}^6$-grading of type $(72,0,0,0,0,1)$, with fixed component a Cartan subalgebra and all the remaining components the corresponding one-dimensional root spaces.

3.7. All the inner fine gradings. As a corollary of Proposition 1, which will be proved in Section 4, we obtain one of the main results of this paper:

Theorem 3. *The MAD-groups of* $\operatorname{Aut}\mathfrak{e}_6$ *contained in* $\operatorname{Int}\mathfrak{e}_6$ *are conjugate to* $\mathcal{Q}_i$ *for* $i = 1, \dots, 5$.

Proof: If A is a MAD-group different from a maximal torus (that is, if A is not conjugate to $\mathcal{Q}_5$), it is non-toral and, according to Proposition 1, A contains a non-toral subgroup $V \leq \operatorname{Int}\mathfrak{e}_6$ isomorphic to either V_2^3, V_3^{2a}, or V_3^{2b}, with the notations in Theorem 2. By Lemma 9, Lemma 11, Lemma 12, and Lemma 13, V must be conjugate to either $\mathcal{P}_1$, $\mathcal{P}_2$ ($\cong \mathcal{P}_4$), or $\mathcal{P}_3$, and we can assume that V is one of them.

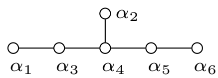
- If $V = \mathcal{P}_3$, then $A \subset \mathfrak{C}(\mathcal{P}_3) = \mathcal{P}_3 \times \Psi'(\mathrm{SL}(3))$, and the problem reduces to calculate MAD-groups of $\Psi'(\mathrm{SL}(3)) \cong \mathrm{PSL}(3) = \operatorname{Int}\mathfrak{a}_2$. There are four fine gradings (up to equivalence) on the algebra sl(3), with grading groups $\mathbb{Z}^2$, $\mathbb{Z}\times\mathbb{Z}_2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_3^2$; that is, there are four MAD-groups (up to conjugation) of $\mathrm{Aut}(\mathrm{sl}(3)) \cong \mathrm{PSL}(3) \rtimes \mathbb{Z}_2$, but only two of them are inner, produced by quasitori of PSL(3), namely, $\langle b, c\rangle \cong \mathbb{Z}_3^2$ and a two-dimensional torus. (This result can be concluded from [**9**], but the gradings are explicitly computed in [**29**].) Hence the only possibility for A is to belong to either $\mathcal{Q}_3 = \mathcal{P}_3 \times \mathcal{P}_4$ or $\mathcal{Q}_4 = \mathcal{P}_3 \times \langle S_{\alpha,\beta}\rangle$.
- If $V = \mathcal{P}_1$, then $A \subset \mathfrak{C}(\mathcal{P}_1) = \mathcal{P}_1 \times \Psi(\mathrm{SL}(3))$, and again it reduces to calculate MAD-groups of $\Psi(\mathrm{SL}(3)) \cong \mathrm{PSL}(3)$, which are conjugate to either the torus $\langle p_{\alpha,\beta}\rangle \cong (\mathbb{F}^*)^2$ or a non-toral $\mathbb{Z}_3^2$ (just $\langle b, c\rangle$). In the first case we obtain $\mathcal{Q}_2 = \mathcal{P}_1 \times \langle T_{\alpha,\beta}\rangle$, and, in the second one, precisely $\mathcal{Q}_1$.
- If $V = \mathcal{P}_2$, then $A \subset \mathfrak{C}(\mathcal{P}_2) = \mathcal{P}_2 \times G_2$, and it reduces to calculate the MAD-groups of $G_2 = \operatorname{Aut}\mathfrak{g}_2$, which are known to be (up to conjugation) the two-dimensional torus and $\mathbb{Z}_2^3$ [**18**]. In the first case we again get $\mathcal{P}_2 \times T_2 = \mathcal{Q}_2$ (see Lemma 11). In the second case (we can take $V = \mathcal{P}_4$), $\mathcal{Q}_3 = \mathcal{P}_3 \times \mathcal{P}_4$ appears again.

This completes the classification. □

4. Technical proofs for the inner gradings

The aim of this section is to prove Proposition 1, which will be done by means of computational tools inspired by [**19**]. We use these computational techniques much less than there in [**19**], because our computations will only make use of the Weyl group of $\mathfrak{e}_6$ and not of the explicit construction of the normalizer of a maximal torus. In particular such computations can be done very easily with any mathematical software. For most of the computations, not even we need all the elements in the Weyl group but it is enough to have representatives of its conjugacy classes, which can be found, for instance, on the web page [**2**]. The auxiliary quasitori that we will need for our argumentation, $\mathcal{T}^{\langle f\rangle}$, $\mathcal{S}^{\langle f\rangle}$ and so on, are easily computed by hand.

4.1. Weyl group. In order to describe the abstract Weyl group of $\mathfrak{e}_6$, we must begin by fixing a basis $\Delta = \{\alpha_i \mid i = 1, \dots, 6\}$ of a root system of $\mathfrak{e}_6$. Its Dynkin diagram is



and its Cartan matrix is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Take the euclidean space $E = \sum_{i=1}^{6} \mathbb{R}\alpha_i$ with the inner product $(\ ,\ )$ described for instance in [**32**, Section 8]. The Weyl group of $\mathfrak{e}_6$ is the subgroup $\mathcal{W}$ of $\mathrm{GL}(E)$ generated by the reflections s_i with $i = 1, \dots, 6$, given by $s_i(x) := x - \langle x, \alpha_i\rangle\alpha_i$, for $\langle x, y\rangle := \frac{2(x,y)}{(y,y)}$ (so that the Cartan integers $\langle \alpha_i, \alpha_j\rangle$ are just the entries of the Cartan matrix). Identify $\mathrm{GL}(E)$ to $\mathrm{GL}(6, \mathbb{R})$ by means of the matrices relative to the $\mathbb{R}$-basis Δ.

We shall consider $\mathcal{W} \subset \mathrm{GL}(6, \mathbb{R})$ lexicographically ordered. That is: first, for any two different couples (i, j), (k, l) such that $i, j, k, l \in \{1, \dots, 6\}$, we define $(i, j) < (k, l)$ if and only if either $i < k$ or $i = k$ and $j < l$; and second, for any two different matrices $\sigma = (\sigma_{ij})$, $\sigma' = (\sigma'_{ij})$ in $\mathcal{W}$, we state $\sigma < \sigma'$ if and only if $\sigma_{ij} < \sigma'_{ij}$ where (i, j) is the least element (with the previous order in the couples) such that $\sigma_{ij} \neq \sigma'_{ij}$. One possible way to compute the Weyl group with this particular enumeration is provided by the following code implemented with `Mathematica`:

```
W=Table[s_i,{i,6}];
a[L_,x_]:=Union[L,
          Table[L[[i]].x,{i,Length[L]}],
          Table[x.L[[i]],{i,Length[L]}]]
Do[W=a[W,s_i],{i,6}]        (6 times repeated).
```

We get a list of $51840 = 2^6 3^4 5$ elements in the table `W` which is nothing but the Weyl group $\mathcal{W}$ of $\mathfrak{e}_6$. We are denoting by σ_i the i-th element of $\mathcal{W}$ with the lexicographical order.

Recall from [**32**, p. 75] that any $\sigma \in \mathcal{W}$ can be extended to an automorphism $\widetilde{\sigma} \in \operatorname{Int} \mathfrak{e}_6$. According to that theorem, if $L = \mathfrak{e}_6 = \mathfrak{h} \oplus (\oplus_{\alpha\in\Phi} L_\alpha)$ is the root decomposition relative to a Cartan subalgebra $\mathfrak{h}$, for any choice $x_{\alpha_i} \in L_{\alpha_i} \setminus \{0\}$ and $x'_{\sigma(\alpha_i)} \in L_{\sigma(\alpha_i)} \setminus \{0\}$ for $i = 1, \dots, 6$, there is only one $\widetilde{\sigma} \in \operatorname{Aut} \mathfrak{e}_6$ such that $\widetilde{\sigma}(t_{\alpha_i}) = t_{\sigma(\alpha_i)}$ and $\widetilde{\sigma}(x_{\alpha_i}) = x'_{\sigma(\alpha_i)}$ for every $i = 1, \dots, 6$, where t_α is the only element in $\mathfrak{h}$ such that $k(t_\alpha, \cdot) = \alpha$, for k the Killing form. For having fixed a precise family of extensions, consider all the choices x_{α_i}, $x'_{\sigma(\alpha_i)}$ in the base B chosen as in Proposition 2. Thus we have extensions $\{\widetilde{\sigma}_i \mid i \leq 51840\} \subset \operatorname{Int} \mathfrak{e}_6$. We are not going to make use of precise descriptions of these extensions.

Denote by $t_{x,y,z,u,v,w}$ the only automorphism of $\mathfrak{e}_6$ which acts diagonally on $\mathfrak{h}, L_{\alpha_1}, \dots, L_{\alpha_6}$, with eigenvalues $\{1, x, y, z, u, v, w\}$ respectively. Take $\mathcal{T} = \{t_{x,y,z,u,v,w} \mid x, y, z, u, v, w \in \mathbb{F}^*\}$, which is a maximal torus of $\mathfrak{G}_0$. Any other extension of $\sigma \in \mathcal{W}$ as in the above paragraph is equal to $\widetilde{\sigma} t$ for some $t \in \mathcal{T}$. Recall that the Weyl group acts in this torus by means of $\mathcal{W} \times \mathcal{T} \to \mathcal{T}$ given by $\sigma \cdot t := \widetilde{\sigma} t \widetilde{\sigma}^{-1}$ for $\sigma \in \mathcal{W}$ and $t \in \mathcal{T}$. This action does not depend on the choice of the extension $\widetilde{\sigma}$. Thus $\sigma \cdot t_{x,y,z,u,v,w} = t_{x',y',z',u',v',w'}$ for

$$
\begin{aligned}
x' &= x^{a_{11}} y^{a_{12}} z^{a_{13}} u^{a_{14}} v^{a_{15}} w^{a_{16}},\\
y' &= x^{a_{21}} y^{a_{22}} z^{a_{23}} u^{a_{24}} v^{a_{25}} w^{a_{26}},\\
z' &= x^{a_{31}} y^{a_{32}} z^{a_{33}} u^{a_{34}} v^{a_{35}} w^{a_{36}},\\
u' &= x^{a_{41}} y^{a_{42}} z^{a_{43}} u^{a_{44}} v^{a_{45}} w^{a_{46}},\\
v' &= x^{a_{51}} y^{a_{52}} z^{a_{53}} u^{a_{54}} v^{a_{55}} w^{a_{56}},\\
w' &= x^{a_{61}} y^{a_{62}} z^{a_{63}} u^{a_{64}} v^{a_{65}} w^{a_{66}}.
\end{aligned} \tag{9}
$$

Take also $\mathfrak{N}(\mathcal{T}) = \{f \in \operatorname{Aut} \mathfrak{e}_6 \mid f t f^{-1} \in \mathcal{T}\ \forall t \in \mathcal{T}\}$ the normalizer of the torus and $\mathfrak{N}_0(\mathcal{T}) := \mathfrak{N}(\mathcal{T}) \cap \operatorname{Int} \mathfrak{e}_6$, and consider the projection $\pi\colon \mathfrak{N}_0(\mathcal{T}) \to \mathfrak{N}_0(\mathcal{T})/\mathcal{T} \cong \mathcal{W}$. From this viewpoint, we will refer to our extension $\widetilde{\sigma}$ as a lifting of σ, which is a more usual term for pre-images under a quotient map. As there does not exist a section of π (see [**37**]), we have an injection $\iota\colon \mathcal{W} \to \mathfrak{N}_0(\mathcal{T})$ given by $\sigma \mapsto \widetilde{\sigma}$, but ι is not a group homomorphism.

Let us consider our previous notations of Equation (1) in these new terms. If $\eta \in \mathcal{W}$ and $s \in \mathcal{T}$, we denote by $\mathcal{T}^{\langle \eta \rangle} := \mathcal{T}^{\langle \widetilde{\eta} \rangle} = \{t \in \mathcal{T} \mid \eta \cdot t = t\}$, and by $Q(\eta, s) := Q(\widetilde{\eta} s)$, that is, the quasitorus generated by $\widetilde{\eta} s$

and $\mathcal{T}^{\langle\eta\rangle}$. We compute $\mathcal{T}^{\langle\eta\rangle}$ for each representative η of some orbit. The 51840 elements of the Weyl group $\mathcal{W}$ of $\mathfrak{e}_6$ are distributed in 25 orbits (= conjugacy classes), whose representatives could be found by using any matrix multiplication software. We extracted such representatives from the public list [**2**], and we identified them to some elements in our ordered list $\mathcal{W}$ in order to make possible to do computations with them.

Order	Rep.	Class size	Centralizer in $\mathcal{T}$	Iso. to
1	40843	1	$xyzuvw \neq 0$	$(\mathbb{F}^*)^6$
2	19	270	$w=u,\ v=\frac{1}{u^2xyz}$	$(\mathbb{F}^*)^4$
2	21	540	$v=1,\ w=u,\ z=\frac{1}{u^2xy}$	$(\mathbb{F}^*)^3$
2	96	45	$z=\frac{1}{xy},\ u=w,\ u^2=v^2=1$	$(\mathbb{F}^*)^2\times\mathbb{Z}_2^2$
2	11323	36	$x=1$	$(\mathbb{F}^*)^5$
4	2	3240	$z=xy,\ u=\frac{1}{z},\ v=1,\ w=u$	$(\mathbb{F}^*)^2$
4	20	1620	$u=w=1,\ v=\frac{1}{xyz}$	$(\mathbb{F}^*)^3$
4	75	540	$u^2=v^2=1,\ y=\frac{v}{x},\ z=v,\ w=u$	$\mathbb{F}^*\times\mathbb{Z}_2^2$
4	140	540	$x=u=w=1,\ v=\frac{1}{yz}$	$(\mathbb{F}^*)^2$
8	1	6480	$u=v=z=w=1,\ y=\frac{1}{x}$	$\mathbb{F}^*$
3	292	480	$z=y,\ v=x,$ $w=u^3x^2y^2,\ (xyu)^3=1$	$(\mathbb{F}^*)^2\times\mathbb{Z}_3$
3	3819	80	$x^3=y^3=z^3=1,$ $u=y,\ v=y^2z,\ w=xy^2$	$\mathbb{Z}_3^3$
3	4079	240	$x=1,\ w=\frac{1}{u^2v^2yz}$	$(\mathbb{F}^*)^4$
6	5	1440	$u=\frac{1}{xy},\ v=\frac{xy}{z},\ w=u$	$(\mathbb{F}^*)^3$
6	15	2160	$y=\frac{1}{xu^2},\ z=v=1,\ w=u$	$(\mathbb{F}^*)^2$
6	22	1440	$u=v=w=1,\ z=\frac{1}{xy}$	$(\mathbb{F}^*)^2$
6	122	4320	$x^3=1,\ z=y,\ u=\frac{1}{y},\ v=x,\ w=ux^2$	$\mathbb{F}^*\times\mathbb{Z}_3$
6	124	720	$x^3=1,\ y=z=u=1,\ v=x,\ w=x^2$	$\mathbb{Z}_3$
6	195	1440	$x^3=1,\ u^2=v^2=1,\ y=z=v,\ w=ux$	$\mathbb{Z}_3\times\mathbb{Z}_2^2$
6	435	1440	$z=y=u,\ v=x^3y^5,\ w=xy,\ (xy^2)^3=1$	$\mathbb{F}^*\times\mathbb{Z}_3$
9	121	5760	$x^3=1,\ y=z=x^2,\ u=v=x,\ w=1$	$\mathbb{Z}_3$
12	4	4320	$z=u=v=t=1,\ y=\frac{1}{x}$	$\mathbb{F}^*$
12	218	4320	$x^3=1,\ y=z=u=1,\ v=x,\ w=x^2$	$\mathbb{Z}_3$
5	3	5184	$z=1,\ u=\frac{1}{xy},\ v=xy,\ w=u$	$(\mathbb{F}^*)^2$
10	135	5184	$x=z=v=1,\ w=u,\ y=\frac{1}{u^2}$	$\mathbb{F}^*$

Table 1. Table of representatives of $\mathcal{W}$.

The second row of this table means the following: the element σ_{19} has order 2, its conjugacy class has 270 elements, and $\mathcal{T}^{\langle\sigma_{19}\rangle} = \{t_{x,y,z,u,v,w} \in \mathcal{T} \mid w = u,\, v = \frac{1}{u^2xyz}\} \cong \mathbb{F}^{*4}$. Such element is the 19th in our list $\mathcal{W}$, and it appears explicitly in [**2**]. For short we also denote by $\mathcal{T}^{\langle i\rangle} := \mathcal{T}^{\langle\sigma_i\rangle}$ and $Q(i,s) := Q(\sigma_i, s) = Q(\widetilde{\sigma_i}s)$.

Remark 8. According to Remark 3, if $Q(\eta, s)$ is non-toral and if

- η has order 2, then η is conjugate to σ_{96};
- η has order 4, then η is conjugate to σ_{75};
- η has order 3, then η is conjugate to either σ_{292} or σ_{3819}.

And, following again Remark 3, it is not possible that $Q(\eta, s)$ is non-toral if η has order 5, so:

Lemma 14. *There is no non-toral* 5*-group of* $\operatorname{Aut}\mathfrak{e}_6$*.*

Proof: Suppose that there is $Q \leq \operatorname{Int}\mathfrak{e}_6$ a non-toral 5-group. Let Q' be a non-toral minimal quasitorus contained in Q (any $Q'' \subsetneq Q'$ is toral). We can assume that $Q' \subset \mathfrak{N}(\mathcal{T})$ and that $Q' \cap \mathcal{T}$ is maximal toral in Q'. Hence there are some element $\eta \in \mathcal{W}$ of order a power of five and some $s \in \mathcal{T}$ such that $Q' \subset Q(\eta, s)$. The contradiction appears since the last quasitorus is toral, and Q' is non-toral. □

(This lemma is also consequence of the same result for the complex field, proved in [**27**], jointly with Proposition 2.)

Remark 9. Again by looking at Table 1, we observe the following useful fact: For $f \in \mathfrak{N}(\mathcal{T})$, fix any subgroup $\mathcal{H}^{\langle f\rangle}$ satisfying the conditions of Lemma 3 (that is, $\mathcal{T}^{\langle f\rangle} = \mathcal{S}^{\langle f\rangle} \times \mathcal{H}^{\langle f\rangle}$). Then denote by $\wp_f\colon \mathcal{T}^{\langle f\rangle} \to \mathcal{H}^{\langle f\rangle}$ the projection. Now, if Q is a non-toral subquasitorus of $Q(f)$ such that $\pi(f)$ is in the orbit of neither σ_{3819} nor σ_{195}, then $\wp_f(Q \cap \mathcal{T}) = \mathcal{H}^{\langle f\rangle}$.

