Publ. Mat. **64** (2020), 353–372
DOI: 10.5565/PUBLMAT6412014

BMO FROM DYADIC BMO FOR NONHOMOGENEOUS MEASURES

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Abstract: The usual one third trick allows to reduce problems involving general cubes to a countable family. Moreover, this covering lemma uses only dyadic cubes, which allows to use nice martingale properties in harmonic analysis problems. We consider alternatives to this technique in spaces equipped with nonhomogeneous measures. This entails additional difficulties which force us to consider martingale filtrations that are not regular. The dyadic covering that we find can be used to clarify the relationship between martingale BMO spaces and the most natural BMO space in this setting, which is the space RBMO introduced by Tolsa.

2010 Mathematics Subject Classification: Primary: 46E30; Secondary: 42B20, 46E40.

Key words: BMO, one third trick, nondoubling measures.

Introduction

Dyadic coverings are useful tools in harmonic analysis. When a problem involves manipulating a family of general cubes or balls, many times it is useful to restrict one's attention to a discrete – countable – family of them. Among such families, the most useful are those formed by dyadic cubes. To relate general cubes to dyadic ones, one uses a version of the so called one third trick. Roughly, this states that there exists a finite number of dyadic families such that given any cube Q, there exists a dyadic cube R in one of them such that $Q \subset R$ and $\ell(Q) \sim \ell(R)$ or, equivalently $|Q| \sim |R|$. The idea behind the one third trick goes back at least to the work of Wolff – see a detailed history in [**6**, p. 8–9] –, but more modern approaches have improved or used variants of it, such as [**11, 2, 9**]. Dyadic covering lemmata have found many applications to harmonic analysis, among which we will highlight two: first, the relationship between BMO and its dyadic (or martingale) counterpart is an almost immediate corollary (see [**11**]). Second, the theory of sparse domination (see [**10**]) initiated by Lerner and which has grown tremendously in the last few years seems to require some version of the one third trick.

The goal of this paper is to study dyadic-like covering arguments in a different context, also motivated by harmonic analysis. In particular, we consider them in the context of nonhomogeneous harmonic analysis. We work on $\mathbb{R}^d$ equipped with a measure μ of n-polynomial growth. This means that μ is a Radon measure that satisfies

$$\mu(B(x,r)) \leq C_\mu r^n, \quad \mu \text{ a.e. } x.$$

Without loss of generality, we will always assume that $C_\mu = 1$. Measures of n-polynomial growth appear naturally in the study of analytic capacity or rectifiability, where harmonic analysis tools have proved to be very useful [**14, 12**]. The main difficulty that measures of n-polynomial growth will pose to us is that they need not be *doubling*, which means that the measure of a ball and the measure of a fixed dilate of it need not be comparable. This means that the usual one third trick is not going to be useful for us, since $\ell(Q) \sim \ell(R)$ may no longer imply $\mu(Q) \sim \mu(R)$. The best we can hope for is to discretize the family of doubling cubes, which is often enough in the applications. Given constants $\alpha > 1$ and $\beta > 0$ we say that a cube or ball Q is (α,β)-doubling if $\mu(\alpha Q) \leq \beta\mu(Q)$. We can certainly cover (α,β)-doubling cubes by dyadic cubes while keeping the key property $\mu(Q) \sim \mu(R)$, as we show in Appendix A. However, this has limited applications because the resulting dyadic families are not complete and the σ-algebras that they generate do not form a filtration of $\mathbb{R}^d$. More specifically, we want our discrete families to satisfy the following properties: if $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$ is a sequence of partitions of $\operatorname{supp}(\mu)$ and $\sigma(\mathcal{A}_k)$ denotes the σ-algebra generated by $\mathcal{A}_k$, we say that they form an admissible filtration if $\{\sigma(\mathcal{A}_k)\}$ is a filtration of $\operatorname{supp}(\mu)$, that is, if the following are satisfied:

- $\sigma(\mathcal{A}_k) \subset \sigma(\mathcal{A}_{k+1})$ for all $k \in \mathbb{Z}$.
- For all $x \in \operatorname{supp}(\mu)$ and all $R > 0$, there exists $k \in \mathbb{Z}$ and $Q \in \mathcal{A}_k$ such that
$$B(x,R) \cap \operatorname{supp}(\mu) \subset Q.$$
- For μ-almost every point x in $\operatorname{supp}(\mu)$ and any locally integrable function f,
$$\lim_{k\to\infty} \frac{1}{\mu(Q_{k,x})} \int_{Q_{k,x}} f(y)\, d\mu(y) = f(x),$$
where $Q_{k,x} \in \mathcal{A}_k$ is the only element in $\mathcal{A}_k$ that contains x.

We also set $\mathcal{A} = \cup_{k\in\mathbb{Z}}\mathcal{A}_k$. To preserve the above structure in the families that we use in the covering, we resort to sets which are more complicated than cubes with sides parallel to the axes. The result that we obtain can be stated as follows.