Indeed, the quasitorus $\langle f\rangle \times \wp_f(Q \cap \mathcal{T})$ is toral taking into account that there are no non-toral 2-groups with less than 3 factors. By Lemma 8, $\langle f\rangle \times \mathcal{S}^{\langle f\rangle} \times \wp_f(Q \cap \mathcal{T})$ is also toral, as well as the quasitorus Q which is contained in it.

4.2. MAD-groups of $\operatorname{Int}\mathfrak{e}_6$ in computational terms.

Proposition 3. *The MAD-groups described in Section 3 can be described in these terms as follows.*

- *The quasitorus* $Q(3819, \mathrm{id}) \cong \mathbb{Z}_3^4$ *is conjugate to* $\mathcal{Q}_1$*.*
- *The quasitorus* $Q(292, \mathrm{id}) \cong (\mathbb{F}^*)^2 \times \mathbb{Z}_3^2$ *is conjugate to* $\mathcal{Q}_2$*.*
- *The quasitorus* $Q(195, \mathrm{id}) \cong \mathbb{Z}_3^2 \times \mathbb{Z}_2^3$ *is conjugate to* $\mathcal{Q}_3$*.*

- *The quasitorus* $Q(96,\mathrm{id}) \cong (\mathbb{F}^*)^2 \times \mathbb{Z}_2^3$ *is conjugate to* $\mathcal{Q}_4$.
- *The quasitorus* $Q(\mathrm{id}) = \mathcal{T}$ *is the maximal torus conjugate to* $\mathcal{Q}_5$.

Proof: We use the notations in Section 3.

$\mathcal{Q}_1$) As $\langle F_2, F_3, F_4\rangle \cong \mathbb{Z}_3^3$ is toral as in Remark 4 but $\mathcal{Q}_1$ is a non-toral quasitorus, there is a maximal torus (we can conjugate to choose such torus equal to $\mathcal{T}$) such that $\langle F_2, F_3, F_4\rangle \subset \mathcal{T}$ and $F_1 \in \mathfrak{N}(\mathcal{T})$ by Lemma 2. Moreover, $F_1 \in \mathfrak{N}_0(\mathcal{T})$ because $F_1 \in \mathrm{Int}\,\mathfrak{e}_6$. As F_1 has order 3, also $\pi(F_1)$ has order 3 (not 1 because $\mathcal{Q}_1$ would be toral), so that we can conjugate without changing the torus to get $\pi(F_1) = \sigma_j$ with $j \in \{292, 3819, 4079\}$ (recall that if σ and τ are conjugate in $\mathcal{W}$, then some element in $\pi^{-1}(\sigma)$ is conjugate to some element in $\pi^{-1}(\tau)$ in $\mathfrak{N}(\mathcal{T})$). The possibility $j = 4079$ is ruled out as in Remark 8. Moreover, $\mathcal{Q}_1 = \langle F_2, F_3, F_4, F_1\rangle \subset Q(F_1)$ (since $\langle F_2, F_3, F_4\rangle \subset \mathcal{T}^{\langle F_1\rangle}$) and $\mathcal{Q}_1$ is a MAD-group, so that $\mathcal{Q}_1 = Q(F_1)$. Thus $j \neq 292$, since $Q(292) \cong (\mathbb{F}^*)^2 \times \mathbb{Z}_3^2$. Therefore $F_1 = \widetilde{\sigma}_{3819} s$ for some $s \in \mathcal{T}$ and $\mathcal{Q}_1 = Q(3819, s)$, which is conjugate to $Q(3819, \mathrm{id})$ as in Remark 1, since $\mathcal{T}^{\langle 3819\rangle}$ is finite.

$\mathcal{Q}_2$) F_1 is a toral element because it is an inner automorphism, so that $\langle F_1, T_{\alpha,\beta} \mid \alpha,\beta \in \mathbb{F}^*\rangle$ is also toral by Lemma 8 and, as before, we can assume that $\langle F_1, T_{\alpha,\beta} \mid \alpha,\beta \in \mathbb{F}^*\rangle \subset \mathcal{T}$ and that $F_2 = \widetilde{\sigma}_j s \in \mathfrak{N}_0(\mathcal{T})$ for some $j \in \{292, 3819\}$ and $s \in \mathcal{T}$. It is clear that $j = 292$, because a two-dimensional torus is not contained in $\mathcal{T}^{\langle 3819\rangle} \cong \mathbb{Z}_3^3$, so that $\mathcal{Q}_2 = Q(292, s)$, which is conjugate to $Q(292, \mathrm{id})$ since $\mathcal{T}^{\langle 292\rangle} \cap \mathfrak{S}^{\langle 292\rangle} = \{t_{x,u,u,u,x,x^2u^2} \mid x^3 = u^3 = 1\} \cong \mathbb{Z}_3^2$ is finite.

$\mathcal{Q}_3$) Now note that $\langle H_2, G_1\rangle$ is toral (by Lemma 5, since it is isomorphic to $\mathbb{Z}_2^2 \times \mathbb{Z}_3$), so that we can assume that $\langle H_2, G_1\rangle \subset \mathcal{T}$ and $H_1 \in \mathfrak{N}_0(\mathcal{T})$. Hence $\mathcal{Q}_3 = \langle H_2, G_1, H_1\rangle \subset Q(H_1)$, and, as $\mathcal{Q}_3$ is maximal, then $\mathcal{Q}_3 = Q(H_1)$. That fact forces $\mathcal{Q}_3$ to be $Q(195, s)$ for some $s \in \mathcal{T}$, which is conjugate to $Q(195, \mathrm{id})$ as in Remark 1, since $\mathcal{T}^{\langle 195\rangle}$ is finite.

$\mathcal{Q}_4$) Finally, $\{G_1, G_2\}$ is toral (two 2-factors are always toral) and we can assume that $\langle G_1, G_2, S_{\alpha,\beta}\rangle \subset \mathcal{T}$ and that $G_3 \in \mathfrak{N}_0(\mathcal{T})$ projects in some element of the orbit of σ_{96}, by Remark 8. Thus $\mathcal{Q}_4$ is conjugate to some $Q(96, s)$ and hence to $Q(96, \mathrm{id})$ because $\mathcal{T}^{\langle 96\rangle} \cap \mathfrak{S}^{\langle 96\rangle} = \{t_{yz,y,z,u,v,u} \mid y^2 = z^2 = u^2 = v^2 = 1\} \cong \mathbb{Z}_2^4$ is finite.

□

4.3. Order of the liftings. Note that the order of $f \in \mathfrak{N}_0(\mathcal{T})$ is a multiple of the order of $\pi(f) \in \mathcal{W}$, and that both numbers may not coincide. For instance, in [**19**, Remark 1] it is observed that $\widetilde{\sigma}_3 \in \mathrm{Aut}\,\mathfrak{f}_4$

has order 8, while its projection σ_3 on the Weyl group of $\mathfrak{f}_4$ has order 4. Moreover, any element in $\pi^{-1}(\sigma_3)$ has also order 8, as in Remark 2. None of them has the same order as its projection.

Remark 10. We can extend the results in Lemma 3 a little bit. If $f \in \mathfrak{N}_0(\mathcal{T})$ verifies that its projection $\pi(f)$ has order r' (in general, r' divides the order of f), then $\mathcal{H}^{\langle f\rangle} \subset \{t \in \mathcal{T} \mid t^{r'} = 1_G\}$ and the torus $\mathcal{S}^{\langle f\rangle} = \{(tf)^{r'}(f)^{-r'} \mid t \in \mathcal{T}\} = \{\pi_{i=0}^{r'-1} f^i \cdot t \mid t \in \mathcal{T}\}$.

In the previous example about $\mathfrak{f}_4$, what happens is $(\widetilde{\sigma}_3 s)^8 \in \mathcal{S}^{\langle 3\rangle} = \mathrm{id}$ but $\mathrm{id} \neq (\widetilde{\sigma}_3 s)^4 \in \mathcal{T}^{\langle 3\rangle}$ for every s in the corresponding torus.

It is difficult in general to know the order of a lifting of a concrete element of the Weyl group by applying only the isomorphism theorem in [**32**, Section 14.2], but in this case we have extra-information extracted from Proposition 3. Using, also, the results in Subsection 2.2, we conclude that there are elements $s_{292} \in \mathcal{T}^{\langle 292\rangle}$ and $s_{96} \in \mathcal{T}^{\langle 96\rangle}$ such that the order of $\widetilde{\sigma}_{96} s_{96}$ is 2, the order of the liftings $\widetilde{\sigma}_{292} s_{292}$ and $\widetilde{\sigma}_{3819}$ is 3, and the order of the lifting $\widetilde{\sigma}_{195}$ is 6, and, up to conjugation, $\mathcal{P}_2 \cong \mathcal{P}_4$ is conjugate to the set of order three elements in $Q(195)$ and

$$\mathcal{P}_1 \cong \langle \widetilde{\sigma}_{292} s_{292}\rangle \times \mathcal{H}^{\langle 292\rangle}, \quad \mathcal{P}_3 \cong \langle \widetilde{\sigma}_{96} s_{96}\rangle \times \mathcal{H}^{\langle 96\rangle},$$

for $\mathcal{H}^{\langle 292\rangle} = \langle t_{1,1,1,\omega,1,1}\rangle$ and $\mathcal{H}^{\langle 96\rangle} = \{t_{1,1,1,u,v,u} \mid u^2 = v^2 = 1\}$, that we fix for the rest of this section.

We obtain the same conclusions as in the above paragraph by reading the appendix, in which natural liftings of representatives of the order two elements in $\mathcal{W}$ are constructed.

For technical purposes, note that, according to Remark 10, $(\widetilde{\sigma}_{292} t)^3 \in \mathcal{S}^{\langle 292\rangle}$ and $(\widetilde{\sigma}_{96} t)^2 \in \mathcal{S}^{\langle 96\rangle}$ for any $t \in \mathcal{T}$, since there exist liftings of σ_{96} and of σ_{292} of orders 2 and 3 respectively.

4.4. On the elementary p-group.

Proof of Proposition 1: Now suppose that Q is in the conditions of the Proposition 1. Then $Q = P \times \prod_i P_{p_i}$, where P is a torus and P_{p_i}'s are p_i-groups (p_i prime), at least one of them non-toral (by Lemma 5) for some $p_i \in \{2, 3\}$. We want to prove that either P_2 contains a non-toral $\mathbb{Z}_2^3$-subgroup or P_3 contains a non-toral $\mathbb{Z}_3^2$-subgroup. By Lemma 2, we can assume that Q is contained in $\mathfrak{N}_0(\mathcal{T})$ for some maximal torus $\mathcal{T}$ in such a way that $Q \cap \mathcal{T}$ is maximal-toral in Q: that is, if $Q \cap \mathcal{T} \subset Q' \subset Q$ with Q' toral, then $Q \cap \mathcal{T} = Q'$. Observe that $P_{p_i} \cap \mathcal{T}$ is maximal-toral in P_{p_i}: otherwise certain $h \in P_{p_i} \setminus \mathcal{T}$ would verify that $\langle P_{p_i} \cap \mathcal{T}, h\rangle$ would be toral, and then $\langle Q \cap \mathcal{T}, h\rangle$ would be toral too, by [**15**, Corollary 1]. In particular $P \times P_5 \times P_7 \times \cdots \subset \mathcal{T}$ holds, $P_2 = (P_2 \cap \mathcal{T}) \times \langle f_1, \dots, f_n \rangle$ with each $f_j \in \mathfrak{N}_0(\mathcal{T})$ of order a power of 2 and $P_3 = (P_3 \cap \mathcal{T}) \times \langle g_1, \dots, g_m \rangle$ with each $g_j \in \mathfrak{N}_0(\mathcal{T})$ of order a power of 3. Besides, as Q is a MAD-group, then $Q \cap \mathcal{T} = \mathcal{T}^{\langle f_1 \rangle} \cap \cdots \cap \mathcal{T}^{\langle f_n \rangle} \cap \mathcal{T}^{\langle g_1 \rangle} \cap \cdots \cap \mathcal{T}^{\langle g_m \rangle}$. Of course, we can take $f_i \notin \langle Q \cap \mathcal{T}, f_1, \dots, f_{i-1} \rangle$, since otherwise $\langle Q \cap \mathcal{T}, f_1, \dots, f_{i-1} \rangle = \langle Q \cap \mathcal{T}, f_1, \dots, f_i \rangle$. Hence $f_i \notin \langle \mathcal{T}, f_1, \dots, f_{i-1} \rangle$ and $\pi(f_i)$ does not belong to the group generated by $\{\pi(f_1), \dots, \pi(f_{i-1})\}$.

As $\langle f_i, P_2 \cap \mathcal{T} \rangle$ is a non-toral subquasitorus of $Q(f_i)$, we know by Remark 8 that $\pi(f_i)$ (an element of order either 2, 4, or 8) belongs to the orbit of either σ_{96} or σ_{75}. The same argument shows that $\pi(g_i)$ (an element of order either 3 or 9) belongs to the orbit of either σ_{292}, σ_{3819}, or σ_{121}. We can rule out the possibilities 75 and 121:

- In the first case, note that $\mathcal{T}^{\langle \sigma_{75}^2 \rangle} \cong (\mathbb{F}^*)^4$, so that $\langle P_2 \cap \mathcal{T}, f_i^2 \rangle$ is toral and contained in P_2, which is a contradiction with the choice of $P_2 \cap \mathcal{T}$ as maximal-toral in P_2.
- In the second case, we can change the element g_i of order $9l$ by its conjugate element $\widetilde{\sigma}_{121} s$ with $s \in \mathcal{T}$ (we conjugate by means of an element in the normalizer, so that $Q \cap \mathcal{T}$ is still contained in $\mathcal{T}$). As $Q \cap \mathcal{T} \subset \mathcal{T}^{\langle g_i \rangle} \cong \mathbb{Z}_3$, then $Q \cap \mathcal{T} = \mathcal{T}^{\langle g_i \rangle}$ (since jointly with g_i is non-toral) and $Q \subset \mathfrak{C}_{\mathfrak{N}_0(\mathcal{T})}(Q(121, s))$. This centralizer is known to coincide with itself (a direct computation with the computer), so that, by maximality, $Q = Q(121, s)$. We have got a contradiction because this set is not a MAD-group: Inspired by [**19**, Proposition 7], there is $h \in \operatorname{Aut} \mathfrak{e}_6$ such that $h g_i h^{-1} \in \mathcal{T}$ and $h t h^{-1} \in \mathfrak{N}(\mathcal{T})$ for $t = t_{\omega,\omega^2,\omega^2,\omega,\omega,1}$ a generator of $\mathcal{T}^{\langle 121 \rangle}$. Thus $h Q h^{-1} \subset Q(k, s')$ for certain σ_k of order 3 and $s' \in \mathcal{T}$, but in no case the quasitorus $Q(k, s')$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_9$ (according to Table 1) and the contained is proper.

We have proved in particular that the order of any $\pi(f_i)$ is 2 and the order of any $\pi(g_i)$ is 3.

Observe the following technical fact (which saves a lot of computations). There are 113 order 2 elements σ_j (including $j = 96$) in $\mathcal{W}$ commuting with σ_{96}. Only 13 of them are in the orbit of σ_{96}, and for all these, the element $\sigma_j \sigma_{96}$ is not in the orbit of σ_{96}. This proves that necessarily $n \leq 1$.

★ If $m \leq 1$, then by maximality $Q \in \{Q(f_1), Q(g_1), Q(f_1 g_1)\}$. The only possible non-toral $Q(j, s)$ for some σ_j of order 2, 3, or 6 are those ones with $j \in \{96, 292, 3819, 122, 124, 195, 435\}$ by Lemma 8. If $j \in \{96, 292, 3819, 195\}$, then Q is conjugate to some quasitorus in the list

$\{\mathcal{Q}_i \mid i = 1,2,3,4\}$ by Proposition 3. In such a case we are done because all these quasitori contain one of the required elementary 2-groups or 3-groups (since $\mathcal{P}_1 \subset \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $\mathcal{P}_3 \subset \mathcal{Q}_3 \cap \mathcal{Q}_4$). If $j \in \{122, 435\}$, then σ_j^3 is not conjugate to σ_{96}, so that $\langle (\widetilde{\sigma}_j s)^3, \mathcal{T}^{\langle j \rangle}\rangle$ is toral, which is a contradiction with the choice of $Q \cap \mathcal{T}$. And if $j = 124$, although σ_j^3 is conjugate to σ_{96}, again $\langle (\widetilde{\sigma}_j s)^3, \mathcal{T}^{\langle j \rangle}\rangle$ is toral. To see it, it is enough to note that $\mathcal{T}^{\langle j \rangle} \cong \mathbb{Z}_3$, and, once we have changed $(\widetilde{\sigma}_j s)^3$ with $\widetilde{\sigma}_{96} s'$, the image of the non-toral quasitorus by the projection $\wp_{96}$ should be $\mathcal{H}^{\langle 96 \rangle}$, as in Remark 9, but it is the trivial group.

★ Finally suppose that $m \geq 2$. Again we can check that there are 26 order 3 elements σ_j (including σ_{292} and σ_{292}^2) in $\mathcal{W}$ commuting with σ_{292}, 8 of them belonging to the orbit of σ_{3819}, 12 in the orbit of σ_{292}, and 6 of them in the orbit of σ_{4079}, the “toral orbit” (the computation of the stabilizer $\mathcal{T}^{\langle \sigma_j \rangle}$ is sufficient in this case for distinguishing the conjugacy class). But what it is useful is that either σ_j, $\sigma_j\sigma_{292}$, or $\sigma_j\sigma_{292}^2$ is not conjugate to σ_{292}. Besides $\mathcal{T}^{\langle 292 \rangle} \cap \mathcal{T}^{\langle j \rangle} \cong \mathbb{Z}_3^2$ in all such cases when σ_j is conjugate to σ_{3819}.

Hence we can assume that $\pi(g_1) = \sigma_{3819}$. In particular g_1 has order 3 and $Q \cap \mathcal{T} \subset \mathcal{T}^{\langle 3819 \rangle} \cong \mathbb{Z}_3^3$. If $Q \cap \mathcal{T}$ is isomorphic to $\mathbb{Z}_3$, we are done because $\langle Q \cap \mathcal{T}, g_1 \rangle$ is a non-toral elementary 3-group of rank 2. Also $Q \cap \mathcal{T}$ is not isomorphic to $\mathbb{Z}_3^3$, because in such a case $Q \supsetneq Q(g_1)$, but $Q(g_1)$ is a MAD-group. Hence $Q \cap \mathcal{T} \cong \mathbb{Z}_3^2$. If g_2 has order 3, then $Q \supset Q' = \langle g_1, g_2 \rangle \cdot \mathcal{T}^{\langle g_1 \rangle} \cap \mathcal{T}^{\langle g_2 \rangle} \cong \mathbb{Z}_3^4$. According again to Theorem 2 and Proposition 2, Q' is either of type $V_3^{4a} \equiv \mathcal{Q}_1 \supset \mathcal{P}_1$ or of type V_3^{4b}, which is identified with the order three elements in $\mathcal{Q}_2$, so that it also contains $\mathcal{P}_1$. Otherwise, $g_2^3 \in \mathcal{T} \cap P_3 \cong \mathbb{Z}_3^2$, hence g_2 has order 9 (and $\pi(g_2)$ is necessarily conjugate to σ_{292}). In particular there is $t \in \mathcal{T} \cap P_3$ of order 3 such that $P_3 \supset P_3' = \langle g_1, g_2, t \rangle \cong \mathbb{Z}_3^2 \times \mathbb{Z}_9$, which is non-toral. As g_2 is an inner automorphism of order 9, there is $h \in \operatorname{Aut} \mathfrak{e}_6$ such that $hg_2h^{-1} \in \mathcal{T}$ and $hg_1h^{-1}, hth^{-1} \in \mathfrak{N}_0(\mathcal{T})$. Thus $\pi(hg_1h^{-1})$ and $\pi(hth^{-1})$ are conjugate to σ_{292}, because both contain some $\mathbb{Z}_9$ in the stabilizer and P_3' is non-toral. We can move again with another automorphism, this time in $\mathfrak{N}(\mathcal{T})$, such that $\pi(hg_1h^{-1}) = \sigma_{292}$ and $\pi(hth^{-1}) = \sigma_j$. As in the paragraph above there is some $l = 0,1,2$ such that $\sigma_{292}^l\sigma_j$ is conjugate to either σ_{3819} or σ_{4079}. In the first case, the order 9 element $hg_2h^{-1} \in \mathcal{T}^{\langle 292 \rangle} \cap \mathcal{T}^{\langle j \rangle} = \mathcal{T}^{\langle \sigma_{292} \rangle} \cap \mathcal{T}^{\langle \sigma_{292}^l \sigma_j \rangle} \cong \mathbb{Z}_3^2$, a contradiction. In the second case, $\langle g_2, g_1^l t \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_9$ would be toral, so that we could conjugate P_3' to a subgroup of some $Q(k,s)$. But P_3' cannot fill $Q(k,s)$, because according to Table 1, there are no $Q(k,s)$

isomorphic to $\mathbb{Z}_3^2 \times \mathbb{Z}_9$. Thus $P_3' \neq Q$ and either $m > 2$ or $n > 0$. If $m > 2$, then $\langle \pi(g_1), \pi(g_2), \pi(g_3) \rangle$ is just the whole set of elements of order divisor of 3 which commute with σ_{292} (the 26 elements described before commute among them). But $\mathcal{T}^{\langle g_1 \rangle} \cap \mathcal{T}^{\langle g_2 \rangle} \cap \mathcal{T}^{\langle g_3 \rangle} = \{t_{\omega,\omega,\omega,\omega,\omega,\omega}\} \cong \mathbb{Z}_3$, so $Q \cap \mathcal{T} \neq \mathbb{Z}_3^2$. And if $n \neq 0$, $\pi(f_1)$ commutes with $\langle \sigma_{292}, \sigma_j \rangle$ for one of the 8 σ_j's commuting with σ_{292} and in the orbit of σ_{3819} (in fact, the number of candidates to σ_j can be reduced with considerations about orbits fixing σ_{292} or also taking into account that several of them generate the same groups). For some cases there does not exist an order two element in $\mathcal{W}$ commuting with $\langle \sigma_{292}, \sigma_j \rangle$, and, in all the cases in which there exists such an element, $\mathcal{T}^{\langle 292 \rangle} \cap \mathcal{T}^{\langle j \rangle} \cap \mathcal{T}^{\langle \pi(f_1) \rangle} \cong \mathbb{Z}_3$.

This finishes the proof of Proposition 1. $\square$

Now we can exhibit an example of a non-toral quasitorus of $\operatorname{Int} \mathfrak{e}_6$ which does not contain a non-toral elementary p-group of $\operatorname{Aut} \mathfrak{e}_6$. The quasitorus

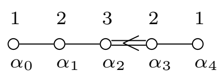
$$Q = \langle \widetilde{\sigma}_{292} s_{292}, t_{\xi,\xi,\xi,\xi^4,\xi,\xi^7} \rangle,$$

isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_9$ ($\xi^3 = \omega^2$), satisfies such condition, since $t_{\xi,\xi,\xi,\xi^4,\xi,\xi^7} \in \mathcal{T}^{\langle 292 \rangle} \setminus \mathcal{S}^{\langle 292 \rangle}$ but $(t_{\xi,\xi,\xi,\xi^4,\xi,\xi^7})^3 \in \mathcal{S}^{\langle 292 \rangle}$.

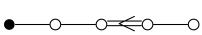
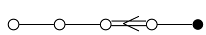
5. Description of the outer gradings

5.1. Outer automorphisms of finite order. The outer automorphisms of finite order m (necessarily even) can be obtained from the affine diagram

$E_6^{(2)}$:



by assigning weights $\bar{p} = (p_0, \dots, p_4)$ ($p_i \in \mathbb{Z}_{\geq 0}$) such that $2(p_0 + 2p_1 + 3p_2 + 2p_3 + p_4) = m$. Obviously the only possibilities for outer order two automorphisms (up to conjugation) are obtained when $\bar{p} = (1, 0, 0, 0, 0)$ and $\bar{p} = (0, 0, 0, 0, 1)$, choices which provide outer automorphisms with fixed subalgebras $\mathfrak{f}_4$ and $\mathfrak{c}_4$, related respectively to

 and  .

We will say that $F \in 2C$ (respectively $F \in 2D$), being F an order two outer automorphism, if its fixed subalgebra is isomorphic to $\mathfrak{f}_4$ (respectively to $\mathfrak{c}_4$) (we were taking the notation from [**27**], but that reference does not use these automorphisms, so that we have taken simply a notation compatible with Subsection 3.1).

Note that there are 3 conjugacy classes of outer automorphisms of order 4, corresponding to $(1,0,0,0,1)$, $(0,1,0,0,0)$, and $(0,0,0,1,0)$. The fixed subalgebras are of types $\mathfrak{c}_3 \oplus Z$, $\mathfrak{b}_3 \oplus \mathfrak{a}_1$, and $\mathfrak{a}_3 \oplus \mathfrak{a}_1$ respectively, and hence of dimensions 22, 24, and 18 respectively. This implies that the conjugacy class of an outer order 4 automorphism is distinguished only by the dimension of its fixed subalgebra. An element which will be useful for us is Υ_1: any (necessarily outer) automorphism of order 4 whose fixed subalgebra is of type $\mathfrak{a}_3 \oplus \mathfrak{a}_1$.

Recall that $\mathfrak{G} = \operatorname{Aut}\mathfrak{e}_6 = \operatorname{Int}\mathfrak{e}_6 \cup F \operatorname{Int}\mathfrak{e}_6$ for any $F \in \mathfrak{G} \setminus \mathfrak{G}_0$. In this section we study the maximal quasitori of $\operatorname{Aut}\mathfrak{e}_6$ not contained in the identity component $\mathfrak{G}_0 = \operatorname{Int}\mathfrak{e}_6$. First we describe those ones containing some automorphism of the class $2C$ and next those ones containing some automorphism of the class $2D$ (some of them coincide). Afterwards we consider the case when the MAD-group of $\mathfrak{G}$ does not contain any outer order two automorphism, although is not contained in $\mathfrak{G}_0$. In this case we will prove that the MAD-group contains necessarily an automorphism conjugate to Υ_1.

5.2. Gradings based on a $\mathfrak{f}_4$-model. Let $\mathcal{J}$ be the Albert algebra, and $\operatorname{Der}\mathcal{J}$ its derivation algebra, which is a Lie algebra of type $\mathfrak{f}_4$. Denote by $\mathcal{J}_0$ the set of zero trace elements of the Albert algebra. Take $\mathcal{N} := \mathcal{J}_0 \oplus \operatorname{Der}\mathcal{J}$ with the product given by

- the restriction of the bracket to $\operatorname{Der}\mathcal{J}$ is the usual bracket;
- if $d \in \operatorname{Der}\mathcal{J}$ and $x \in \mathcal{J}_0$, take $[d,x] := d(x) \in \mathcal{J}$;
- if $x, y \in \mathcal{J}_0$, take $[x,y] := [R_x, R_y] \in \operatorname{Der}\mathcal{J}$, where R_x denotes the multiplication operator in $\mathcal{J}$.

It is well-known that $\mathcal{N}$ is a Lie algebra of type $\mathfrak{e}_6$. Consider G_4 the order two automorphism producing the grading $\mathcal{N}_{\bar{0}} := \operatorname{Der}\mathcal{J}$ and $\mathcal{N}_{\bar{1}} := \mathcal{J}_0$.

It is also well-known that every automorphism of the Albert algebra can be extended to an automorphism of $\mathfrak{e}_6$. Namely, if $f \in \operatorname{Aut}\mathcal{J}$, take $f^\bullet \in \operatorname{Aut}\mathcal{N}$ given by $f^\bullet(d) = \operatorname{Ad} f(d) := f d f^{-1}$ if $d \in \operatorname{Der}\mathcal{J}$, and $f^\bullet(x) = f(x)$ if $x \in \mathcal{J}_0$. Moreover, $\mathfrak{C}_{\operatorname{Aut}\mathfrak{e}_6}(G_4) = \{f^\bullet, f^\bullet G_4 \mid f \in \operatorname{Aut}\mathcal{J}\} \cong \operatorname{Aut}\mathcal{J} \times \mathbb{Z}_2$. Indeed, if $\varphi \in \mathfrak{C}_{\operatorname{Aut}\mathfrak{e}_6}(G_4)$, then $\varphi(\mathcal{N}_{\bar{0}}) \subset \mathcal{N}_{\bar{0}}$, so that $\varphi|_{\mathcal{N}_{\bar{0}}} \in \operatorname{Aut}\operatorname{Der}\mathcal{J} = \operatorname{Ad}(\operatorname{Aut}\mathcal{J})$ and there is $f \in \operatorname{Aut}\mathcal{J}$ such that $\varphi|_{\mathcal{N}_{\bar{0}}} = \operatorname{Ad} f = f^\bullet|_{\mathcal{N}_{\bar{0}}}$. Now $\varphi(f^\bullet)^{-1}|_{\mathcal{N}_{\bar{1}}} \in \operatorname{Hom}_{\mathcal{N}_{\bar{0}}}(\mathcal{N}_{\bar{1}}, \mathcal{N}_{\bar{1}})$, so there is certain $\alpha \in \mathbb{F}^*$ such that $\varphi(f^\bullet)^{-1}|_{\mathcal{N}_{\bar{1}}} = \alpha\,\mathrm{id}$ by Schur's Lemma. As $[\mathcal{N}_{\bar{1}}, \mathcal{N}_{\bar{1}}] = \mathcal{N}_{\bar{0}}$, then $\alpha^2 = 1$ and $\varphi(f^\bullet)^{-1} \in \{\mathrm{id}, G_4\}$.

Hence any MAD-group of $\operatorname{Aut}\mathcal{N}$ containing G_4 is the direct product of a MAD-group of $\operatorname{Aut}\mathcal{J}$ (its copy by means of $\bullet$) with $\langle G_4 \rangle$, and conversely, any direct product of a MAD-group of $\operatorname{Aut}\mathcal{J}$ with $\langle G_4 \rangle$ is a MAD-group of $\operatorname{Aut}\mathcal{N}$.

The MAD-groups of $\operatorname{Aut}\mathcal{J}$ are completely described in [**19**]. According to it and with its notations, there are four MAD-groups, described by:

- $\{t_{x,y,z,u} \mid x,y,z,u \in \mathbb{F}^*\} \cong (\mathbb{F}^*)^4$, which produces fine gradings on $\mathcal{J}$ and $\operatorname{Der}\mathcal{J}$ of types $(24,0,1)$ and $(48,0,0,1)$ respectively.
- $\{t_{x,y,z,u} \mid x^2 = y^2 = z^2 = u^2 = 1\} \times \langle\widetilde{\sigma}_{405}\rangle \cong \mathbb{Z}_2^5$, which produces fine gradings on $\mathcal{J}$ and $\operatorname{Der}\mathcal{J}$ of types $(24,0,1)$ and $(24,0,0,7)$ respectively.
- $\{t_{x,y,xy,u} \mid x^2 = y^2 = 1, u \in \mathbb{F}^*\} \times \langle\widetilde{\sigma}_{105}\rangle \cong \mathbb{Z}_2^3 \times \mathbb{F}^*$, which produces fine gradings on $\mathcal{J}$ and $\operatorname{Der}\mathcal{J}$ of types $(25,1)$ and $(31,0,7)$ respectively.
- $\langle\{t_{\omega,1,\omega^2,\omega^2}, t_{1,\omega,\omega,1}, \widetilde{\sigma}_{15}\}\rangle \cong \mathbb{Z}_3^3$, which produces fine gradings on $\mathcal{J}$ and $\operatorname{Der}\mathcal{J}$ of types (26) and $(0,26)$ respectively.

Besides $\dim\mathcal{J}_e = 3,3,2,1$ respectively, so the type of the graded subspace $\mathcal{J}_0$ is $(24,1)$ in the first and second cases, and (26) in the third and fourth ones (taking into account that $1 \in \mathcal{J}$ belongs always to $\mathcal{J}_e$). Consequently, the MAD-groups of $\operatorname{Aut}\mathcal{N}$ containing G_4 (equivalently, the MAD-groups of $\operatorname{Aut}\mathfrak{e}_6$ containing an order two automorphism fixing a subalgebra of type $\mathfrak{f}_4$) are:

- $\mathcal{Q}_6 := \{t^\bullet_{x,y,z,u} \mid x,y,z,u \in \mathbb{F}^*\} \times \langle G_4\rangle \cong \mathbb{Z}_2 \times (\mathbb{F}^*)^4$, which produces a fine grading on $\mathcal{N}$ of type $(72,1,0,1)$.
- $\mathcal{Q}_7 := \{t^\bullet_{x,y,z,u} \mid x^2 = y^2 = z^2 = u^2 = 1\} \times \langle G_4, \widetilde{\sigma}^\bullet_{405}\rangle \cong \mathbb{Z}_2^6$, which produces a fine grading on $\mathcal{N}$ of type $(48,1,0,7)$.
- $\mathcal{Q}_8 := \{t^\bullet_{x,y,xy,u} \mid x^2 = y^2 = 1, u \in \mathbb{F}^*\} \times \langle G_4, \widetilde{\sigma}^\bullet_{105}\rangle \cong \mathbb{Z}_2^4 \times \mathbb{F}^*$, which produces a fine grading on $\mathcal{N}$ of type $(57,0,7)$.
- $\mathcal{Q}_9 := \langle\{G_4, t^\bullet_{\omega,1,\omega^2,\omega^2}, t^\bullet_{1,\omega,\omega,1}, \widetilde{\sigma}^\bullet_{15}\}\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_3^3$, which produces a fine grading on $\mathcal{N}$ of type $(26,26)$.

It is easy to give an automorphism conjugate to G_4 in terms of the model in Equation (2). If we take G_4' the automorphism interchanging V_1 with V_2, then the fixed subalgebra is $\{f_1+f_2 \mid f \in \mathrm{sl}(V)\} \oplus \mathrm{sl}(V_3) \oplus \{(v_1 \otimes v_2 + v_2 \otimes v_1) \otimes v_3 \mid v_i \in V\} \oplus S^2(V^*) \otimes V_3^*$, which is isomorphic to $\mathfrak{f}_4$ (see, for instance, [**14**]). With this terminology, $\langle G_4', F_1, F_3, F_4\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_3^3$ is a MAD-group conjugate to $\mathcal{Q}_9$. (Thus there are two fine gradings refining the Jordan grading produced by $\langle F_1, F_3, F_4\rangle$ described in Remark 5.) Observe that this MAD-group only contains one order two element, in particular it does not contain automorphisms of type $2D$.

5.3. Gradings based on a $\mathfrak{c}_4$-model. Take $G_5 \in 2D$. There are some 8-dimensional vector space V and a non-degenerate symplectic bilinear form $b\colon V \times V \to \mathbb{F}$ such that the subalgebra of $\mathcal{L} = \mathfrak{e}_6$ fixed by G_5 is (isomorphic to) $\mathfrak{sp}(V,b) = \{f \in \operatorname{End} V \mid b(f(x),y) + b(x,f(y)) = 0\ \forall\, x,y \in V\}$, a simple Lie algebra of type $\mathfrak{c}_4$. Recall from [**34**, Chapter 8] that, if $\mathcal{L}_{\bar 0} \oplus \mathcal{L}_{\bar 1}$ is the $\mathbb{Z}_2$-grading induced by G_5, then $\mathcal{L}_{\bar 1}$ is an irreducible $\mathcal{L}_{\bar 0}$-module. But the only irreducible $\mathfrak{c}_4$-module of dimension $78-36 = 42$ is $V(\lambda_4)$ (λ_i's the fundamental weights as in [**32**]). A suitable way of describing it is as a submodule of $\wedge^4 V$, which has dimension $\binom{8}{4} = 70$. The decomposition of this module into its irreducible summands is $\wedge^4 V \cong V(\lambda_4) \oplus V(\lambda_2) \oplus V(0)$. Thus, if we consider the contraction

$$\begin{array}{llll} c\colon & \bigwedge^4 V & \to & \bigwedge^2 V \ \ (\cong V(\lambda_2) \oplus V(0)) \\ & v_1 \wedge v_2 \wedge v_3 \wedge v_4 & \mapsto & \sum_{\substack{\sigma \in S_4 \\ \sigma(1)<\sigma(2) \\ \sigma(3)<\sigma(4)}} (-1)^\sigma b(v_{\sigma(1)}, v_{\sigma(2)}) v_{\sigma(3)} \wedge v_{\sigma(4)} \end{array}$$

for S_4 the group of permutations of $\{1,2,3,4\}$, its kernel $\ker c$ is isomorphic to $\mathcal{L}_{\bar 1}$. This construction seems not to be natural, but one only has to recall that the Lie algebra of type $\mathfrak{e}_7$ can be modeled as $\mathrm{sl}(V) \oplus \wedge^4 V$ and that $\mathfrak{e}_6$ lives there. Moreover, it has been described, for instance, in [**38**, Chapter 5, §2, Example 2]. The main reason for using it is that it is quite easy to extend the automorphisms of $\mathfrak{c}_4$ until the whole $\mathfrak{e}_6$. Recall that $\operatorname{Aut} \mathfrak{c}_4 \cong \mathrm{SP}(V)$ and take the map

$$\mathrm{SP}(V) = \{f \in \operatorname{End} V \mid b(f(x), f(y)) = b(x,y)\ \forall\, x,y \in V\} \to \operatorname{Aut} \mathcal{L},\ f \mapsto f^\diamondsuit$$

given by $f^\diamondsuit(g) = f^{-1} g f$ if $g \in \mathcal{L}_{\bar 0} = \mathfrak{sp}(V,b)$ and $f^\diamondsuit(v) = \sum f(v_{i_1}) \wedge f(v_{i_2}) \wedge f(v_{i_3}) \wedge f(v_{i_4})$ if $v = \sum v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} \in \mathcal{L}_{\bar 1} = \ker c$. It is a computation similar to that one in Subsection 5.2 that $\mathfrak{C}_{\operatorname{Aut} \mathfrak{e}_6}(G_5) = \{f^\diamondsuit \mid f \in \mathrm{SP}(V)\} \times \langle G_5 \rangle \cong \mathrm{SP}(V) \times \mathbb{Z}_2$. This implies that every MAD-group of $\operatorname{Aut} \mathfrak{e}_6$ containing G_5 (that is, containing some automorphism of the isomorphy class $2D$) is of the form $\{f^\diamondsuit \mid f \in Q\} \times \langle G_5 \rangle$ for some MAD-group Q of $\mathrm{SP}(V)$. There are seven MAD-groups of $\operatorname{Aut} \mathfrak{c}_4$, according to [**28, 23**]. The induced fine gradings can also be extracted from [**7**], although in such paper there is one missing grading. Although we know that we will obtain just seven MAD-groups of $\operatorname{Aut} \mathfrak{e}_6$ by means of this procedure, we don't know a priori how many of these MAD-groups have appeared before (equivalently, how many of these MAD-groups contain an outer order two automorphism of type $2C$). Thus we are going to recall the descriptions of these quasitori, and extend each automorphism of $\mathfrak{c}_4$ to $\mathfrak{e}_6$ to get the complete simultaneous diagonalizations. We will make use of the notations of [**28**] for giving the MAD-groups of $\mathrm{SP}(V)$.