Theorem A. *Fix $\alpha > 60$, $C_0 > (6\sqrt{d}\alpha)^d$, and set $\alpha_0 = 6\sqrt{d}\alpha$. There exist $N = N(d)$ families $\Sigma^1, \Sigma^2, \dots, \Sigma^N$ of $\operatorname{supp}(\mu) \subset \mathbb{R}^d$ with the following properties:*

(1) *For all $1 \le \ell \le N$, we have $\Sigma^\ell = \cup_{k\in\mathbb{Z}} \Sigma^\ell_k$, where Σ^ℓ_k is a partition of $\operatorname{supp}(\mu)$ and $\{\sigma(\Sigma^\ell_k)\}_{k\in\mathbb{Z}}$ is an admissible filtration.*

(2) *For each atom $T \in \Sigma^k$, $1 \le k \le N$, there exists a ball B_T such that*

$$B_T \cap \operatorname{supp}(\mu) \subset T \subset 30B_T \cap \operatorname{supp}(\mu) \quad \text{and} \quad \mu(\alpha B_T) \le C_0\mu(B_T).$$

(3) *For each (α_0, C_0)-doubling cube Q in $\mathbb{R}^d$, there exists $1 \le k \le N$ and $T \in \Sigma^k$ such that*

$$Q \cap \operatorname{supp}(\mu) \subset T, \quad \ell(Q) \sim r(B_T), \quad \text{and} \quad \mu(Q) \sim \mu(T).$$

The number $N = N(d)$ in the statement does not depend on the value of α_0, but the filtrations themselves do. As the statement suggests, the sets T that belong to each of the partitions Σ^ℓ_k will play the role of dyadic cubes of the same dimension as the measure (recall that they are similar to balls). The families Σ^ℓ satisfy additional properties (see Section 1 for precise definitions and details). The structure of each Σ^ℓ is known to be well adapted to the study of problems related to Calderón–Zygmund operators (see [**5**]), properties of the harmonic measure (see [**1**]), or rectifiability of sets (see [**4, 7**]), to name a few potential scenarios where our result may be of use. To prove Theorem A we will elaborate on the construction of [**5**], where a version of the celebrated lattice by David and Mattila [**7**] is tweaked to get further properties. The main challenge here is to be able to construct several dyadic-like lattices at the same time so that all doubling balls in the space are well adapted to them, something we believe has not been considered before.

As we said above, coverings by cubes in finitely many dyadic filtrations have found many different applications and are nowadays part of the standard set of tools in harmonic analysis. As an application of our methods, we shall prove versions of *BMO from dyadic BMO* results (see [**8**]) for nondoubling measures. In the nonhomogeneous setting, there are natural candidates for both spaces. The role of BMO is usually played by the space RBMO that was introduced by Tolsa in [**13**]. The definition of this space is postponed to Section 2, like the ones below. Its dyadic counterpart is denoted by RBMO_Σ and was introduced in [**5**]. It was shown there that this space satisfies

$$\mathrm{RBMO} \subsetneq \mathrm{RBMO}_\Sigma,$$

an analogous result to the obvious inclusion of BMO into dyadic BMO in the classical case. Ideally, one would like to prove a statement similar to

$$\mathrm{RBMO} = \bigcap_{j=1}^{N} \mathrm{RBMO}_{\Sigma^j},$$

with equivalent norms. This would be a direct generalization of the main result in [**11**]. However, this seems to be false for reasons that will become clear later in the text. What we will do is to define slight variations RBMO_Σ^* of RBMO_Σ (whose precise definition we postpone again to Section 2) for which the result holds. This is our second result:

Theorem B. *Let* $\{\Sigma^j\}_{1\leq j\leq N}$ *be the families given by Theorem A. We have*

$$\mathrm{RBMO} = \bigcap_{j=1}^{N} \mathrm{RBMO}^*_{\Sigma^j},$$

with equivalent norms. Moreover, for each family Σ^j *we have*

$$\mathrm{RBMO}^*_{\Sigma^j} \subset \mathrm{RBMO}_{\Sigma^j}.$$

The implications of Theorem B regarding the interpolation structure of the spaces involved in its statement and their relation with boundedness of Calderón–Zygmund operators are deferred to Section 2. Finally, we remark that in the same spirit as in [**5**], all our results generalize directly to the operator valued setting with very minor changes that we omit and that can be easily figured out by the interested reader. In any case, we expect that our main result, Theorem A, will be useful in other contexts where David–Mattila cubes naturally replace regular cubes in the nondoubling setting, such as in the aforementioned one of sparse domination.

Remark about notation. In this paper we use different kinds of partitions that generate martingale filtrations. To avoid confusion, we shall use the letter $\mathcal{D}$ (maybe with superscripts) to denote usual dyadic systems. We will use $\mathscr{D}$ for David–Mattila families, while we keep Σ for David–Mattila families where all the David–Mattila cubes are doubling (see Section 1 for details).

Acknowledgments. The author wants to thank Jill Pipher for initially pointing out the main question addressed in this note and Alex Barron for finding a key mistake in an earlier version of this manuscript.

1. The one third trick for nondoubling measures

1.1. David–Mattila families. The sets that we are going to deal with are the ones that appear in the statement of Theorem A and replace dyadic cubes in our covering argument. We will call them David–Mattila cubes in what follows, and they are sets Q which have the following property: there exists a ball B_Q, that we call the ball associated with Q, such that

$$B_Q \cap \operatorname{supp}(\mu) \subset Q \subset 30B_Q \cap \operatorname{supp}(\mu).$$

Given $\alpha > 30$, we say that a David–Mattila cube Q is (α,β)-doubling if its associated ball B_Q is (α,β)-doubling, that is, if $\mu(\alpha B_Q) \le \beta\mu(B_Q)$.