According to it, the MAD-groups are

$$\Xi_1 = T^{(0)}_{8,0}; \quad \Xi_2 = T^{(1)}_{4,0}(I_4, I_4); \quad \Xi_3 = T^{(1)}_{2,2}(I_4, I_4); \quad \Xi_4 = T^{(1)}_{0,4}(I_4, I_4);$$

$$\Xi_5 = T^{(2)}_{0,2}(((\begin{smallmatrix} 1 & 0 \\ 0 & \mathfrak{i} \end{smallmatrix}), I_2), (I_2, I_2));$$

$$\Xi_6 = T^{(2)}_{2,0}((I_2, I_2), (I_2, I_2)); \quad \Xi_7 = T^{(3)}_{0,1}((1,1), (1,1), (1,1)),$$

for $\mathfrak{i} \in \mathbb{F}$ such that $\mathfrak{i}^2 = -1$. We try to avoid that the reader has to dive into the details of such paper by providing the exact descriptions of the automorphisms in the Ξ_i's. We will use the so called *Pauli's matrices*, given by

$$\theta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta_2 = \theta_3\theta_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As usual, if $A = (a_{ij}) \in \mathrm{Mat}_{m\times n}(\mathbb{F})$ and $B \in \mathrm{Mat}_{p\times q}(\mathbb{F})$, the Kronecker product $A \otimes B$ denotes the block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix} \in \mathrm{Mat}_{mp\times nq}(\mathbb{F}).$$

- Ξ_1. This is the case of the maximal 4-dimensional torus of $\mathrm{SP}(V)$, which produces a $\mathbb{Z}^4$-grading on $\mathfrak{c}_4$ of type $(32, 0, 0, 1)$.
- Ξ_2. We fix some basis $\mathcal{B} = \{w_1, \dots, w_8\}$ of V such that the matrix of the form b relative to $\mathcal{B}$ is the skew-symmetric matrix $I_2 \otimes \theta_1 \otimes \theta_2$ (take into account that $(A \otimes B)^t = A^t \otimes B^t$). Take in $\mathrm{SP}(V, b)$ the automorphisms of V whose related matrices in the basis $\mathcal{B}$ are

  $$\Xi_2 = \langle r_{\alpha,\beta}, I_4 \otimes \theta_1, I_4 \otimes \theta_3 \mid \alpha, \beta \in \mathbb{F}^* \rangle \cong (\mathbb{F}^*)^2 \times \mathbb{Z}_2^2,$$

  for $r_{\alpha,\beta} = \mathrm{diag}\{\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}\} \otimes I_2$. The induced $\mathbb{Z}^2 \times \mathbb{Z}_2^2$-grading on $\mathfrak{c}_4$ is of type $(28, 4)$.
- Ξ_3. We fix some basis of V such that the matrix of the form b relative to it is the skew-symmetric matrix $\mathrm{diag}\{I_2, \theta_1\} \otimes \theta_2$, and take

$$\Xi_3 = \langle r_{1,\alpha}, I_4 \otimes \theta_1, I_4 \otimes \theta_3, \mathrm{diag}\{-I_2, I_2, I_2, I_2\} \mid \alpha \in \mathbb{F}^* \rangle \cong \mathbb{F}^* \times \mathbb{Z}_2^3.$$

  The induced $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on $\mathfrak{c}_4$ is of type $(27, 0, 3)$.

- Ξ_4. We fix some basis of V such that the matrix of the form b relative to it is $I_4 \otimes \theta_2$, and take

$$\Xi_4 = \langle I_4 \otimes \theta_1, I_4 \otimes \theta_3, \mathrm{diag}\{-1,1,1,1\} \otimes I_2, \mathrm{diag}\{1,-1,1,1\} \otimes I_2, \\ \mathrm{diag}\{1,1,-1,1\} \otimes I_2\rangle \cong \mathbb{Z}_2^5.$$

  The induced $\mathbb{Z}_2^5$-grading on $\mathfrak{c}_4$ is of type $(24,0,0,3)$.
- Ξ_5. We fix a basis of V relative to which the matrix of the form b is $\mathrm{diag}\{\theta_2 \otimes I_2, \theta_2 \otimes \theta_1\}$, and take

$$\Xi_5 = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{smallmatrix}\right) \otimes I_2 \otimes \theta_3, I_2 \otimes \theta_1 \otimes I_2, I_2 \otimes \theta_3 \otimes I_2, I_4 \otimes \theta_1\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2^3.$$

  The induced $\mathbb{Z}_4 \times \mathbb{Z}_2^3$-grading on $\mathfrak{c}_4$ is of type $(24,6)$.
- Ξ_6. We fix some basis relative to which the matrix of the form b is $\theta_1 \otimes \theta_2 \otimes I_2$, and take

$$\Xi_6 = \langle \mathrm{diag}\{\alpha, \frac{1}{\alpha}\} \otimes I_4, I_4 \otimes \theta_1, I_4 \otimes \theta_3, I_2 \otimes \theta_1 \otimes I_2, \\ I_2 \otimes \theta_3 \otimes I_2 \mid \alpha \in \mathbb{F}^*\rangle \cong \mathbb{F}^* \times \mathbb{Z}_2^4.$$

  The induced $\mathbb{Z} \times \mathbb{Z}_2^4$-grading on $\mathfrak{c}_4$ is of type (36).
- Ξ_7. We fix some basis relative to which the matrix of the form b is $\theta_2 \otimes I_4$, and take

$$\Xi_7 = \langle I_4 \otimes \theta_1, I_4 \otimes \theta_3, I_2 \otimes \theta_1 \otimes I_2, I_2 \otimes \theta_3 \otimes I_2, \theta_1 \otimes I_4, \theta_3 \otimes I_4\rangle \cong \mathbb{Z}_2^6.$$

  The induced $\mathbb{Z}_2^6$-grading on $\mathfrak{c}_4$ is of type (36).

Take into consideration now that $G_5 \cdot \mathrm{diag}\{-I_2, I_2, I_2, I_2\}^\diamond$ is an automorphism of the class $2C$, and hence that we do not get anything new from Ξ_1, Ξ_3, and Ξ_4.

Extending the automorphisms and making the simultaneous diagonalization is a tedious task, although straighforward. Let us provide some details of the first of our significant cases, Ξ_2. Denote by $w_{ijkl} = w_i \wedge w_j \wedge w_k \wedge w_l \in \wedge^4 V$, so that $\{w_{ijkl} \mid 1 \leq i < j < k < l \leq 8\}$ is a basis of $\wedge^4 V$. Denote by $L_{(i,j,k)} = \{x \in \ker c \mid r_{\alpha,\beta}(x) = ix, I_4 \otimes \theta_1(x) = jx, I_4 \otimes \theta_3(x) = kx\}$ the homogeneous components of the grading produced by G_5 and Ξ_2, restricted to the odd part. Then all the homogeneous components are one-dimensional (for instance, $L_{(\alpha^2\beta^2,1,1)} = \langle w_{1256}\rangle$ and so on) except for one four-dimensional component

$$L_{(1,1,1)} = \langle w_{1368} + w_{2457}, w_{1458} + w_{1467} + w_{2358} + w_{2367}, \\ w_{1357} + w_{2468}, w_{1234} - w_{1467} - w_{2358} + w_{5678}\rangle,$$

and three two-dimensional components

$$L_{(1,-1,1)} = \langle w_{1357} - w_{2468}, w_{1368} - w_{2457} \rangle,$$
$$L_{(1,1,-1)} = \langle w_{1358} + w_{1367} + w_{2458} + w_{2467}, w_{1457} + w_{2357} + w_{2368} + w_{1468} \rangle,$$
$$L_{(1,-1,-1)} = \langle w_{1358} + w_{1367} - w_{2458} - w_{2467}, w_{1457} + w_{2357} - w_{2368} - w_{1468} \rangle.$$

The type of the induced grading on $\mathfrak{e}_6$ is hence $(28,4,0,0)+(32,3,0,1) = (60,7,0,1)$.

In the same way, we check that the type of the grading induced by Ξ_6 restricted to $\mathcal{L}_{\bar{1}} = \ker c$ is $(37,0,0,0,1)$ because all the homogeneous components have dimension 1 except for one of dimension 5:

$$\mathfrak{fix}(\Xi_6) \cap \mathcal{L}_{\bar{1}} = \langle w_{1368} + w_{2457}, w_{1458} + w_{2367}, w_{1357} - w_{1467} - w_{2358} + w_{2468}, \\ w_{1256} + w_{3478}, w_{1278} - w_{1467} - w_{2358} + w_{3456} \rangle.$$

Observe that in this case we are not dealing with the same contraction c than for Ξ_2 (since it depends of the bilinear form b), although we use the same notation.

Finally, it is trivial to compute the types of the gradings induced by Ξ_5 and Ξ_7 restricted to $\mathcal{L}_{\bar{1}} = \ker c$, which are, respectively, $(24,7,0,1)$ and $(36,0,0,0,0,1)$.

Consequently, the MAD-groups of $\operatorname{Aut} \mathcal{L}$ containing G_5 (equivalently, the MAD-groups of $\operatorname{Aut} \mathfrak{e}_6$ containing an order two automorphism fixing a subalgebra of type $\mathfrak{c}_4$) but not conjugate to any MAD-group in Subsection 5.2 are:

- $\mathcal{Q}_{10} := \{f^\diamondsuit \mid f \in \Xi_2\} \times \langle G_5 \rangle \cong (\mathbb{F}^*)^2 \times \mathbb{Z}_2^3$, which induces a fine grading of type $(60,7,0,1)$.
- $\mathcal{Q}_{11} := \{f^\diamondsuit \mid f \in \Xi_5\} \times \langle G_5 \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2^4$, which induces a fine grading of type $(48,13,0,1)$.
- $\mathcal{Q}_{12} := \{f^\diamondsuit \mid f \in \Xi_6\} \times \langle G_5 \rangle \cong \mathbb{F}^* \times \mathbb{Z}_2^5$, which induces a fine grading of type $(73,0,0,0,1)$.
- $\mathcal{Q}_{13} := \{f^\diamondsuit \mid f \in \Xi_7\} \times \langle G_5 \rangle \cong \mathbb{Z}_2^7$, which induces a fine grading of type $(72,0,0,0,0,1)$.

5.4. A MAD-group containing outer automorphisms but without outer order two automorphisms. As was mentioned in the introduction, there are simple Lie algebras L such that there exist MAD-groups of $\operatorname{Aut} L$ containing outer automorphisms but without outer order two automorphisms. In this section we will describe how this situation occurs again for L of type $\mathfrak{e}_6$.

We are going to describe a fine $\mathbb{Z}_4^3$-grading on $\mathfrak{e}_6$. Take Υ_1 any outer automorphism of order 4 which fixes a subalgebra of type $\mathfrak{a}_3 \oplus \mathfrak{a}_1$, as in Subsection 5.1.

Remark 11. We can find an explicit automorphism of this conjugacy class with our descriptions in Subsection 5.2. For instance, the automorphism $t = t_{-\mathrm{i},\mathrm{i},-1,\mathrm{i}}$ produces a $\mathbb{Z}_4$-grading on $\mathcal{J}$ with components of dimensions 9, 6, 6, and 6, and $\operatorname{Ad} t$ produces a $\mathbb{Z}_4$-grading on $\operatorname{Der}\mathcal{J}$ with components of dimensions 12, 14, 12, and 14. Thus the subalgebra fixed by $G_4 t^\bullet \in \mathcal{Q}_6$ is the sum of the elements of $\operatorname{Der}\mathcal{J}$ fixed by $\operatorname{Ad} t$ with the elements of $\mathcal{J}_0$ antifixed by t, whose dimension is $12 + 6 = 18$, so that $G_4 t^\bullet_{-\mathrm{i},\mathrm{i},-1,\mathrm{i}}$ can be taken as Υ_1. And in terms of the notations used in Section 3, it is conjugate to the automorphism $G_4'' T_{1,\mathrm{i}}$ for G_4'' the automorphism interchanging V_1 with V_3 in Equation (2).

Consider now the $\mathbb{Z}_4$-grading on $\mathfrak{e}_6$ induced by Υ_1, $\mathcal{L} = \mathcal{L}_{\bar 0} \oplus \mathcal{L}_{\bar 1} \oplus \mathcal{L}_{\bar 2} \oplus \mathcal{L}_{\bar 3}$. Recall that $\mathcal{L}_{\bar i}$ must be an $\mathcal{L}_{\bar 0}$-irreducible module (if $i \neq 0$), that is, a tensor product of an irreducible $\mathfrak{a}_3$-module with an irreducible $\mathfrak{a}_1$-module. If we also take into account that $\dim \operatorname{Hom}_{\mathcal{L}_{\bar 0}}(\mathcal{L}_{\bar i} \otimes \mathcal{L}_{\bar j}, \mathcal{L}_{\bar i + \bar j}) = 1$, then the unique possibility for the decomposition of $\mathcal{L}$ as a sum of $\mathcal{L}_{\bar 0}$-modules is

$$\mathfrak{a}_3 \oplus \mathrm{sl}(V) \oplus V(2\lambda_1) \otimes V \oplus V(2\lambda_2) \otimes \mathbb{F} \oplus V(2\lambda_3) \otimes V$$

for V a two-dimensional vector space, and $\{\lambda_i \mid i = 1, 2, 3\}$ the set of fundamental weights for $\mathfrak{a}_3$. The dimension of each non-identity homogeneous component is 20.

Take W a four-dimensional vector space, so that $\mathfrak{a}_3 \cong \mathrm{sl}(W)$. The natural module W is isomorphic to $V(\lambda_1)$ and its second symmetric power $S^2(W)$ is isomorphic to $V(2\lambda_1)$. Their dual $\mathrm{sl}(W)$-modules, W^* and $S^2(W^*)$, are respectively of types $V(\lambda_3)$ and $V(2\lambda_3)$. Finally $\bigwedge^2 W$ is of type $V(\lambda_2)$ and its second symmetric power $S^2(\bigwedge^2 W) \cong V(2\lambda_2) \oplus V(0)$. So consider $S^2(\bigwedge^2 W)'$ its only non-trivial submodule (of type $V(2\lambda_2)$). We have an isomorphism of $\mathrm{sl}(W) \oplus \mathrm{sl}(V)$-modules between $\mathfrak{e}_6$ and

$$\mathcal{N} := \mathrm{sl}(W) \oplus \mathrm{sl}(V) \oplus S^2(W) \otimes V \oplus S^2\Big(\bigwedge^2 W\Big)' \otimes \mathbb{F} \oplus S^2(W^*) \otimes V^*$$

which endows $\mathcal{N}$ with a $\mathbb{Z}_4$-graded Lie algebra structure such that $\mathcal{N}_{\bar 1} = S^2(W) \otimes V$, $\mathcal{N}_{\bar 2} = S^2(\bigwedge^2 W)' \otimes \mathbb{F} = [\mathcal{N}_{\bar 1}, \mathcal{N}_{\bar 1}]$, and $\mathcal{N}_{\bar 3} = S^2(W^*) \otimes V^* = [\mathcal{N}_{\bar 2}, \mathcal{N}_{\bar 1}]$. Thus Υ_1 can be considered as the automorphism of $\mathcal{N} \cong \mathfrak{e}_6$ producing this $\mathbb{Z}_4$-grading.

Now, for each $f \in \operatorname{End} W$ and $g \in \operatorname{End} V$, denote by $\operatorname{ext}(f \otimes g) \in \operatorname{End} \mathcal{N}_{\bar{1}}$ the map given by $\operatorname{ext}(f \otimes g)(w \cdot w' \otimes v) = f(w) \cdot f(w') \otimes g(v)$, for all $w, w' \in W$ and $v \in V$, where $\cdot$ denotes the symmetric product.

Define the automorphisms Υ_2 and Υ_3 as the only automorphisms of $\mathcal{N}$ ($\cong \mathfrak{e}_6$) whose restrictions to $\mathcal{N}_{\bar{1}}$ are:

$$\Upsilon_2|_{\mathcal{N}_{\bar{1}}} = \operatorname{ext}\left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}_W \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_V\right),$$

$$\Upsilon_3|_{\mathcal{N}_{\bar{1}}} = \operatorname{ext}\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathfrak{i} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\mathfrak{i} \end{pmatrix}_W \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_V\right),$$

where we have chosen $\{w_0, w_1, w_2, w_3\}$ and $\{v_0, v_1\}$ basis of W and V respectively and we have identified the endomorphisms of W and V with their matrices in such bases. Thus $\{w_j \cdot w_k \otimes v_l \mid 1 \le j \le k \le 4,\ l = 0, 1\}$ is a basis of $\mathcal{N}_{\bar{1}}$, and the action of the automorphism in it is $\Upsilon_2(w_j \cdot w_k \otimes v_l) = w_{j+1} \cdot w_{k+1} \otimes v_{l+1}$ (j and k summed modulo 4 and l summed modulo 2) and $\Upsilon_3(w_j \cdot w_k \otimes v_l) = (\mathfrak{i})^{j+k+2l} w_j \cdot w_k \otimes v_l$. It is a straightforward computation that the extensions Υ_2 and Υ_3 are Lie algebra automorphisms.

Now $\Upsilon_2\Upsilon_3(w_j \cdot w_k \otimes v_l) = (\mathfrak{i})^{j+k+2l} w_{j+1} \cdot w_{k+1} \otimes v_{l+1} = \Upsilon_3\Upsilon_2(w_j \cdot w_k \otimes v_l)$, so that Υ_2 and Υ_3 commute. Consider, then, the quasitorus of $\operatorname{Aut} \mathfrak{e}_6$ given by

$$\mathcal{Q}_{\mathbf{14}} := \langle \Upsilon_1, \Upsilon_2, \Upsilon_3 \rangle \cong \mathbb{Z}_4^3.$$

Let us compute the simultaneous diagonalization of $\mathcal{N}$ relative to $\mathcal{Q}_{14}$. In Proposition 4 we will prove that it is a maximal quasitorus and hence the induced grading is fine. Again the common diagonalization is long but straightforward. Denote by $L_{(\bar{i},\bar{j},\bar{k})} = \{x \in \mathcal{N} \mid \Upsilon_1(x) = \mathfrak{i}^i x,\ \Upsilon_2(x) = \mathfrak{i}^j x,\ \Upsilon_3(x) = \mathfrak{i}^k x\}$ if $i, j, k \in \{0, 1, 2, 3\}$. Of course $\langle \Upsilon_2|_{\mathfrak{fix}\Upsilon_1}, \Upsilon_3|_{\mathfrak{fix}\Upsilon_1} \rangle$ produce the non-toral $\mathbb{Z}_4^2$-grading on $\mathfrak{a}_3$ with trivial identity component and all the remaining 15 components of dimension 1, and produce the non-toral $\mathbb{Z}_2^2$-grading on $\mathfrak{a}_1$ given by Pauli's matrices. Thus $\dim L_{(\bar{0},\bar{0},\bar{0})} = 0$, $\dim L_{(\bar{0},\bar{2},\bar{0})} = \dim L_{(\bar{0},\bar{0},\bar{2})} = \dim L_{(\bar{0},\bar{2},\bar{2})} = 2$, and $\dim L_{(\bar{0},\bar{i},\bar{j})} = 1$ for each $(\bar{i}, \bar{j}) \in \mathbb{Z}_4^2$ with $2(\bar{i}, \bar{j}) \neq (\bar{0}, \bar{0})$. So the type of the restriction to $\mathcal{N}_{\bar{0}} \cong \mathfrak{a}_3 \oplus \mathfrak{a}_1$ is $(12, 3)$. Now consider the restriction to $\mathcal{N}_{\bar{1}}$. Observe that $\Upsilon_2^2(x = w_0 \cdot w_0 \otimes v_k) \neq w_0 \cdot w_0 \otimes v_k$, so that Υ_2 acts with eigenvalue $\varepsilon \in \{1, i, -1, -i\}$ in $x + \varepsilon^3\Upsilon_2(x) + \varepsilon^2\Upsilon_2^2(x) + \varepsilon\Upsilon_2^3(x)$, and the same happens to $x = w_0 \cdot w_1 \otimes v_k$. On the contrary, $\Upsilon_2^2(x = w_0 \cdot w_2 \otimes v_k) = w_0 \cdot w_2 \otimes v_k$, and hence Υ_2 acts with eigenvalue ± 1 in $x \pm \Upsilon_2(x)$. Hence $\dim L_{(\bar{1},\bar{0},\bar{0})} =$

$\dim L_{(\bar1,\bar2,\bar0)} = \dim L_{(\bar1,\bar0,\bar2)} = \dim L_{(\bar1,\bar2,\bar2)} = 2$ and $\dim L_{(\bar1,\bar i,\bar j)} = 1$ for the remaining i, j, and the type of the restriction to $\mathcal{N}_{\bar1}$ is $(12,4)$. It is not difficult to work with the other two components to conclude that both $\mathcal{L}_{\bar2}$ and $\mathcal{L}_{\bar3}$ break into 12 one-dimensional components and 4 two-dimensional components, therefore the type of the grading induced by $\mathcal{Q}_{14}$ in $\mathfrak{e}_6$ is $(12,3)+3(12,4) = (48,15)$. More precisely, the identity component is trivial, $\dim L_{(\bar i,\bar j,\bar k)} = 2$ if $j,k \in \{0,2\}$ and all the remaining homogeneous components are one-dimensional. In particular, $\dim \mathfrak{fix}(\langle \Upsilon_1^2, \Upsilon_2, \Upsilon_3\rangle) = \dim L_{(\bar2,\bar0,\bar0)} = 2 < 6$, and hence the quasitorus $\langle \Upsilon_1^2, \Upsilon_2, \Upsilon_3\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4^2$ is non-toral. Observe also that $\mathfrak{fix}(\langle \Upsilon_2, \Upsilon_3\rangle)$ is just a Cartan subalgebra and hence $\langle \Upsilon_2, \Upsilon_3\rangle$ is contained in a torus. These considerations will be useful to argument later with $\mathcal{Q}_{14}$.

5.5. All the outer fine gradings. As a corollary of Propositions 4 and 5 in the next technical section, we will obtain the other main result of the paper:

Theorem 4. *The MAD-groups of* $\operatorname{Aut}\mathfrak{e}_6$ *not contained in* $\operatorname{Int}\mathfrak{e}_6$ *are, up to conjugation,* $\mathcal{Q}_i$ *for* $i = 6,\dots,14$.

Proof: If A is a MAD-group of $\operatorname{Aut}\mathfrak{e}_6$ such that $A \not\subseteq \operatorname{Int}\mathfrak{e}_6$, then either A has an outer automorphism of type $2C$, and in such case A is conjugate to some $\mathcal{Q}_i$ with $i \in \{6,7,8,9\}$; or A has an outer automorphism of type $2D$, and in such case A is conjugate to some $\mathcal{Q}_i$ with $i \in \{6,7,8,10,11,12,13\}$; or A has not any outer automorphism of order two, and in such case A is conjugate to $\mathcal{Q}_{14}$ by Proposition 5. The quasitorus $\mathcal{Q}_{14}$ appears in this list by Proposition 4. $\square$

6. Technical proofs for the outer gradings

6.1. Extended Weyl group. The diagram automorphism interchanging α_3 with α_5, and α_1 with α_6, has matrix relative to Δ:

$$\sigma = \begin{pmatrix} 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}$$

and the extended Weyl group is $\mathcal{V} := \mathcal{W} \cup \mathcal{W}\sigma \cong \mathcal{W} \rtimes \mathbb{Z}_2$, which is the set of automorphisms of the root system.

We will make an extensive use of those orbits in $\mathcal{V}$ whose representatives have order a power of 2. We find 10 new orbits (those that are not contained in $\mathcal{W}$ and hence they did not appear in Table 1), and we summarize the information related to them in the following table:

Order	Rep.	Centralizer in $\mathcal{T}$	Iso. to
2	$\sigma \equiv \eta_1$	$t_{x,y,z,u,z,x}$	$(\mathbb{F}^*)^4$
2	$\sigma\sigma_{555} \equiv \eta_2$	$t_{\frac{z}{w^2y^4},y,z,\frac{1}{z},y,w}$	$(\mathbb{F}^*)^3$
2	$\sigma\sigma_{458} \equiv \eta_3$	$t_{x,vx,z,x,v,\frac{\alpha}{zv^2}} \mid x^2=\alpha^2=1$	$(\mathbb{F}^*)^2 \times \mathbb{Z}_2^2$
2	$\sigma\sigma_{2402} \equiv \eta_4$	$t_{x,y,z,u,xy,\frac{\alpha}{z}} \mid x^2=y^2=u^2=\alpha^2=1$	$\mathbb{F}^* \times \mathbb{Z}_2^4$
2	$-\operatorname{id} = \eta_5$	$t_{x,y,z,u,v,w} \mid x^2=y^2=z^2=u^2=v^2=w^2=1$	$\mathbb{Z}_2^6$
4	$\sigma\sigma_{15} \equiv \mu_1$	$t_{x,y,\frac{1}{x^3y},x,x^3y,\frac{1}{x^2y}}$	$(\mathbb{F}^*)^2$
4	$\sigma\sigma_{52} \equiv \mu_2$	$t_{x,\frac{zx^2}{w^2},z,\frac{w^2}{x^3z^2},\frac{zx^2}{w^2},w}$	$(\mathbb{F}^*)^3$
4	$\sigma\sigma_{460} \equiv \mu_3$	$t_{x,y,x,x,xy,\frac{\alpha}{y^2}} \mid x^2=\alpha^2=1$	$\mathbb{F}^* \times \mathbb{Z}_2^2$
4	$\sigma\sigma_{484} \equiv \mu_4$	$t_{x,y,x,x,xy,x^2} \mid x^4=y^4=1$	$\mathbb{Z}_4^2$
8	$\sigma\sigma_{17}$	$t_{x,\frac{1}{x^2},\frac{1}{x},x,x,1}$	$\mathbb{F}^*$

TABLE 2. Table of representatives of $\mathcal{W}\sigma$ of order a power of 2.

We use the notations η_i and μ_i for the order 2 and 4 representatives respectively, in order to shorten the usage of indices. The only restriction about the scalars in the third column when nothing is said is that they are non-zero.

This time the projection $\pi\colon \mathfrak{N}(\mathcal{T}) \to \mathfrak{N}(\mathcal{T})/\mathcal{T} \cong \mathcal{V}$ is an extension of the projection π considered in Subsection 4.1. For each $\nu \in \mathcal{W}\sigma$ we choose some element $\widetilde{\nu} \in \pi^{-1}(\nu)$ of minimum order among the elements in $\pi^{-1}(\nu)$. (Perhaps our choice of $\widetilde{\nu}$ when $\nu \in \mathcal{W}$ does not coincide with that one in Subsection 4.1, where we made a more concrete election not based in the order, but this does not interfere with our next arguments.)

Remark 12. A consequence of the appendix, and of our elections of extensions, is that all those extensions $\widetilde{\eta}_j$ have order 2. It will be quite useful in the next proofs that this fact (the existence of some order two extensions) jointly with Remark 10 imply that $(\widetilde{\eta}_j s)^2 \in \mathcal{S}^{\langle \eta_j \rangle}$ for all $s \in \mathcal{T}$ and for all j.

6.2. The $\mathbb{Z}_4^3$-grading in computational terms. Recall that in Subsection 5.4 we found a quasitorus $\mathcal{Q}_{14} = \langle \Upsilon_1, \Upsilon_2, \Upsilon_3 \rangle$, isomorphic as abstract group to $\mathbb{Z}_4^3$, satisfying the following conditions:

- Υ_1 is an outer automorphism;
- $\langle \Upsilon_2, \Upsilon_3\rangle$ is toral;
- $\langle \Upsilon_1^2, \Upsilon_2, \Upsilon_3\rangle \subset \operatorname{Int}\mathfrak{e}_6$ is non-toral.

A first consequence is that

Proposition 4. *$\mathcal{Q}_{14}$ is conjugate to $Q(\widetilde{\mu}_4)$. Moreover, $\mathcal{Q}_{14}$ is a MAD-group.*

Proof: By Lemma 2, we can assume that $\langle \Upsilon_2, \Upsilon_3\rangle \subset \mathcal{T}$ and that $\Upsilon_1 \in \mathfrak{N}(\mathcal{T})$ (by conjugating, if necessary). So $\pi(\Upsilon_1)$ is an element in $\mathcal{V}\setminus\mathcal{W}$ of order a divisor of 4. But such order cannot be 2, since in such a case we could suppose that $\Upsilon_1 = \widetilde{\eta}_j s$ for some $j = 1,\dots,5$ and $s \in \mathcal{T}$ and then $\Upsilon_1^2 \in \mathcal{T}$, so that $\langle \Upsilon_1^2, \Upsilon_2, \Upsilon_3\rangle \subset \mathcal{T}$ would be toral. Thus there is some $j = 1,\dots,4$ and some $s \in \mathcal{T}$ such that $\Upsilon_1 = \widetilde{\mu}_j s$ (again after conjugating) and $\langle \Upsilon_2, \Upsilon_3\rangle \subset \mathcal{T}^{\langle \mu_j\rangle}$. Note that $j \neq 3$ because there is no subgroup isomorphic to $\mathbb{Z}_4^2$ contained in $\mathcal{T}^{\langle \mu_3\rangle} \cong \mathbb{F}^* \times \mathbb{Z}_2^2$. Moreover, $j \neq 1,2$ because in such cases $\mathcal{T}^{\langle \mu_j\rangle}$ is a torus, so that $\langle \Upsilon_1^2, \Upsilon_2, \Upsilon_3\rangle \subset \langle \Upsilon_1^2, \mathcal{T}^{\langle \mu_j\rangle}\rangle$ would be toral (we can apply Lemma 8 because Υ_1^2 is inner). Hence $\mathcal{Q}_{14} \subset Q(\widetilde{\mu}_4 s)$ and they must coincide ($\widetilde{\mu}_4 s$ has order 4 and both quasitori are then isomorphic to $\mathbb{Z}_4^3$). Besides $Q(\widetilde{\mu}_4 s) \cong Q(\widetilde{\mu}_4)$ by Remark 1, since $\mathcal{T}^{\langle \mu_4\rangle}$ is finite.

Now we are going to check that $Q(\widetilde{\mu}_4)$ coincides with its own centralizer. Let us take $f \in \mathfrak{C}_{\operatorname{Aut}\mathfrak{e}_6}(Q(\widetilde{\mu}_4))$. Consider $Z = \mathfrak{C}_{\operatorname{Aut}\mathfrak{e}_6}(\mathcal{T}^{\langle \mu_4\rangle})$. There exists $\mathcal{T}'$ a maximal torus of Z such that $Q(\widetilde{\mu}_4)\cup\{f\}$ is contained in the normalizer of $\mathcal{T}'$. As all the maximal tori of Z are conjugate, there is $p \in Z$ such that $p\mathcal{T}'p^{-1} = \mathcal{T}$, hence $ptp^{-1} = t$ for all $t \in \mathcal{T}^{\langle \mu_4\rangle}$ and $p(Q(\widetilde{\mu}_4)\cup\{f\})p^{-1} \subset \mathfrak{N}(\mathcal{T})$. Take into account that $\{\nu \in \sigma\mathcal{W} \mid \nu\cdot t = t\ \forall t \in \mathcal{T}^{\langle \mu_4\rangle}\} = \{\mu_4, \mu_4^3\}$ and therefore there are $l \in \{1,3\}$ and $s \in \mathcal{T}$ such that $p\widetilde{\mu}_4 p^{-1} = \widetilde{\mu}_4^l s$. Hence $pQ(\widetilde{\mu}_4)p^{-1} = Q(\widetilde{\mu}_4^l s)$. As in Remark 1, there is $s' \in \mathcal{T}$ such that $s'(\widetilde{\mu}_4^l s)(s')^{-1} \in \widetilde{\mu}_4^l\mathcal{T}^{\langle \mu_4\rangle}$, so that $\operatorname{Ad}p'$ for $p' = s'p \in \operatorname{Aut}\mathfrak{e}_6$ leaves invariant $Q(\widetilde{\mu}_4)$ (and fixes $\mathcal{T}^{\langle \mu_4\rangle}$ pointwise). As $p'fp'^{-1}$ also belongs to $\mathfrak{N}(\mathcal{T})$ and commutes with $\mathcal{T}^{\langle \mu_4\rangle}$, and taking into account that $\{\nu \in \mathcal{W} \mid \nu\cdot t = t\ \forall t \in \mathcal{T}^{\langle \mu_4\rangle}\} = \{\mathrm{id}, \mu_4^2\}$, then $p'fp'^{-1} = \widetilde{\mu}_4^r s''$ for some $r \in \{0,1,2,3\}$ and $s'' \in \mathcal{T}$. But $p'fp'^{-1}$ commutes with $p'\widetilde{\mu}_4 p'^{-1} \in \widetilde{\mu}_4^l\mathcal{T}^{\langle \mu_4\rangle}$ and with $\mathcal{T}^{\langle \mu_4\rangle}$, so that $p'fp'^{-1}$ commutes with $\widetilde{\mu}_4$ and hence s'' also does, in other words $s'' \in \mathcal{T}^{\langle \mu_4\rangle}$. This means that $p'fp'^{-1} = \widetilde{\mu}_4^r s'' \in Q(\widetilde{\mu}_4) = p'Q(\widetilde{\mu}_4)p'^{-1}$ and hence that $f \in Q(\widetilde{\mu}_4)$. $\square$

In particular, the outer automorphism $\widetilde{\mu}_4$ has order 4.

It is clear that $\mathcal{Q}_{14}$ does not contain any order two outer automorphism, since any outer automorphism in $Q(\mu_4, \mathrm{id})$ belongs to the set

$\{\widetilde{\mu}_4 s, \widetilde{\mu}_4^3 s \mid s \in \mathcal{T}^{\langle\mu_4\rangle}\}$ and its square does not belong to $\mathcal{T}$ (in particular, its square is not the identity).

Finally, another property satisfied by $\mathcal{Q}_{14}$ is the following:

- $\langle \Upsilon_1^2, \Upsilon_2^2, \Upsilon_3\rangle \subset \operatorname{Int}\mathfrak{e}_6$ is toral.

Remark 13. This fact implies that there are $j \in \{1,\dots,5\}$, σ_i in the orbit of σ_{96} and $s, s' \in \mathcal{T}$ such that $\mathcal{Q}_{14}$ is conjugate to $\langle \widetilde{\eta}_j s, \widetilde{\sigma}_i s', \mathcal{T}^{\langle\sigma_i\rangle} \cap \mathcal{T}^{\langle\eta_j\rangle}\rangle$. This is only remarkable for technical purposes.