Proposition 1.1 (Theorem 3.2 in [**7**])**.** *Let μ be a Radon measure on $\mathbb{R}^d$ of n-polynomial growth. Fix $\alpha > 1$. Then there exist $C_0 = C_0(\alpha) > 1$ and $A_0 > 5000\, C_0$ such that, for each choice of $1 \le \tilde{C}_0 < C_0$, there exists a sequence $\mathscr{D} = \cup_{k\ge 0}\mathscr{D}_k$ of partitions of* $\operatorname{supp}(\mu)$ *into Borel subsets Q with the following properties:*

- *If $k < \ell$, $Q \in \mathscr{D}_k$, and $R \in \mathscr{D}_\ell$, then either $Q \cap R = \varnothing$ or else $Q \subset R$.*
- *For each k and each cube $Q \in \mathscr{D}_k$, there is a ball $B_Q = B(x_Q, r(Q))$ such that*
$$x_Q \in \operatorname{supp}(\mu), \quad C_0^{-1}\tilde{C}_0 A_0^{-k} \le r(Q) \le \tilde{C}_0\, A_0^{-k},$$
$$\begin{aligned}\operatorname{supp}(\mu) \cap B(Q) \subset Q \subset \operatorname{supp}(\mu) \cap 30B(Q)& \\ = \operatorname{supp}(\mu) \cap B(x_Q, 30r(Q))&,\end{aligned}$$
and the balls $5B(Q)$, $Q \in \mathscr{D}_k$, are disjoint.
- *The balls $\frac{1}{2}B_Q$ and $\frac{1}{2}B_{Q'}$ associated with $Q \ne Q'$ are disjoint unless $Q \subset Q'$ or $Q' \subset Q$.*
- *The cubes $Q \in \mathscr{D}_k$ have small boundaries: for each $Q \in \mathscr{D}_k$ and each integer $\ell \ge 0$, set*
$$N_\ell^{\mathrm{ext}}(Q) = \{x \in \operatorname{supp}(\mu) \setminus Q : \operatorname{dist}(x, Q) < A_0^{-k-\ell}\},$$
$$N_\ell^{\mathrm{int}}(Q) = \{x \in Q : \operatorname{dist}(x, \operatorname{supp}(\mu) \setminus Q) < A_0^{-k-\ell}\},$$
and
$$N_\ell(Q) = N_\ell^{\mathrm{ext}}(Q) \cup N_\ell^{\mathrm{int}}(Q).$$
Then
$$\mu(N_\ell(Q)) \le (C^{-1}C_0^{-3d-1}A_0)^{-\ell}\, \mu(90B_Q).$$
- *If $Q \in \mathscr{D}_k$ is not (α, C_0)-doubling, then $r(Q) = A_0^{-k}$ and*
$$\mu(\alpha B_Q) \le C_0^{-\ell}\, \mu(\alpha^{\ell+1}B_Q) \quad \text{for all } \ell \ge 1 \text{ with } \alpha^\ell \le C_0.$$

The roles of the various constants above are the following: A_0 is the quotient between the respective side lengths of a David–Mattila cube and any of its children. The constants α and C_0 are the relevant doublingness constants and $\widetilde{C}_0$ is just a parameter that one can vary to make sure that any given side length can be realized in a David–Mattila family. The last item in Proposition 1.1 implies that if $Q \in \mathscr{D}$ is (α, C_0)-doubling and $\widehat{Q}$ is the smallest (α, C_0)-doubling cube in $\mathscr{D}$ that contains it properly, then

$$\int_{\alpha B_{\widehat{Q}} \setminus \alpha B_Q} \frac{1}{|x_Q - y|^n} \, d\mu(y) \lesssim_{\alpha, A_0} 1. \tag{1.1}$$

This property is useful in applications to harmonic analysis, as we will see in Section 2. In [**5**] the David–Mattila construction is modified to yield a family of partitions that generates a filtration with a few additional properties. The first one is that the new filtration is two sided, which turns it into an admissible one (in the terminology of the introduction). In addition, the construction of [**5**] allows one to choose a particular (α, β)-doubling ball to be a ball B_Q associated to some David–Mattila cube. The result is the following:

Proposition 1.2 (Theorem A in [**5**]). *Let μ be a measure of n-polynomial growth on $\mathbb{R}^d$. Fix $\alpha > 1$. Then there exists a positive constant $C_0 = C_0(\alpha)$ such that, for any fixed (α, C_0)-doubling ball B, there exists a sequence of nested partitions $\{\Sigma_k\}_{k \in \mathbb{Z}}$ that satisfies the following properties, setting $\Sigma = \cup_{k \in \mathbb{Z}} \Sigma_k$:*

- *The sequence $\{\sigma(\Sigma_k)\}_{k \in \mathbb{Z}}$ is an admissible filtration.*
- *If $Q \in \Sigma$, then Q is a David–Mattila cube and there exists an (α, C_0)-doubling ball B_Q with $B_Q \subset Q \subset 30 B_Q$.*
- *There exists $Q \in \Sigma$ such that $B_Q = B$.*

Remark 1.3. Proposition 1.2 is proved using a similar construction as in Proposition 1.1. As a result, the thin boundaries of the David–Mattila cubes are preserved, and so is (1.1), which justifies our notation $\widehat{Q}$ for the smallest doubling ancestor of Q (it is strictly the father in Σ).

Remark 1.4. To prove Theorem A, one needs to modify the proof of Proposition 1.1 in a similar way as it was done to prove Proposition 1.2. The main change that one needs to implement affects only the first step of the proof, which is where the balls B_Q associated to the David–Mattila cubes are selected. This will be done so that the resulting family satisfies all the properties in the original construction by David and Mattila, so that we can just apply the remaining steps in the construction both from [**7**] and [**5**]. We cannot directly use the statement of Proposition 1.2

to prove Theorem A, although we show below how a weaker version can be obtained if one does precisely that.