In order to prove such property, an easy computation says that $\mathcal{T}^{\langle\mu_4\rangle} = \langle t_1, t_2\rangle$ for $t_1 = t_{\mathrm{i},1,\mathrm{i},\mathrm{i},\mathrm{i},-1}$ and $t_2 = t_{1,\mathrm{i},1,1,\mathrm{i},1}$. If we take $\mathcal{H}^{\langle\mu_4^2\rangle} = \{t_{1,1,\alpha,1,\alpha\beta,\alpha} \mid \alpha^2 = \beta^2 = 1\}$, then $\wp_{\mu_4^2}(t_{x,y,\frac{\alpha}{x},x,\frac{\alpha\beta}{xy},\alpha}) = t_{1,1,\alpha,1,\alpha\beta,\alpha}$ and hence $\wp_{\mu_4^2}(t_1) = t_{1,1,-1,1,-1,-1}$, $\wp_{\mu_4^2}(t_1^2) = \mathrm{id}$, $\wp_{\mu_4^2}(t_2) = t_{1,1,1,1,-1,1}$, and $\wp_{\mu_4^2}(t_2^2) = \mathrm{id}$. Thus, the projection by $\wp_{\mu_4^2}$ of any proper quasitorus of $\mathcal{T}^{\langle\mu_4\rangle}$ is not the whole $\mathcal{H}^{\langle\mu_4^2\rangle}$, in particular the projection of $\langle \Upsilon_2^2, \Upsilon_3\rangle$, which is equivalent to the fact that $\langle \Upsilon_1^2, \Upsilon_2^2, \Upsilon_3\rangle \subset \operatorname{Int}\mathfrak{e}_6$ is toral.

Here another example of the situation described in Subsection 2.3 appears: $\langle \Upsilon_1^2, \Upsilon_2, \Upsilon_3\rangle \subset \operatorname{Int}\mathfrak{e}_6$ is a non-toral quasitorus isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2$ which does not contain an elementary non-toral 2-group.

6.3. MAD-groups without order two outer automorphisms.

Lemma 15. *There are two order two commuting automorphisms in* $\mathfrak{N}(\mathcal{T})$ *whose projections on the extended Weyl group are* η_3 *and* σ_{11127} *respectively.*

Proof: Start with $\mathcal{Q}_{11}$ the MAD-group isomorphic to $\mathbb{Z}_2^4 \times \mathbb{Z}_4$ obtained after combining an automorphism of type $2D$ with a copy of Ξ_5, the MAD-group of $\operatorname{Aut}\mathfrak{c}_4$ isomorphic to $\mathbb{Z}_2^3 \times \mathbb{Z}_4$ described in Subsection 5.3. Of course this copy of Ξ_5 is a non-toral quasitorus not only of $\operatorname{Aut}\mathfrak{c}_4$ but of $\operatorname{Aut}\mathfrak{e}_6$. Besides the subquasitorus $\langle (\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{i}\end{smallmatrix}) \otimes I_2 \otimes \theta_3, I_2 \otimes \theta_3 \otimes I_2, I_4 \otimes \theta_1\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2^2$ is toral. This implies that, after conjugating, we can find two order two commuting automorphisms f and g, being f inner, in $\mathfrak{N}(\mathcal{T})$ such that $\mathcal{T}^{\langle f\rangle} \cap \mathcal{T}^{\langle g\rangle} \cong \mathbb{Z}_4 \times \mathbb{Z}_2^2$. As $\pi(g)$ cannot be the identity, this implies the existence of $j = 1,\dots,5$, σ_i in the orbit of σ_{96} and $s, s' \in \mathcal{T}$ such that $\mathcal{Q}_{11}$ is conjugate to $\langle \widetilde{\eta}_j s, \widetilde{\sigma}_i s', \mathcal{T}^{\langle\sigma_i\rangle} \cap \mathcal{T}^{\langle\eta_J\rangle}\rangle$. With the help of a computer, we study the elements in the orbit of σ_{96} (there are only 45) which commute with each of the η_j's and divide them in orbits, getting the following possibilities:

$$\begin{aligned}
&j=1, \quad i=25470, \mathcal{T}^{\langle\sigma_i\rangle}\cap\mathcal{T}^{\langle\eta_j\rangle}\cong\mathbb{Z}_2^4,\\
&j=1, \quad i=2416, \mathcal{T}^{\langle\sigma_i\rangle}\cap\mathcal{T}^{\langle\eta_j\rangle}\cong\mathbb{Z}_2^2\times\mathbb{F}^*,\\
&j=2, \quad i=11127, \mathcal{T}^{\langle\sigma_i\rangle}\cap\mathcal{T}^{\langle\eta_j\rangle}\cong\mathbb{Z}_2^3,\\
&j=2, \quad i=11104, \mathcal{T}^{\langle\sigma_i\rangle}\cap\mathcal{T}^{\langle\eta_j\rangle}\cong\mathbb{Z}_2\times\mathbb{F}^*,\\
&j=3, \quad i=11127, \mathcal{T}^{\langle\sigma_i\rangle}\cap\mathcal{T}^{\langle\eta_j\rangle}\cong\mathbb{Z}_2^2\times\mathbb{Z}_4,\\
&j=3, \quad i=11104, \mathcal{T}^{\langle\sigma_i\rangle}\cap\mathcal{T}^{\langle\eta_j\rangle}\cong\mathbb{Z}_2^2\times\mathbb{F}^*,\\
&j=4, \quad i=11007, \mathcal{T}^{\langle\sigma_i\rangle}\cap\mathcal{T}^{\langle\eta_j\rangle}\cong\mathbb{Z}_2^2,\\
&j=5, \quad \text{any } i, \mathcal{T}^{\langle\sigma_i\rangle}\cap\mathcal{T}^{\langle\eta_j\rangle}\subset\mathcal{T}^{\langle\eta_5\rangle}\cong\mathbb{Z}_2^6.
\end{aligned}$$ □

Proposition 5. *If Q is a MAD-group of* $\operatorname{Aut}\mathfrak{e}_6$*, not contained in* $\operatorname{Int}\mathfrak{e}_6$*, such that there is not an order two outer automorphism in Q, then Q is conjugate to $\mathcal{Q}_{14}$.*

Proof: Take Q a MAD-group of $\operatorname{Aut}\mathfrak{e}_6$, not contained in $\operatorname{Int}\mathfrak{e}_6$, such that there is not an order two outer automorphism in Q. Take $\mathcal{T}$ a maximal torus of $\operatorname{Aut}\mathfrak{e}_6$ such that Q is contained in the normalizer of such torus, $\mathfrak{N}(\mathcal{T})$, and $Q\cap\mathcal{T}$ is maximal toral in Q. Hence there are indices $i, i_1, \dots, i_l \in \{1, \dots, 51480\}$ and toral elements $s, s_1, \dots, s_l \in \mathcal{T}$ such that $Q\cap\mathcal{T} = \mathcal{T}^{\langle\sigma\sigma_i\rangle}\cap\mathcal{T}^{\langle\sigma_{i_1}\rangle}\cap\dots\cap\mathcal{T}^{\langle\sigma_{i_l}\rangle}$ and

$$Q = \langle\widetilde{\sigma}\widetilde{\sigma}_i s\rangle\times\langle\widetilde{\sigma}_{i_1}s_1, \dots, \widetilde{\sigma}_{i_l}s_l\rangle\times Q\cap\mathcal{T},$$

and such that the quasitorus generated by $Q\cap\mathcal{T}\cup\{\widetilde{\sigma}_{i_j}s_j\}$ is non-toral for all j and $\widetilde{\sigma}_{i_j}s_j\notin\langle\widetilde{\sigma}_{i_1}s_1, \dots, \widetilde{\sigma}_{i_{j-1}}s_{j-1}\rangle\times Q\cap\mathcal{T}$.

We can assume that $\widetilde{\sigma}\widetilde{\sigma}_i s$ is an outer automorphism with order minimum in Q. This order is $2^h m$ for m an odd number, but note that $m=1$, because otherwise $(\widetilde{\sigma}\widetilde{\sigma}_i s)^m$ would be outer with order 2^h. Besides $h>1$, by hypothesis.

Take r the order of $\sigma\sigma_i$. It divides 2^h, so that $r\in\{2,4,8\}$ according to Table 2.

★ If $r=8$, we can assume that $i=17$ and so $Q\cap\mathcal{T}\subset\mathcal{T}^{\langle\sigma\sigma_{17}\rangle}\cong\mathbb{F}^*$. By Lemma 8, the quasitorus $\langle(\widetilde{\sigma}\widetilde{\sigma}_{17}s)^2, Q\cap\mathcal{T}\rangle$ is toral ($(\widetilde{\sigma}\widetilde{\sigma}_{17}s)^2$ is an inner automorphism), and, according to our choice of $\mathcal{T}$, we have $(\widetilde{\sigma}\widetilde{\sigma}_{17}s)^2\in Q\cap\mathcal{T}$. Hence $(\sigma\sigma_{17})^2=\mathrm{id}$, which is a contradiction.

★ Suppose now that $r=4$. Hence we can assume that $\sigma\sigma_i\in\{\mu_1,\mu_2,\mu_3,\mu_4\}$. If $\sigma\sigma_i$ were μ_1 or μ_2, we would obtain a contradiction as in case $r=$

8, since $\mathcal{T}^{\langle\mu_1\rangle} \cong (\mathbb{F}^*)^2$ and $\mathcal{T}^{\langle\mu_2\rangle} \cong (\mathbb{F}^*)^3$, so that $\langle(\widetilde{\sigma}\widetilde{\sigma}_i s)^2, Q\cap\mathcal{T}\rangle$ would be in the conditions of Lemma 8. If $\sigma\sigma_i = \mu_3$, as $\mathcal{T}^{\langle\mu_3^2\rangle} = \{t_{x,y,\frac{1}{x},x,v,w} \mid x,y,v,w \neq 0\} \cong (\mathbb{F}^*)^4$, then $\langle(\widetilde{\sigma}\widetilde{\sigma}_i s)^2, Q\cap\mathcal{T}\rangle$ is contained in the toral quasitorus $\langle(\widetilde{\sigma}\widetilde{\sigma}_i s)^2, \mathcal{T}^{\langle\mu_3^2\rangle}\rangle$. Hence we can assume that $\sigma\sigma_i = \mu_4$. The case $l \neq 0$ never happens: as $\langle\widetilde{\sigma}_{i_1}s_1\rangle \times Q\cap\mathcal{T} \subset \langle\widetilde{\sigma}_{i_1}s_1\rangle \times \mathcal{T}^{\langle\mu_4\rangle}$ is non-toral, and $\mathcal{T}^{\langle\mu_4\rangle} \cong \mathbb{Z}_4^2$, then σ_{i_1} is in the orbit of either σ_{96} or σ_{75}. The first possibility does not occur because the only element in the orbit of σ_{96} which commutes with μ_4 is μ_4^2. The second one is also impossible because σ_{75}^2 is not conjugate to σ_{96} (otherwise $\langle(\widetilde{\sigma}_{i_1}s_1)^2\rangle \times Q\cap\mathcal{T}$ would be toral and we would get a contradiction by the same arguments as in case $r = 8$). Thus $l = 0$, $Q = Q(\widetilde{\mu}_4 s)$, and, as $\mathcal{T}^{\langle\mu_4\rangle} \cong \mathbb{Z}_4^2$ is finite, then Q is conjugate to $Q(\widetilde{\mu}_4)$ by Remark 1.

Our purpose now is to check that this quasitorus $Q(\widetilde{\mu}_4)$ of type $\mathbb{Z}_4^3$ is the only possible MAD-group satisfying the required conditions. By Remark 13, we have to wait for its appearance in the case $r = 2$.

★ Thus suppose that $r = 2$ and that (perhaps by changing the element s in the torus) an outer automorphism in Q of minimum order 2^h is $\widetilde{\eta}_j s$ for some $j \in \{1, \dots, 5\}$. Observe first some useful facts:

a) $l \neq 0$.

Otherwise $Q = Q(\widetilde{\eta}_j s) = \langle\widetilde{\eta}_j s, \mathcal{T}^{\langle\eta_j\rangle}\rangle$, but this quasitorus contains outer automorphisms of order just 2, a contradiction: Indeed, the automorphism $(\widetilde{\eta}_j s)^2$ belongs to $\mathcal{S}^{\langle\eta_j\rangle}$ by Remark 12, and it has order 2^{h-1} (a multiple of 2), so that $\mathcal{S}^{\langle\eta_j\rangle} \neq \mathrm{id}$ is a non-trivial torus. A torus contains a square root of each of its elements, so there is $s' \in \mathcal{S}^{\langle\eta_j\rangle} \subset \mathcal{T}^{\langle\eta_j\rangle} \subset Q$ such that $(s')^2 = (\widetilde{\eta}_j s)^2$, so that $\widetilde{\eta}_j s(s')^{-1}$ is an outer order two automorphism in Q.

b) $(\widetilde{\eta}_j s)^2$ belongs to $\mathcal{S}^{\langle\eta_j\rangle}$ but it is not contained in any subtorus of $Q\cap\mathcal{T}$. In particular, $j \neq 5$ since $\mathcal{S}^{\langle\eta_5\rangle} = \mathrm{id}$ ($\mathcal{T}^{\langle\eta_5\rangle} = \mathbb{Z}_2^6$).

c) There is no $k \in \{1, \dots, l\}$ such that σ_{i_k} has order three.

In order to check it, take into account two facts. First, the quasitorus $\langle\widetilde{\sigma}_{i_k}s_k, \mathcal{T}^{\langle\eta_j\rangle}\cap\mathcal{T}^{\langle\sigma_{i_k}\rangle}\rangle$ is non-toral since it contains $\langle\widetilde{\sigma}_{i_k}s_k, Q\cap\mathcal{T}\rangle$. This implies, by Lemma 8, that $\mathcal{T}^{\langle\eta_j\rangle} \cap \mathcal{T}^{\langle\sigma_{i_k}\rangle}$ is the direct product of a torus with a finite group, say P. This P contains a non-trivial 3-group, since $\langle\widetilde{\sigma}_{i_k}s_k, P\rangle$ is non-toral but $\langle(\widetilde{\sigma}_{i_k}s_k)^3, P\rangle \subset \mathcal{T}$ is toral. Second, $(\widetilde{\eta}_j s)^{2^{h-1}}$ is an order two element in $\mathcal{S}^{\langle\eta_j\rangle} \cap \mathcal{T}^{\langle\sigma_{i_k}\rangle}$. Now we check that these conditions do not happen for any value of $j = 1, 2, 3, 4$.

For $j = 1$, there are 80 order three elements in $\mathcal{W}$ commuting with η_1, but they can be divided in three orbits under conjugation by some element which preserves η_1, with representatives 2920, 12406, and 3826. We can look only at these representatives because if $p \in \mathcal{W}$ such that $p\eta p^{-1} = \eta$ and $p\nu p^{-1} = \nu'$, then $\mathcal{T}^{\langle\eta\rangle} \cap \mathcal{T}^{\langle\nu\rangle} \cong \mathcal{T}^{\langle\eta\rangle} \cap \mathcal{T}^{\langle\nu'\rangle}$. But $\mathcal{T}^{\langle\eta_1\rangle} \cap \mathcal{T}^{\langle\sigma_{2920}\rangle} \cong (\mathbb{F}^*)^2$ and $\mathcal{T}^{\langle\eta_1\rangle} \cap \mathcal{T}^{\langle\sigma_{12406}\rangle} \cong (\mathbb{F}^*)^2$ have not direct factors $\mathbb{Z}_3$, and $\mathcal{T}^{\langle\eta_1\rangle} \cap \mathcal{T}^{\langle\sigma_{3826}\rangle} \cong \mathbb{Z}_3^2$ has not order two elements.

For $j = 2$, there are 8 order three elements (one is 3026) in $\mathcal{W}$ commuting with η_2, all of them conjugated by some element which commutes with η_2. Note that $\mathcal{T}^{\langle\eta_2\rangle} \cap \mathcal{T}^{\langle\sigma_{3026}\rangle} = \{t_{x,x^{-1},x^{-3},x^3,x^{-1},1} \mid x \neq 0\} \cong \mathbb{F}^*$.

Again for $j = 3$, there are 8 order three elements (one is 4796) in $\mathcal{W}$ commuting with η_3, all of them conjugate preserving fixed η_3. We see that $\mathcal{T}^{\langle\eta_3\rangle} \cap \mathcal{T}^{\langle\sigma_{4796}\rangle} = \{t_{1,y,z,1,y,\frac{1}{y^2 z}} \mid y, z \neq 0\} \cong (\mathbb{F}^*)^2$.

Finally, for $j = 4$, there are 80 order three elements in $\mathcal{W}$ commuting with η_4, but they can be grouped in two conjugation orbits fixing η_4, with representatives 3839, 4079. Now we check that $\mathcal{T}^{\langle\eta_4\rangle} \cap \mathcal{T}^{\langle\sigma_{3839}\rangle} = \{t_{1,1,z,1,1,\frac{1}{z}} \mid z \neq 0\} \cong \mathbb{F}^*$ and $\mathcal{T}^{\langle\eta_4\rangle} \cap \mathcal{T}^{\langle\sigma_{4079}\rangle} = \{t_{1,y,z,u,y,\frac{y}{z}} \mid y^2 = u^2 = 1,\ z \neq 0\} \cong \mathbb{F}^* \times \mathbb{Z}_2^2$.

d) The matrix σ_{i_k} has order just 2 for all $k \in \{1, \dots, l\}$ and it belongs to the orbit of σ_{96}.

Indeed, if σ_{i_k} has order 5 or 10, then $\langle \widetilde{\sigma}_{i_k} s_k, \mathcal{T}^{\langle\eta_k\rangle}\rangle$ is toral, and, if the order σ_{i_k} is multiple of 3, then one of its powers $\sigma_{i_k}^m$ has order just three, and the arguments in item c) work for $\sigma_{i_k}^m$. Thus σ_{i_k} has order a power of 2. As in Remark 8, the element σ_{i_k} is conjugate to either σ_{96} or σ_{75}. But in the latter case, we argue again that $\langle (\widetilde{\sigma}_{i_k} s_k)^2, Q \cap \mathfrak{T}\rangle$ is toral since $\mathcal{T}^{\langle(\sigma_{75})^2\rangle} \cong (\mathbb{F}^*)^4$, which is a contradiction with the maximal-torality of $Q \cap \mathcal{T}$.

Keeping in mind items a), b), and d), we proceed to a detailed analysis of the possible cases.

• *Case* $j = 4$. There are 15 elements in the orbit of σ_{96} which commute with η_4, but all of them are conjugated by means of an element of $\mathcal{W}$ which fixes η_4. So we can assume that σ_{i_1} is any of them, for instance, σ_{11007}. As $\langle \widetilde{\sigma}_{i_1} s_1, \mathcal{T}^{\langle\sigma_{i_1}\rangle} \cap \mathcal{T}^{\langle\eta_4\rangle}\rangle$ is non-toral (it contains $\langle \widetilde{\sigma}_{i_1} s_1, Q \cap \mathcal{T}\rangle$), then by Lemma 8 $\langle \widetilde{\sigma}_{i_1} s_1, \wp_{i_1}(\mathcal{T}^{\langle\sigma_{i_1}\rangle} \cap \mathcal{T}^{\langle\eta_4\rangle})\rangle$ is non-toral too and $\wp_{i_1}(\mathcal{T}^{\langle\sigma_{i_1}\rangle} \cap \mathcal{T}^{\langle\eta_4\rangle}) = \mathcal{H}^{\langle i_1\rangle}$ as in Remark 9. Note that

$$\mathcal{T}^{\langle 11007\rangle} = \{t_{x,x,\frac{u}{z},z,x,w} \mid u^2 = x^2 = 1,\ z, w \neq 0\} \cong (\mathbb{F}^*)^2 \times \mathbb{Z}_2^2,$$
$$\mathcal{S}^{\langle 11007\rangle} = \{t_{1,1,z,\frac{1}{z},1,w} \mid z, w \neq 0\} \cong (\mathbb{F}^*)^2,$$

so that a complement satisfying the conditions in Lemma 3 is, for instance,

$$\mathcal{H}^{\langle 11007\rangle} = \{t_{x,x,u,1,x,1} \mid u^2 = x^2 = 1\} \cong \mathbb{Z}_2^2.$$

Now $\mathcal{T}^{\langle\eta_4\rangle} \cap \mathcal{T}^{\langle\sigma_{11007}\rangle} = \{t_{1,1,z,u,1,w} \mid z^2 = u^2 = w^2 = 1\} \cong \mathbb{Z}_2^3$, so we have to compute $\wp_{11007}(t_{1,1,z,u,1,w})$. As $t_{1,1,z,u,1,w} = t_{1,1,\frac{1}{u},u,1,w} t_{1,1,uz,1,1,1}$, with the first factor in $\mathcal{S}^{\langle\sigma_{11007}\rangle}$ and the second one in $\mathcal{H}^{\langle\sigma_{11007}\rangle}$, then $\wp_{11007}(t_{1,1,z,u,1,w}) = t_{1,1,uz,1,1,1}$. In this way we obtain a contradiction, since $\wp_{11007}(\mathcal{T}^{\langle\sigma_{11007}\rangle} \cap \mathcal{T}^{\langle\eta_4\rangle}) \cong \mathbb{Z}_2$ (roughly speaking, although $\mathcal{T}^{\langle\sigma_{11007}\rangle} \cap \mathcal{T}^{\langle\eta_4\rangle}$ contains three $\mathbb{Z}_2$'s, it only contains one of the two *bad* required $\mathbb{Z}_2$'s).

• *Case* $j = 2$. There are 7 elements in the orbit of σ_{96} which commute with η_2, divided in two orbits of $\mathcal{W}$ under conjugation by an element fixing η_2, with representatives, for instance, 11127 and 11104. If $\sigma_{i_1} = \sigma_{11127}$, then $\mathcal{T}^{\langle\eta_2\sigma_{11127}\rangle} \cong \mathbb{F}^* \times \mathbb{Z}_2^4$, so that $\eta_2\sigma_{11127}$ is conjugate to η_4 and such case has already been studied. If $\sigma_{i_1} = \sigma_{11104}$, then $\mathcal{T}^{\langle\sigma_{11104}\rangle} \cap \mathcal{T}^{\langle\eta_2\rangle} = \{t_{x,y,x,x,y,\frac{1}{y^2}} \mid x^2 = 1,\, y \neq 0\} \cong \mathbb{F}^* \times \mathbb{Z}_2$, so that $\langle\widetilde{\sigma}_{11104}, \mathcal{T}^{\langle\sigma_{11104}\rangle} \cap \mathcal{T}^{\langle\eta_2\rangle}\rangle$ is toral by Remark 3, and again we have found a contradiction.

• *Case* $j = 1$. There are 13 elements in the orbit of σ_{96} which commute with η_1, divided in two orbits of $\mathcal{W}$ under conjugation by an element fixing η_1, with representatives, for instance, 25470 and 2416. A simple computation shows us that $\mathcal{T}^{\langle\eta_1\sigma_{25470}\rangle} \cong \mathbb{Z}_2^6$ and that $\mathcal{T}^{\langle\eta_1\sigma_{2416}\rangle} = \{t_{x,y,z,u,\frac{y}{x},\frac{y}{z}} \mid y^2 = u^2 = 1,\, x, z \neq 0\} \cong (\mathbb{F}^*)^2 \times \mathbb{Z}_2^2$, hence $\eta_1\sigma_{25470}$ is conjugate to η_5 and $\eta_1\sigma_{2416}$ is conjugate to η_4, and both cases have been previously considered.

• *Case* $j = 3$. There are 5 elements in the orbit of σ_{96} which commute with η_3, divided in two orbits of $\mathcal{W}$ under conjugation by an element fixing η_3, $O_1 = \{k_1 = 10850, k_2 = 11104\}$ and $O_2 = \{k_3 = 23234, k_4 = 11127, k_5 = 28154\}$. Thus we can take $i_1 \in \{k_2, k_4\}$. It is important to note that $l = 1$, because there are no $k_i, k_j, k_m \in O_1 \cup O_2$ such that $\sigma_{k_i}\sigma_{k_j} = \sigma_{k_m}$, that is, $\sigma_{k_i}\sigma_{k_j}$ is not in the orbit of σ_{96}.

In the first subcase, $(\widetilde{\eta}_3 s)^2 \in \mathcal{S}^{\langle\eta_3\rangle} \cap \mathcal{T}^{\langle 10850\rangle} = \{t_{1,y,\frac{1}{y^2},1,y,1} \mid y \neq 0\} \cong \mathbb{F}^*$, which is a contradiction with item b), since $\mathcal{S}^{\langle\eta_3\rangle} \cap \mathcal{T}^{\langle 10850\rangle} \subset Q \cap \mathcal{T}$ $(l = 1)$. Consider then that $i_1 = 11127$. For our convenience, we take the liftings $\widetilde{\eta}_3$ and $\widetilde{\sigma}_{11127}$ such that not only $\widetilde{\eta}_3^2 = \widetilde{\sigma}_{11127}^2 = \mathrm{id}$ but besides $\widetilde{\eta}_3\widetilde{\sigma}_{11127} = \widetilde{\sigma}_{11127}\widetilde{\eta}_3$. Note that we can make such choice by Lemma 14. As $l = 1$, there are $s, s' \in \mathcal{T}$ such that $Q = \langle\widetilde{\eta}_3 s, \widetilde{\sigma}_{11127} s', \mathcal{T}^{\langle\sigma_{i_1}\rangle} \cap \mathcal{T}^{\langle\eta_3\rangle}\rangle$.

Take $f_1 = \widetilde{\eta}_3 s$, $f_2 = \widetilde{\sigma}_{11127}s'$, and $f_3 = t_{-1,\mathrm{i},1,-1,-\mathrm{i},1} \in \mathcal{T}^{\langle\sigma_{11127}\rangle} \cap \mathcal{T}^{\langle\eta_3\rangle} = \{t_{y^2,y,z,y^2,y^3,w} \mid y^4 = z^2 = w^2 = 1\} \cong \mathbb{Z}_4 \times \mathbb{Z}_2^2$.

Our first aim is to check that f_1 and f_2 are order 4 automorphisms (with $f_1^2 \neq f_2^2$), what implies that $Q = \langle f_1, f_2, f_3\rangle$. Note that if we could prove that:

i) f_1 is an outer automorphism,
ii) $\langle f_2, f_3\rangle$ is toral,
iii) $\langle f_1^2, f_2, f_3\rangle \subset \operatorname{Int}\mathfrak{e}_6$ is non-toral,

hence, we could apply the same arguments than in Proposition 4 to conclude that Q is conjugate to $Q(\widetilde{\mu}_4)$ and then it would be also conjugate to $\mathcal{Q}_{14}$.

Thus, our second aim is to prove items ii) and iii). In order to make such comprobations about torality, take $\mathcal{H}^{\langle 11127\rangle} = \{t_{x,1,z,x,1,1} \mid x^2 = z^2 = 1\}$. The corresponding projection is hence given by

$$\wp_{11127}\colon \mathcal{T}^{\langle 11127\rangle} \to \mathcal{H}^{\langle 11127\rangle},$$
$$\wp_{11127}(t_{x,y,z,x,\frac{1}{y},w}) = t_{x,1,z,x,1,1},$$

if $x^2 = z^2 = 1$, $y, w \neq 0$. We will check that $\wp_{11127}(f_2^2) = \mathrm{id}$, which guarantees that ii) is verified, since

$$\wp_{11127}(\langle f_2^2, f_3\rangle) = \langle \wp_{11127}(f_3)\rangle = \langle t_{-1,1,1,-1,1,1}\rangle \neq \mathcal{H}^{\langle 11127\rangle} \cong \mathbb{Z}_2^2.$$

And finally we will check that $\wp_{11127}(f_1^2) \notin \langle t_{-1,1,1,-1,1,1}\rangle$, so that $\wp_{11127}(f_1^2, f_3) = \mathcal{H}^{\langle 11127\rangle}$ and $\langle f_1^2, f_2, f_3\rangle \subset \operatorname{Int}\mathfrak{e}_6$ is non-toral.

Let us begin by noting that

$$(\widetilde{\eta}_3 s)^2 \in \mathcal{S}^{\langle\eta_3\rangle} \cap \mathcal{T}^{\langle 11127\rangle} = \{t_{1,y,z,1,y,z} \mid y^2 = z^2 = 1\} \cong \mathbb{Z}_2^2,$$
$$(\widetilde{\sigma}_{11127}s')^2 \in \mathcal{T}^{\langle\eta_3\rangle} \cap \mathcal{S}^{\langle 11127\rangle} = \{t_{1,y,1,1,y,w} \mid y^2 = w^2 = 1\} \cong \mathbb{Z}_2^2,$$

so that f_1 has order 4 (not 2 by hypothesis) and f_2 has order either 2 or 4. Besides $f_1^2 \neq t_{1,-1,1,1,-1,1} = f_3^2$, because otherwise $f_1 f_3^{-1}$ would be an outer order two automorphism in Q. If $f_1^2 = t_{1,-1,-1,1,-1,-1}$, then $(f_1 f_3)^2 = t_{1,1,-1,1,1,-1}$, so that we can assume that $f_1^2 = t_{1,1,-1,1,1,-1}$. In particular, $\wp_{11127}(f_1^2 = t_{1,1,-1,1,1,-1}) = t_{1,1,-1,1,1,1} \notin \langle t_{-1,1,1,-1,1,1}\rangle$.

Besides this implies that $s = t_{a,b,c,\frac{-1}{ac^2},\frac{-ac^2}{b},d}$ for some $a, b, c, d \in \mathbb{F}^*$. Now take the element $r_1 = t_{u,\frac{b}{uc},1,\frac{1}{uc},1,w}$ for some $u, w \in \mathbb{F}^*$ such that $u^2 = -a$ and $w^2 = \frac{-adc^3}{b^2}$. It is a straightforward computation that $r_1(\widetilde{\eta}_3 s)r_1^{-1} = \widetilde{\eta}_3 t_{-1,1,1,1,1,1}$, so that we can assume that $s = t_{-1,1,1,1,1,1}$. Now the commutativity of $\widetilde{\eta}_3 s$ with $\widetilde{\sigma}_{11127}s'$ means that $s' = t_{x,y,z,x,xy,\frac{\alpha\mathrm{i}}{y^2 z}}$ for some $x, \alpha \in \{\pm 1\}$ and $y, z \in \mathbb{F}^*$. Then $f_2^2 = t_{1,\frac{1}{x},1,1,x,-x^6\alpha^2} = t_{1,\frac{1}{x},1,1,x,-1} \neq \mathrm{id}$, so that f_2 has order 4 and in any case $f_2^2 \neq f_1^2$. Besides the projection $\wp_{11127}(t_{1,\frac{1}{x},1,1,x,-1}) = \mathrm{id}$, what finishes the proof. $\square$

In fact the power of this kind of arguments is strong, and we could have described all the MAD-groups of $\mathrm{Aut}\,\mathfrak{e}_6$ in computational terms: It is possible to take good choices of the extensions $\widetilde{\sigma}_{25470}$, $\widetilde{\sigma}_{3826}$, $\widetilde{\sigma}_{11104}$, $\widetilde{\sigma}_{11127}$ (the two latter ones as in Remark 14 and in Lemma 14 respectively) such that

$$(10)\qquad \begin{array}{ll} \mathcal{Q}_6 \cong Q(\widetilde{\eta}_1), & \mathcal{Q}_7 \cong \langle \widetilde{\eta}_1, \widetilde{\sigma}_{25470}, \mathcal{T}^{\langle \eta_1 \rangle} \cap \mathcal{T}^{\langle \sigma_{25470} \rangle} \rangle, \\ \mathcal{Q}_{10} \cong Q(\widetilde{\eta}_3), & \mathcal{Q}_9 \cong \langle \widetilde{\eta}_1, \widetilde{\sigma}_{3826}, \mathcal{T}^{\langle \eta_1 \rangle} \cap \mathcal{T}^{\langle \sigma_{3826} \rangle} \rangle, \\ \mathcal{Q}_{12} \cong Q(\widetilde{\eta}_4), & \mathcal{Q}_8 \cong \langle \widetilde{\eta}_3, \widetilde{\sigma}_{10850}, \mathcal{T}^{\langle \eta_3 \rangle} \cap \mathcal{T}^{\langle \sigma_{11104} \rangle} \rangle, \\ \mathcal{Q}_{13} \cong Q(\widetilde{\eta}_5), & \mathcal{Q}_{11} \cong \langle \widetilde{\eta}_3, \widetilde{\sigma}_{11127}, \mathcal{T}^{\langle \eta_3 \rangle} \cap \mathcal{T}^{\langle \sigma_{11127} \rangle} \rangle. \end{array}$$

The proof can be made with analogous arguments to those ones in the proof of Proposition 5. Furthermore, with those reasonings we could have proved that $\{\mathcal{Q}_i \mid i = 1, \dots, 14\}$ are all the MAD-groups up to conjugation without using the description of the non-toral elementary p-groups. The paper would have then been practically self-contained, but, for evident reasons (to avoid most of the such unpleasant computational arguments) we have preferred a combined option.

Appendix

In this section we will provide natural descriptions of liftings $\widetilde{\eta} \in \mathrm{Aut}\,\mathfrak{e}_6$ for all the representatives of the conjugacy classes of order two elements η in the extended Weyl group $\mathcal{V} = \mathcal{W} \cup \sigma\mathcal{W}$. This may help the reader to have a better understanding of the situation. All the computations here are made by hand.

But this section has also a practical objective: to assure Remark 12 (a key piece in the proof of Proposition 5) thanks to the existence of outer order two automorphisms which project on η_j for all $j = 1, \dots, 5$.

We do not want to construct explicit expressions of these liftings by following the lines in [**32**, Section 14.2] as in Subsection 4.1. We prefer the following procedure, also constructive: Choose a maximal torus, take some distinguished automorphisms and compute their projections on $\mathfrak{N}(T)/T$. Take into account that only the computation of $\mathcal{T}^{\langle f \rangle}$ is enough to distinguish the orbit of the element (among the elements in $\mathcal{W}$ and also in $\sigma\mathcal{W}$).

We work again with the model described in Equation (2). We choose $\{E_1, E_2, E_3\}$ a basis of V and $\{e_1, e_2, e_3\}$ its dual basis (of V^*). We call e_{ij}^k the element in $\mathrm{sl}(V_k)$ which sends $E_j \in V_k$ to $E_i \in V_k$ (and V_l to 0 if $k \neq l$). Take $\mathfrak{h}$ the abelian subalgebra spanned by $\{h_1 = e_{11}^1 - e_{33}^1$, $h_2 =$

$e^1_{22}-e^1_{33}$, $h_3=e^2_{11}-e^2_{33}$, $h_4=e^2_{22}-e^2_{33}$, $h_5=e^3_{11}-e^3_{33}$, $h_6=e^3_{22}-e^3_{33}\}$, which is a Cartan subalgebra of $\mathcal{L}$. Now, if $h=\sum_{i=1}^6 w_ih_i$ is an arbitrary element in $\mathfrak{h}$, then $[h,E_i\otimes E_j\otimes E_k]=(\alpha_{1,i}+\alpha_{2,j}+\alpha_{3,k})E_i\otimes E_j\otimes E_k$ for all $i,j,k\in\{1,2,3\}$, for the scalars $\alpha_{l,1}=w_{2l-1}$, $\alpha_{l,2}=w_{2l}$, and $\alpha_{l,3}=-w_{2l-1}-w_{2l}$, $l\in\{1,2,3\}$. Define $\alpha_i\colon\mathfrak{h}\to\mathbb{F}$ by

$$\begin{aligned}
&\alpha_1(h)=w_1-w_2, &&\alpha_4(h)=-w_1-w_2-w_3-w_4-w_5-w_6,\\
&\alpha_2(h)=2w_3+w_4, &&\alpha_5(h)=w_5+2w_6,\\
&\alpha_3(h)=w_1+2w_2, &&\alpha_6(h)=w_5-w_6.
\end{aligned}$$

It is a simple computation that $\{\alpha_i\}_{i=1}^6$ is a set of simple roots of Φ, the root system relative to $\mathfrak{h}$. Moreover, the root spaces corresponding to the simple roots are

$$\begin{aligned}
&L_{\alpha_1}=\langle e^1_{12}\rangle, &&L_{\alpha_3}=\langle e^1_{23}\rangle, &&L_{\alpha_5}=\langle e^3_{23}\rangle,\\
&L_{\alpha_2}=\langle e^2_{13}\rangle, &&L_{\alpha_4}=\langle E_3\otimes E_3\otimes E_3\rangle, &&L_{\alpha_6}=\langle e^3_{12}\rangle.
\end{aligned}$$

If $\mathfrak{T}$ is the torus of the automorphisms fixing pointwise $\mathfrak{h}$, then an automorphism $f\in\operatorname{Aut}\mathfrak{e}_6$ belongs to $\mathfrak{N}(\mathfrak{T})$ if and only if f permutes the root spaces. Precisely, if there is $\eta\in\mathrm{GL}(E)$ for $E=\sum_{i=1}^6\mathbb{F}\alpha_i$ such that $f(L_\alpha)\subset L_{\eta(\alpha)}$ for all $\alpha\in\Phi$, then $\pi(f)=\eta\in\mathcal{V}$. Identify $\mathrm{GL}(E)$ with $\mathrm{GL}(6)$ by mapping each η to its matrix relative to the basis $\{\alpha_i\}_{i=1}^6$.

If $f,g,h\in\mathrm{GL}(3)\equiv\mathrm{GL}(V)$, we call $f\otimes g\otimes h$ the unique automorphism of $\mathfrak{e}_6$ whose action in $\mathcal{L}_{\bar 1}$ is $u\otimes v\otimes w\mapsto f(u)\otimes g(v)\otimes h(w)$ (thus $\Psi(A)$ in Subsection 3.2 coincides with $A\otimes A\otimes A$).