1.2. The proof of Theorem A. We are going to give the proof first in an easier and much less interesting case, which is admitting countably many David–Mattila families. We do so because in that case we can directly apply Proposition 1.2, and therefore the proof is much simpler and may help to understand better the proof of the more complicated case.

Proposition 1.5. *Fix $\alpha > 60$, $C_0 > (6\sqrt{d}\alpha)^d$, and set $\alpha_0 = 6\sqrt{d}\alpha$. There exists a sequence $\{\Sigma^\ell\}_{\ell\in\mathbb{N}}$ of David–Mattila families with the properties of Proposition 1.2 such that for each (α_0, C_0)-doubling cube Q in $\mathbb{R}^d$, there exists $\ell \in \mathbb{N}$ and $T \in \Sigma^\ell$ such that*

$$Q \cap \operatorname{supp}(\mu) \subset T, \quad \ell(Q) \sim r(B_T), \quad \textit{and} \quad \mu(Q) \sim \mu(T).$$

Proof: The families Σ^ℓ that we shall construct will be given by directly applying Proposition 1.2. In order to prove the covering property we first discretize the family of cubes that we deal with. This is done via the usual one third trick. We use the version in Appendix A: given an (α_0, C_0)-doubling cube Q we can use Lemma A.1 to find a dyadic cube Q' belonging to one of 3^d dyadic systems $\mathcal{D}^1, \dots, \mathcal{D}^{3^d}$ such that $Q \subset Q' \subset 6Q$. These inclusions imply

$$\mu\left(\frac{\alpha_0}{6}Q'\right) \leq \mu(\alpha_0 Q) \leq C_0\mu(Q) \leq C_0\mu(Q'), \tag{1.2}$$

so Q' is (α_0', C_0)-doubling, with $\alpha_0' = \alpha_0/6$. Therefore, we have reduced to proving the covering property for (α_0', C_0)-doubling cubes belonging to the union of 3^d dyadic families. Note that this is a countable family.

Next we choose the families $\{\Sigma_\ell\}_{\ell\in\mathbb{N}}$. For each (α_0', C_0)-doubling cube Q belonging to one of the 3^d dyadic families $\mathcal{D}^1, \dots, \mathcal{D}^{3^d}$, consider the ball $B(Q)$ with the same center as Q and radius $r = \sqrt{d}\,\ell(Q)/2$. By definition, we have

$$Q \subset B(Q) \subset \sqrt{d}Q$$

and so $B(Q)$ is (α_0'', C_0)-doubling with $\alpha_0'' = \alpha_0'/\sqrt{d}$, by a computation similar to (1.2). Therefore, we may apply Proposition 1.2 with $\alpha = \alpha_0''$ and $B = B(Q)$ to get a family of David–Mattila cubes that we denote Σ^Q. Then our countable list of families is just $\{\Sigma^Q\}_Q$.

We can check the covering property now. Fix an (α_0', C_0)-doubling cube Q belonging to one of the systems $\mathcal{D}^1, \dots, \mathcal{D}^{3^d}$. Let T be the David–Mattila cube that belongs to Σ^Q and such that $B_T = B(Q)$. We then have

$$Q \cap \operatorname{supp}(\mu) \subset B(Q) \cap \operatorname{supp}(\mu) \subset T \subset 30B(Q) \cap \operatorname{supp}(\mu).$$

On the other hand, Q is (α_0', C_0)-doubling and $\alpha_0' = \alpha_0/6 = \sqrt{d}\alpha > 60\sqrt{d}$, and so

$$\mu(30B(Q)) \leq \mu(30\sqrt{d}Q) \leq \mu(\alpha_0' Q) \leq C_0 \mu(Q).$$

Therefore, $\mu(Q) \sim \mu(T)$ as desired. □

In order to prove Theorem A in the general case, we need to make sure that each of the families that we use covers more than one of the doubling dyadic cubes in $\mathbb{R}^d$, so that we only need a finite amount of them. This forces us to modify the first step of the construction in [**5**], which is the one in which the balls associated to the cubes in each generation are chosen. This choice is made via the $5R$ covering lemma. Since we need to ensure that certain balls are chosen in our construction, we will employ the following modification:

Lemma 1.6. *Let $E \subset \mathbb{R}^d$ be a set and $\mathcal{B}_0$ a countable family of balls of the same radius R and which are pairwise disjoint. Assume that for each $x \in E \setminus \cup_{B \in \mathcal{B}_0} B$ there exists a ball B_x with radius $r_x \leq R$. Then there exists a countable subcollection $\mathcal{B}_1 \subset \{B_x\}_{x \in (E \setminus \cup_{B \in \mathcal{B}_0} B)}$ satisfying:*

- *The balls in $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1$ are pairwise disjoint.*
- *$E \subset \cup_{B \in \mathcal{B}} 5B$.*

Proof: The proof is just the usual one of the classical $5R$ covering theorem applied to the family of balls

$$\mathcal{B}_0 \cup \{B_x\}_{x \in (E \setminus \cup_{B \in \mathcal{B}_0} B)},$$

with the only modification that we always pick the balls in $\mathcal{B}_0$ first. This can be done because their radii are maximal and they are disjoint. □

Proof of Theorem A: As in Proposition 1.5, items (1) and (2) in the statement will follow from the construction: the families that we are going to obtain satisfy all the properties in Proposition 1.2. To prove item (3), we divide the proof in several steps:

Step 1: Reduction to dyadic cubes. This is done exactly as in the proof of Proposition 1.5. We set $\alpha_0' = \alpha_0/6$ and we conclude that we can reduce ourselves to showing that there exist families $\Sigma^1, \ldots, \Sigma^N$ of David–Mattila cubes that generate admissible filtrations with the following additional property: for each (α_0', C_0)-doubling cube Q belonging to the union of 3^d dyadic families $\mathcal{D}^1, \ldots, \mathcal{D}^{3^d}$, there exists $1 \leq k \leq N$ and $T \in \Sigma^k$ such that

$$Q \cap \operatorname{supp}(\mu) \subset T, \quad \ell(Q) \sim r(B_T), \quad \text{and} \quad \mu(Q) \sim \mu(T).$$

Step 2: Splitting the doubling dyadic cubes of one dyadic system into N_0 families. Fix now $1 \leq \ell \leq 3^d$, and denote by $\mathcal{F}^\ell$ the family of (α_0', C_0)-doubling cubes that belong to $\mathcal{D}^\ell$. We want to partition $\mathcal{F}^\ell$ into finitely many families $\{\mathcal{F}_j^\ell\}_{1 \leq j \leq N_0}$ according to the following two rules:

(i) If two different cubes Q and Q' with the same side length 2^{-k} belong to $\mathcal{F}_j^\ell$, the distance between them is at least $5 \cdot 2^{-k}\sqrt{d}$.

(ii) If two different cubes Q and Q' with different respective side lengths 2^{-k} and 2^{-s}, $k < s$, belong to $\mathcal{F}_j^\ell$, then $2^{-k+s} = A_0^m$ for some positive integer m.

Recall that the constant A_0 is precisely the ratio of the side lengths of David–Mattila cubes of consecutive generations in Proposition 1.1. We are assuming that it is of the form $A_0 = 2^b$ for some positive integer b. The above splitting is obviously possible, and the smallest number N_0 depends on the dimension d and A_0. Also, without loss of generality, we are assuming that the constant C_0 is of the form $C_0 = 2^a$ for some positive integer a.

Step 3: Construction of a David–Mattila family associated with $\mathcal{F}_j^\ell$. Fix now an index $1 \leq j \leq N_0$. For each $Q \in \mathcal{F}_j^\ell$, denote by $B(Q)$ the ball with the same center as Q and radius $r = \sqrt{d}\,\ell(Q)/2$. We are going to construct a David–Mattila family $\Sigma^{j,\ell}$ such that for each cube $Q \in \mathcal{F}_j^\ell$, the ball $B(Q)$ is the ball B_T associated to some $T \in \Sigma^{j,\ell}$.

To construct $\Sigma^{j,\ell}$, we follow the proof of Proposition 1.2 in [**5**]. This means that we first want to build a David–Mattila system $\mathscr{D}^{j,\ell}$. This is done in Proposition 2.1 in [**5**], and that is the step that we need to modify. Later, $\mathscr{D}^{j,\ell}$ is used to construct $\Sigma^{j,\ell}$ in a way that we do not need to alter. The step that we must modify is the one in which the balls B_T associated with each $T \in \mathscr{D}^{j,\ell}$ are selected. After that, the original David and Mattila construction can be used directly. We do so separately for each generation. To that end, we further split

$$\mathcal{F}_j^\ell = \bigcup_{m \in \mathbb{Z}} \mathcal{F}_j^\ell(m) := \bigcup_{m \in \mathbb{Z}} \{Q \in \mathcal{F}_j^\ell : \ell(Q) = 2^{-m}\}.$$

For each nonempty $\mathcal{F}_j^\ell(m)$ we also set

$$\mathcal{B}(\mathcal{F}_j^\ell(m)) = \{B(Q) : Q \in \mathcal{F}_j^\ell(m)\}.$$

By property (ii) of the cubes in $\mathcal{F}_j^\ell$, there exists $m_0 = m_0(j,\ell)$ such that for each m there exists an integer $s = s(m)$ such that

$$r(B) = 2^{-m}\sqrt{d} = 2^{m_0}\sqrt{d}A_0^s, \quad \text{for all } B \in \mathcal{B}(\mathcal{F}_j^\ell(m)).$$

Our goal is to select the family of balls

$$\mathcal{B}^{j,\ell}(s) = \{B : B = B_T \text{ for some } T \in \mathscr{D}^{j,\ell},\ \ell(T) \sim A_0^s\}.$$

In order to be able to follow the scheme explained above to construct $\Sigma^{j,\ell}$, we must check that $\mathcal{B}^{j,\ell}(s)$ satisfies the following properties:

(a) If $B \in \mathcal{B}^{j,\ell}(s)$, then

$$C_0^{-1}2^{m_0}\sqrt{d}A_0^s \le r(B) \le 2^{m_0}\sqrt{d}A_0^s.$$

(b) If $B, B' \in \mathcal{B}^{j,\ell}(s)$, then $5B \cap 5B' = \emptyset$.