Take $\varphi_1=\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)\otimes I_3\otimes I_3\in\operatorname{Aut}\mathfrak{e}_6$, which obviously is an order two automorphism. As $\varphi_1(e^1_{12})=e^1_{13}\in L_{\alpha_1+\alpha_3}$, $\varphi_1(e^1_{23})=e^1_{32}\in L_{-\alpha_3}$ and $\varphi_1(E_3\otimes E_3\otimes E_3)=E_2\otimes E_3\otimes E_3\in L_{\alpha_3+\alpha_4}$, that means that $\pi(\varphi_1)=\varrho_1$ for ϱ_1 the automorphism of the root system mapping α_1 into $\alpha_1+\alpha_3$, α_3 into $-\alpha_3$, α_4 into $\alpha_3+\alpha_4$, and α_2, α_5, and α_6 into themselves, whose related matrix is:

$$\varrho_1=\begin{pmatrix}1&0&1&0&0&0\\0&1&0&0&0&0\\0&0&-1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}.$$

Hence $\mathfrak{T}^{\langle\varrho_1\rangle}=\{t\in\mathfrak{T}\mid\varrho_1\cdot t=t\}=\{t_{x,y,1,u,v,w}\mid x,y,u,v,w\in\mathbb{F}^*\}\cong(\mathbb{F}^*)^5$ and $\pi(\varphi_1)$ is conjugate to σ_{11323}. In particular $\varphi_1\in\operatorname{Int}\mathfrak{e}_6$.

Take $\varphi_2 = I_3 \otimes \left(\begin{smallmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{smallmatrix}\right) \otimes I_3 \in \operatorname{Aut}\mathfrak{e}_6$. By analogy with the previous case, φ_2 is also an inner automorphism. It satisfies

$$\varphi_2(e^2_{13}) = e^2_{12} \in L_{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6},$$
$$\varphi_2(E_3 \otimes E_3 \otimes E_3) = E_3 \otimes E_2 \otimes E_3 \in L_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6},$$

and it does not move the other simple root spaces, so that $\varrho_2 = \pi(\varphi_2)$ is the element in the Weyl group

$$\varrho_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 2 & 2 & 3 & 2 & 1\\ 0 & 0 & 1 & 0 & 0 & 0\\ -1 & -1 & -2 & -2 & -2 & -1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is a hand-computation that $\mathfrak{T}^{\langle\varrho_2\rangle}$ is again isomorphic to $(\mathbb{F}^*)^5$, but that $\mathfrak{T}^{\langle\varrho_1\varrho_2\rangle}$ is isomorphic to $(\mathbb{F}^*)^4$, hence implying that $\varphi_1\varphi_2$ is an order two automorphism (φ_1 commutes with φ_2) such that $\pi(\varphi_1\varphi_2)$ is conjugate to σ_{19}.

Third take $\varphi_3 = I_3 \otimes I_3 \otimes \left(\begin{smallmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{smallmatrix}\right) \in \operatorname{Int}\mathfrak{e}_6$ and check that it only moves the simple root vectors $\varphi_3(e^3_{12}) = e^3_{13} \in L_{\alpha_5+\alpha_6}$, $\varphi_3(e^3_{23}) = e^3_{32} \in L_{-\alpha_5}$, and $\varphi_3(E_3 \otimes E_3 \otimes E_3) = E_3 \otimes E_3 \otimes E_2 \in L_{\alpha_4+\alpha_5}$, so that

$$\pi(\varphi_3) = \varrho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Hence $t_{x,y,z,u,v,w} \in \mathfrak{T}^{\langle\varrho_1\varrho_2\varrho_3\rangle}$ whenever $z = v = 1 = xyu^3w$, that is, $\mathfrak{T}^{\langle\varrho_1\varrho_2\varrho_3\rangle} \cong (\mathbb{F}^*)^3$ and $\pi(\varphi_1\varphi_2\varphi_3)$ is conjugate to σ_{21}. Note that there is no confusion with the orbit of η_2, because all the φ_i's until this moment are inner, so that $\varphi_1\varphi_2\varphi_3$ is also.

Take φ_4 the order two automorphism of $\mathfrak{e}_6$ mapping $E_i \otimes E_j \otimes E_k$ into $e_i \otimes e_j \otimes e_k \in (V^*)^{\otimes 3}$. Note that it sends any element in $\mathrm{sl}(V_i)$ to the opposite of its transpose. Hence $\varrho_4 = \pi(\varphi_4) = -\mathrm{id} = \eta_5$ because it sends each α_i to $-\alpha_i$. Note that φ_4 is an outer automorphism, more precisely, $\varphi_4 \in 2D$ since $\dim \mathfrak{fix}\varphi_4 = \frac{72}{2}$.

Another order two outer automorphism, which will be denoted by φ_5, is that one interchanging V_1 with V_3 (which fixes a subalgebra isomorphic to $\mathfrak{f}_4$, as we saw in Subsection 5.2). As it applies e^1_{ij} into e^3_{ij}, its projection

$\varrho_5 = \pi(\varphi_5)$ interchanges α_1 with α_6 and α_3 with α_5, not moving α_2 and α_4. Hence $\varphi_5 \in 2C$ is an order two outer automorphism with $\pi(\varphi_5) = \sigma$.

Observe that $\varphi_4\varphi_5$ is an inner order two automorphism (φ_4 and φ_5 commute) whose projection is in the orbit of σ_{96}.

The remaining descriptions of representatives of the outer orbits can be found by combining the previous elements. Notice that φ_5 commutes with φ_2 (not with φ_1 or φ_3) and φ_4 commutes with all the φ_i's. Hence, by looking again at the $\mathfrak{T}^{\langle j\rangle}$'s we get that

$$\pi(\varphi_5\varphi_2) \sim \eta_2, \quad \pi(\varphi_4\varphi_2) \sim \eta_4, \quad \pi(\varphi_4\varphi_1\varphi_2) \sim \eta_3.$$

It is interesting to observe that the MAD-groups $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are just contained in the normalizer of this maximal torus $\mathfrak{T} = \{f \in \operatorname{Aut}\mathfrak{e}_6 \mid f|_{\mathfrak{h}} = \mathrm{id}\}$. More precisely, $F_1 = t_{\omega^2,\omega,\omega^2,1,\omega^2,\omega^2}$, $\pi(F_2) = \sigma_{51529}$ (in the 292-orbit), $F_3 = t_{1,1,1,\omega,1,1}$, $\pi(F_4) = \sigma_{30245}$ (in the 3819-orbit), and $T_{\alpha,\beta} = t_{\frac{\alpha}{\beta},\alpha^2\beta,\alpha\beta^2,\frac{1}{\alpha^3\beta^3},\alpha\beta^2,\frac{\alpha}{\beta}}$.

Remark 14. Note that again most of the MAD-groups can be described in these natural terms starting from Equation (2). If we denote by $Q(f;g)$ the quasitori generated by f, g, and $\mathcal{T}^{\langle f\rangle} \cap \mathcal{T}^{\langle g\rangle}$ for each pair of commuting automorphisms $f, g \in \mathfrak{N}(\mathcal{T})$, then

$$\begin{array}{ll}
\mathcal{Q}_1 = Q(F_4), & \mathcal{Q}_8 \cong Q(\varphi_4\varphi_1\varphi_3;\varphi_4\varphi_5),\\
\mathcal{Q}_2 = Q(F_2), & \mathcal{Q}_9 \cong Q(F_4\varphi_5),\\
\mathcal{Q}_4 \cong Q(\varphi_4\varphi_5), & \mathcal{Q}_{10} \cong Q(\varphi_4\varphi_1\varphi_3),\\
\mathcal{Q}_5 \cong Q(\mathrm{id}), & \mathcal{Q}_{12} \cong Q(\varphi_4\varphi_1),\\
\mathcal{Q}_6 \cong Q(\varphi_5), & \mathcal{Q}_{13} \cong Q(\varphi_4).\\
\mathcal{Q}_7 \cong Q(\varphi_4;\varphi_5), &
\end{array}$$

Observe that, although $\varphi_4\varphi_1\varphi_3$ and $\varphi_4\varphi_5$ are two order two commuting automorphisms such that $\pi(\varphi_4\varphi_1\varphi_3) \sim \eta_3$ and $\pi(\varphi_4\varphi_5) \sim \sigma_{96} \sim \sigma_{11127}$, this does not imply the existence of order two commuting liftings $\widetilde{\eta}_3$ and $\widetilde{\sigma}_{11127}$, as was necessary in Subsection 6.3, because the conjugating element has to be the same. In other words, what have been found are two order two commuting liftings $\widetilde{\eta}_3$ and $\widetilde{\sigma}_{11104}$, what is checked by computing the stabilizer.

Recent advances

During the time passed until the publication of this manuscript, some progress on the topic has been realized. Most of our results have just been enclosed in the AMS monograph [**25**, Chapter 6], which gives a comprehensive survey of the classification of gradings on simple finite-dimensional Lie algebras over algebraically closed fields. In particular, it contains alternative descriptions of the fine gradings on $\mathfrak{e}_6$, although without including the proofs of Propositions 1 and 5. The paper [**5**] is based in our work and it also contains alternative descriptions of our gradings, chosen in order to compute their Weyl groups. Finally, the work [**17**] is a recent survey of the gradings on the "E-series", where the authors interpret all the known gradings on $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$ in a homogeneous way, relating the gradings on exceptional algebras with some others on composition, Jordan and structurable algebras. One more step has been given in the classification of fine gradings on $\mathfrak{e}_8$, at the moment only for finite groups, in [**16**]. There, a new tool plays a key role, the Brauer invariants of the irreducible modules for graded semisimple Lie algebras studied in [**26**].

Acknowledgements. We would like to thank Professors Cándido Martín González and Alberto Elduque Palomo for their continuous support. The stays of the first author in the University of Zaragoza have been very fruitful for this work.

References

[1] J. F. Adams, *"Lectures on exceptional Lie groups"*, With a foreword by J. Peter May, Edited by Zafer Mahmud and Mamoru Mimura, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996.

[2] J. Adams et al. Atlas of Lie Groups and Representations, `http://atlas.math.umd.edu/weyl/integral/E6/conjugacyClasses`.

[3] A. V. Alekseevskiĭ, Jordan finite commutative subgroups of simple complex Lie groups, (Russian), *Funkcional. Anal. i Priložen* **8(4)** (1974), 1–4; English translation: *Functional Anal. Appl.* **8(4)** (1974), 277–279 (1975).

[4] K. K. S. Andersen, J. Grodal, J. M. Møller, and A. Viruel, The classification of p-compact groups for p odd, *Ann. of Math. (2)* **167(1)** (2008), 95–210. `DOI: 10.4007/annals.2008.167.95`.

[5] D. Aranda Orna, C. Draper Fontanals, and V. Guido, Weyl groups of the fine gradings on $\mathfrak{e}_6$, *J. Algebra* **417(1)** (2014), 353–390. `DOI: 10.1016/j.jalgebra.2014.07.001`.

[6] Y. Bahturin and M. Kochetov, Classification of group gradings on simple Lie algebras of types $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, *J. Algebra* **324(11)** (2010), 2971–2989. `DOI: 10.1016/j.jalgebra.2010.03.003`.

[7] Y. A. Bahturin, I. P. Shestakov, and M. V. Zaicev, Gradings on simple Jordan and Lie algebras, *J. Algebra* **283(2)** (2005), 849–868. DOI: 10.1016/j.jalgebra.2004.10.007.

[8] Y. A. Bahturin and M. V. Tvalavadze, Group gradings on G_2, *Comm. Algebra* **37(3)** (2009), 885–893. DOI: 10.1080/00927870802278529.

[9] Y. A. Bahturin and M. V. Zaicev, Group gradings on simple Lie algebras of type "A", *J. Lie Theory* **16(4)** (2006), 719–742.

[10] P. Benito, C. Draper, and A. Elduque, Models of the octonions and G_2, *Linear Algebra Appl.* **371** (2003), 333–359. DOI: 10.1016/S0024-3795(03)00478-6.

[11] N. Bourbaki, *"Elements of mathematics. Algebra, Part I: Chapters 1–3"*, Translated from the French, Hermann, Paris; Addison-Wesley Publishing Co., Reading Mass., 1974.

[12] A. J. Calderón Martín, C. Draper, and C. Martín González, Gradings on the real forms of the Albert algebra, of g_2, and of f_4, *J. Math. Phys.* **51(5)** (2010), 053516, 21 pp. DOI: 10.1063/1.3326238.

[13] L. Corwin, Y. Ne'eman, and S. Sternberg, Graded Lie algebras in mathematics and physics (Bose–Fermi symmetry), *Rev. Modern Phys.* **47** (1975), 573–603. DOI: 10.1103/RevModPhys.47.573.

[14] C. Draper, Models of the Lie algebra F_4, *Linear Algebra Appl.* **428(11–12)** (2008), 2813–2839. DOI: 10.1016/j.laa.2008.01.009.

[15] C. Draper Fontanals, A non-computational approach to the gradings on $\mathfrak{f}_4$, *Rev. Mat. Iberoam.* **28(1)** (2012), 273–296. DOI: 10.4171/RMI/677.

[16] C. Draper and A. Elduque, Maximal finite abelian subgroups of E_8, Preprint. arXiv:1312.4326.

[17] C. Draper and A. Elduque, Fine gradings on the simple Lie algebras of type E, *Note Mat.* **34(1)** (2014), 53–88. DOI: 10.1285/i15900932v34n1p53.

[18] C. Draper and C. Martín, Gradings on $\mathfrak{g}_2$, *Linear Algebra Appl.* **418(1)** (2006), 85–111. DOI: 10.1016/j.laa.2006.01.017.

[19] C. Draper and C. Martín, Gradings on the Albert algebra and on $\mathfrak{f}_4$, *Rev. Mat. Iberoam.* **25(3)** (2009), 841–908.

[20] C. Draper, C. Martín, and A. Viruel, Fine gradings on the Lie algebra $\mathfrak{d}_4$, *Forum Math.* **22(5)** (2010), 863–877. DOI: 10.1515/FORUM.2010.046.

[21] A. Elduque, Gradings on octonions, *J. Algebra* **207(1)** (1998), 342–354. DOI: 10.1006/jabr.1998.7474.

[22] A. ELDUQUE, Jordan gradings on exceptional simple Lie algebras, *Proc. Amer. Math. Soc.* **137(12)** (2009), 4007–4017. DOI: 10.1090/S0002-9939-09-09994-8.

[23] A. ELDUQUE, Fine gradings on simple classical Lie algebras, *J. Algebra* **324(12)** (2010), 3532–3571. DOI: 10.1016/j.jalgebra.2010.09.018.

[24] A. ELDUQUE AND M. KOCHETOV, Gradings on the exceptional Lie algebras F_4 and G_2 revisited, *Rev. Mat. Iberoam.* **28(3)** (2012), 773–813. DOI: 10.4171/RMI/691.

[25] A. ELDUQUE AND M. KOCHETOV, *"Gradings on simple Lie algebras"*, Mathematical Surveys and Monographs **189**, American Mathematical Society, Providence, RI; Atlantic Association for Research in the Mathematical Sciences (AARMS), Halifax, NS, 2013. DOI: 10.1090/surv/189.

[26] A. ELDUQUE AND M. KOCHETOV, Graded modules over classical simple Lie algebras with a grading, *Israel J. Math.* **207(1)** (2015), 229–280. DOI: 10.1007/s11856-015-1174-8.

[27] R. L. GRIESS, JR., Elementary abelian p-subgroups of algebraic groups, *Geom. Dedicata* **39(3)** (1991), 253–305. DOI: 10.1007/BF00150757.

[28] M. HAVLÍCEK, J. PATERA, AND E. PELANTOVA, On Lie gradings II, *Linear Algebra Appl.* **277(1–3)** (1998), 97–125. DOI: 10.1016/S0024-3795(97)10039-8.

[29] M. HALVÍČEK, J. PATERA, AND E. PELANTOVÁ, Fine gradings of the real forms of $\mathrm{sl}(3, \mathbf{C})$, *Phys. Atomic Nuclei* **61(12)** (1998), 2183–2186; translated from *Yadernaya Fiz.* **61(12)** (1998), 2297–2300 (Russian).

[30] M. HAVLÍČEK, J. PATERA, AND E. PELANTOVÁ, On Lie gradings III. Gradings of the real forms of classical Lie algebras, *Linear Algebra Appl.* **314(1–3)** (2000), 1–47. DOI: 10.1016/S0024-3795(00)00099-9.

[31] W. H. HESSELINK, Special and pure gradings of Lie algebras, *Math. Z.* **179(1)** (1982), 135–149. DOI: 10.1007/BF01173920.

[32] J. E. HUMPHREYS, *"Introduction to Lie algebras and representation theory"*, Second printing, revised, Graduate Texts in Mathematics **9**, Springer-Verlag, New York-Berlin, 1978.

[33] P. JORDAN, Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik, *Nachr. Ges. Wiss. Göttingen* (1933), 209–214.

[34] V. G. Kac, *"Infinite-dimensional Lie algebras"*, Third edition, Cambridge University Press, Cambridge, 1990. DOI: 10.1017/CBO9780511626234.

[35] B. Kostant, Graded manifolds, graded Lie theory, and prequantization, in: *"Differential geometrical methods in mathematical physics"* (Proc. Sympos., Univ. Bonn, Bonn, 1975), Lecture Notes in Math. **570**, Springer, Berlin, 1977, pp. 177–306.

[36] J. M. Møller, N-determined 2-compact groups. II, *Fund. Math.* **196(1)** (2007), 1–90. DOI: 10.4064/fm196-1-1.

[37] F. Neumann, A theorem of Tits, normalizers of maximal tori and fibrewise Bousfield–Kan completions, *Publ. Res. Inst. Math. Sci.* **35(5)** (1999), 711–723. DOI: 10.2977/prims/1195143419.

[38] A. L. Onishchik and È. B. Vinberg, eds., *"Lie groups and Lie algebras, III. Structure of Lie groups and Lie algebras"*, A translation of Current problems in mathematics. Fundamental directions **41** (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990, Translation by V. Minachin, Encyclopaedia of Mathematical Sciences **41**, Springer-Verlag, Berlin, 1994.

[39] J. Patera and H. Zassenhaus, On Lie gradings. I, *Linear Algebra Appl.* **112** (1989), 87–159. DOI: 10.1016/0024-3795(89)90591-0.

Cristina Draper Fontanals:
Departamento de Matemática Aplicada
Escuela de las Ingenierías
Ampliación Campus de Teatinos, S/N
29071 Málaga
Spain
E-mail address: cdf@uma.es

Antonio Viruel Arbáizar:
Departamento de Álgebra, Geometría y Topología
Campus de Teatinos, S/N
Facultad de Ciencias, Ap. 59
29080 Málaga
Spain
E-mail address: viruel@agt.cie.uma.es

Primera versió rebuda el 31 de març de 2014,
darrera versió rebuda el 29 de setembre de 2014.