(c)

$$\operatorname{supp}(\mu) \subset \bigcup_{B \in \mathcal{B}^{j,\ell}(s)} 25B.$$

(d) If $B \in \mathcal{B}^{j,\ell}(s)$, either B is (α, C_0)-doubling or $r(B) = C_0^{-1}2^{m_0}\sqrt{d}A_0^s$ and

$$\mu(\alpha B) \le C_0^{-t}\,\mu(\alpha^{t+1}B) \quad \text{for all } t \ge 1 \text{ with } \alpha^t \le C_0.$$

Additionally, we want that

$$\mathcal{B}(\mathcal{F}_j^\ell(m)) \subset \mathcal{B}^{j,\ell}(s), \tag{1.3}$$

which is what ultimately will yield the desired covering property. We achieve all the properties applying Lemma 1.6 to a suitable family of balls. For each $x \in \operatorname{supp}(\mu) \setminus \cup_{B\in\mathcal{B}(\mathcal{F}_j^\ell(m))}B$, take B_x to be the largest (α, C_0)-doubling ball centered at x of radius $r(x)$ such that

$$C_0^{-1}2^{m_0}\sqrt{d}A_0^s \le r(x) \le 2^{m_0}\sqrt{d}A_0^s.$$

If there is no such doubling ball, then take $r(x) = C_0^{-1}2^{m_0}\sqrt{d}A_0^s$. Then we define

$$\mathcal{B}_0 := \{5B : B \in \mathcal{B}(\mathcal{F}_j^\ell(m))\}, \quad \mathcal{B}_1 := \{5B_x : x \in \operatorname{supp}(\mu) \setminus \cup_{B\in\mathcal{B}(\mathcal{F}_j^\ell(m))}B\}.$$

Notice that by property (i) of the families $\mathcal{F}_j^\ell$, if B and B' both belong to $\mathcal{B}_0$, then they are disjoint. We can therefore apply Lemma 1.6 to the families $\mathcal{B}_0$ and $\mathcal{B}_1$ to obtain a family $\mathcal{B} \subset \mathcal{B}_0 \cup \mathcal{B}_1$. We claim that

$$\mathcal{B}^{j,\ell}(s) := \left\{\frac{1}{5}B : B \in \mathcal{B}\right\}$$

satisfies properties (a)–(d) and (1.3). Indeed, (a) is immediate, (b) and (c) are part of the conclusion of the lemma, while (d) is satisfied by all balls under consideration. Finally, (1.3) also follows from the conclusion of Lemma 1.6. We repeat this process for each value of s to generate all the families $\mathcal{B}^{j,\ell}(s)$, $s \in \mathbb{Z}$. From this point onwards, we follow the rest of the steps in the proof of Theorem A in [**5**] and we obtain a family $\Sigma^{j,\ell} = \cup_{s\in\mathbb{Z}} \Sigma_s^{j,\ell}$ that satisfies the following properties:

- The sequence $\{\sigma(\Sigma_s^{j,\ell})\}_{s\in\mathbb{Z}}$ is an admissible filtration.
- If $Q \in \Sigma$, then Q is a David–Mattila cube and there exists an (α, C_0)-doubling ball B_Q with $B_Q \subset Q \subset 30B_Q$.
- For each m and each $B \in \mathcal{B}(\mathcal{F}_j^\ell(m))$, there exists $Q \in \Sigma$ such that $B_Q = B$.

Step 4: Conclusion. We now apply Steps 2 and 3 above to each of the dyadic systems $\mathcal{D}^\ell$. We get $N := 3^d N_0$ different David–Mattila families $\Sigma^{j,\ell}$, $1 \le \ell \le 3^d$, $1 \le j \le N_0$ that directly satisfy (1) and (2). Therefore, we only need to check (3) applying the reduction in Step 1 to complete the proof. Fix an (α_0, C_0)-doubling cube Q belonging to $\mathcal{D}^\ell$ for some $1 \le \ell \le 3^d$. Then there is j_0 so that $Q \in \mathcal{F}_{j_0}^\ell$. Using the last property proved in Step 3, denote by T the David–Mattila cube of the family $\Sigma^{j_0,\ell}$ such that $B_T = B(Q)$. We then have

$$Q \cap \operatorname{supp}(\mu) \subset B(Q) \cap \operatorname{supp}(\mu) \subset T \subset 30B(Q) \cap \operatorname{supp}(\mu).$$

On the other hand, Q is (α_0', C_0)-doubling and $\alpha_0' = \alpha_0/6 = \sqrt{d}\alpha > 60\sqrt{d}$, and so

$$\mu(30B(Q)) \le \mu(30\sqrt{d}Q) \le \mu(\alpha_0' Q) \le C_0 \mu(Q).$$

Therefore, $\mu(Q) \sim \mu(T)$ as desired. $\square$

2. RBMO from martingale RBMO

2.1. The RBMO space of Tolsa. In order to define the appropriate BMO space for measures of n-polynomial growth we need to recall a way of comparing two cubes independent of their respective side lengths. In particular, given a pair of cubes or balls Q and R, we define

$$\delta(Q, R) = 1 + \int_{2R\setminus 2Q} \frac{d\mu(y)}{|x_Q - y|^n}.$$

$\delta(Q, R)$ is a notion of distance between two cubes or balls Q and R with nontrivial intersection. We will always be considering cubes or balls with nontrivial intersection, and so we will not worry about $\delta(Q, R)$ when they are far away. The following easy properties of δ are going to be useful in the sequel:

Lemma 2.1 (see [**13**]). *The following hold:*

- *If $Q \subset R \subset T$, then* $\max\{\delta(Q, R), \delta(R, T)\} \le \delta(Q, T)$.
- *If $\ell(Q) \sim \ell(R)$, then* $\delta(Q, R) \sim 1$.

Fix two constants $\alpha > 1$, $\beta > \alpha^d$. We say that a function f belongs to RBMO if the following quantity is finite:

$$\|f\|_{\mathrm{RBMO}(\alpha,\beta)} = \max\{\|f\|_{\mathrm{DBMO}(\alpha,\beta)}, \|f\|_{\mathrm{RBMO}_d(\alpha,\beta)}\},$$

where

$$\|f\|_{\mathrm{DBMO}(\alpha,\beta)} = \sup_{Q\ (\alpha,\beta)\text{-doubling}} \frac{1}{\mu(Q)} \int_Q |f - \langle f\rangle_Q|\, d\mu$$

and

$$\|f\|_{\mathrm{RBMO}_d(\alpha,\beta)} = \sup_{\substack{Q\subset R \\ Q,R\ (\alpha,\beta)-\text{doubling}}} \frac{|\langle f\rangle_Q - \langle f\rangle_R|}{\delta(Q,R)}.$$

For us, $\langle f\rangle_Q$ denotes the integral average with respect to the measure μ:

$$\langle f\rangle_Q := \frac{1}{\mu(Q)} \int_Q f(x)\, d\mu(x).$$

It can be seen that the above definition does not really depend on the constants α and β:

Lemma 2.2 (see [**13**])**.** *Let α, α_0, β, β' be such that $\alpha, \alpha_0 > 1$, $\beta > \alpha^d$, and $\beta' > \alpha_0$. Then*

$$\|f\|_{\mathrm{RBMO}(\alpha,\beta)} \sim_{\alpha,\alpha_0,\beta,\beta'} \|f\|_{\mathrm{RBMO}(\alpha_0,\beta')}.$$

Because of Lemma 2.2, in what follows we will simply use the terms $\|f\|_{\mathrm{RBMO}}$, $\|f\|_{\mathrm{DBMO}}$, and $\|f\|_{\mathrm{RBMO}_d}$ without explicitly mentioning the associated constants α and β.

2.2. Dyadic RBMO spaces. In [**5**] a martingale version of RBMO was introduced. Fix a family of partitions $\Sigma = \{\Sigma_k\}_{k\in\mathbb{Z}}$ that generates an admissible filtration, as the one given by Proposition 1.2 or the ones given by Theorem A. We define

$$\mathsf{E}_{\Sigma_k} f := \sum_{Q\in\Sigma_k} \left(\frac{1}{\mu(Q)} \int_Q f\, d\mu\right) 1_Q.$$

E_{Σ_k} denotes the conditional expectation with respect to the σ-algebra generated by Σ_k, $\sigma(\Sigma_k)$. Then we define RBMO_Σ to be the martingale BMO space associated to the filtration generated by Σ, which is the space whose norm is given by

$$\begin{aligned}\|f\|_{\mathrm{RBMO}_\Sigma} &= \sup_k \|\mathsf{E}_{\Sigma_k}|f - \mathsf{E}_{\Sigma_{k-1}} f|\|_\infty \\ &\sim \sup_{Q\in\Sigma} \left[\frac{1}{\mu(Q)} \int_Q |f - \langle f\rangle_Q|\, d\mu + |\langle f\rangle_Q - \langle f\rangle_{\widehat{Q}}|\right].\end{aligned}$$

Above, $\widehat{Q}$ denotes the parent of $Q \in \Sigma_k$ in Σ, that is, the only atom $R \in \Sigma_{k-1}$ that properly contains Q. As we stated in the introduction, RBMO_Σ enjoys two remarkable properties that make it useful (see [**5**]):

– *Interpolation and predual.* RBMO_Σ is a martingale BMO space and so it can be used as an endpoint for interpolation of $L^p(\mu)$ spaces (see, for example, [**15**]). This means that
$$[\mathrm{RBMO}_\Sigma, L^1(\mu)]_{\frac{1}{p}} = L^p(\mu), \quad 1 < p < \infty,$$
with equivalent norms.

– *Suitable endpoint for boundedness of operators.* $\mathrm{RBMO} \subset \mathrm{RBMO}_\Sigma$, which implies that for all functions $f \in \mathrm{RBMO}$,
$$\|f\|_{\mathrm{RBMO}_\Sigma} \lesssim \|f\|_{\mathrm{RBMO}}.$$
Therefore, if an operator T is bounded from $L^\infty(\mu)$ to RBMO, then it is also bounded from $L^\infty(\mu)$ to RBMO_Σ. This happens for example when T is a Calderón–Zygmund operator, as was shown in [**13**].

We now introduce the modified RBMO_Σ spaces that we will use in the proof of Theorem B which will be, in a sense, an intermediate space between the two spaces defined previously. We start by redefining the quantity δ as follows: given David–Mattila cubes $Q \subset R$, we set
$$\delta(Q,R) := 1 + \int_{\alpha B_R \setminus \alpha B_Q} \frac{d\mu(y)}{|x_{B_Q} - y|^n}.$$
The abuse of notation above is justified by the fact that if Q and R belong to Σ, then $\delta(Q,R) \sim \delta(B_Q, B_R)$, which follows directly from the definition and Lemma 2.1. By (1.1), one can see that $\delta(Q,\widehat{Q}) \lesssim 1$ but it may be much smaller. Therefore, averages of a function in RBMO_Σ may oscillate more on two cubes of different generations (in Σ) than a function in RBMO would be allowed to. We have to take that into account to define the right dyadic BMO spaces to prove Theorem B. We do so as follows: given a David–Mattila family Σ as the one given in Proposition 1.2 or the ones given by Theorem A, we set
$$\|f\|_{\mathrm{RBMO}^*_\Sigma} := \sup_{Q\in\Sigma}\left[\frac{1}{\mu(Q)}\int_Q |f - \langle f\rangle_Q|\,d\mu\right] + \sup_{Q\in\Sigma, j>0}\left|\frac{\langle f\rangle_Q - \langle f\rangle_{Q^{(j)}}}{\delta(Q,Q^{(j)})}\right|.$$
In the above formula, $Q^{(j)}$ denotes the j-th dyadic ancestor of $Q \in \Sigma_k$, that is, the only $R \in \Sigma_{k-j}$ such that $Q \subset R$. From the definition, and again by (1.1), it follows that $\mathrm{RBMO}^*_\Sigma \subset \mathrm{RBMO}_\Sigma$, so
$$\|f\|_{\mathrm{RBMO}_\Sigma} \lesssim \|f\|_{\mathrm{RBMO}^*_\Sigma}.$$

We take now $\{\Sigma^j\}_{j=1}^N$ to be the families given by Theorem A, with $\alpha = 480\sqrt{d}$ – the reason of this choice will be clear immediately. We have now defined all the elements in the statement of Theorem B, that we restate here:

Theorem 2.3. *Under the choice of the value of α above, we have*

$$\mathrm{RBMO} = \bigcap_{j=1}^{N} \mathrm{RBMO}^*_{\Sigma^j},$$

with equivalent norms.

The rest of this section is entirely devoted to its proof.

2.3. Proof of Theorem 2.3. We start with the easier inequality, that is,

$$\|f\|_{\mathrm{RBMO}^*_{\Sigma^j}} \lesssim \|f\|_{\mathrm{RBMO}}, \tag{2.1}$$

for any j and any $f \in \mathrm{RBMO}$. Fix $Q \in \Sigma^j$ and $k \geq 0$. Denote by T an Euclidean cube such that $30B_Q \subset T$ and $2T \subset \frac{\alpha}{4}B_Q$. Also, let $\widehat{T}$ be a cube such that $60B_{Q^{(k)}} \subset \widehat{T}$ and $2\widehat{T} \subset \frac{\alpha}{2}B_{Q^{(k)}}$. Then, by our choice of α, both T and $\widehat{T}$ are $(2, C_0)$-doubling. Also, since $Q \subset Q^{(k)}$, then $T \subset \widehat{T}$. According to Lemma 2.2, we may choose the doublingness constants in the RBMO norm to be 2 and C_0. On the other hand, by Lemma 2.1 we see that $\delta(Q, Q^{(k)}) \sim \delta(T, \widehat{T})$. By the preceding discussion, we get

$$\begin{aligned}
\frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_Q| \, d\mu &\leq \frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_T| \, d\mu + |\langle f \rangle_T - \langle f \rangle_Q| \\
&\leq 2 \frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_T| \, d\mu \\
&\leq 2 \frac{\mu(T)}{\mu(Q)} \frac{1}{\mu(T)} \int_T |f - \langle f \rangle_T| \, d\mu \\
&\lesssim \frac{1}{\mu(T)} \int_T |f - \langle f \rangle_T| \, d\mu \leq \|f\|_{\mathrm{RBMO}}.
\end{aligned}$$

On the other hand, using the above computation we can also estimate

$$\begin{aligned}
|\langle f \rangle_Q - \langle f \rangle_{Q^{(k)}}| &\leq |\langle f \rangle_Q - \langle f \rangle_T| + |\langle f \rangle_T - \langle f \rangle_{\widehat{T}}| + |\langle f \rangle_{\widehat{T}} - \langle f \rangle_{Q^{(k)}}| \\
&\leq \frac{1}{\mu(Q)} \int_Q |f - \langle f \rangle_T| \, d\mu \\
&\quad + |\langle f \rangle_T - \langle f \rangle_{\widehat{T}}| + \frac{1}{\mu(Q^{(k)})} \int_{Q^{(k)}} |f - \langle f \rangle_{\widehat{T}}| \, d\mu \\
&\leq 2\|f\|_{\mathrm{RBMO}} + \delta(T, \widehat{T})\|f\|_{\mathrm{RBMO}} \lesssim \delta(Q, Q^{(k)})\|f\|_{\mathrm{RBMO}}.
\end{aligned}$$

Taking a supremum over $Q \in \Sigma^j$ and over $1 \le j \le N$ yields (2.1). We are therefore left with the more difficult inequality, that is,

$$\|f\|_{\mathrm{RBMO}} \lesssim \max_{1\le j\le N} \|f\|_{\mathrm{RBMO}^*_{\Sigma^j}}. \tag{2.2}$$

We use Lemma 2.2 so that we may assume that the cubes that appear in the expression of the RBMO norm are (α_0, C_0)-doubling, where α_0 is such that Theorem A holds with David–Mattila cubes that are (α, C_0)-doubling. That way, we can make sure that we can cover them by David–Mattila cubes in our families Σ^j. We estimate the terms in the norm in turn. Fix an (α_0, C_0)-doubling cube T, let $Q \in \Sigma^j$ be the David–Mattila cube associated to T via Theorem A, and recall that $T \cap \mathrm{supp}(\mu) \subset Q$ and $\mu(T) \sim \mu(Q)$. Then we have, as above,

$$\begin{aligned}\frac{1}{\mu(T)}\int_T |f-\langle f\rangle_T|\,d\mu &\le \frac{1}{\mu(T)}\int_T |f-\langle f\rangle_Q|\,d\mu + |\langle f\rangle_Q - \langle f\rangle_T| \\ &\le 2\frac{1}{\mu(T)}\int_T |f-\langle f\rangle_Q|\,d\mu \\ &\lesssim \frac{1}{\mu(Q)}\int_Q |f-\langle f\rangle_Q|\,d\mu \le \|f\|_{\mathrm{RBMO}}.\end{aligned}$$

We now have to estimate the term $|\langle f\rangle_T - \langle f\rangle_S|\delta(T,S)^{-1}$ for $T \subset S$. In this case we take $Q \in \Sigma^j$ as the one given by Theorem A such that $S \cap \mathrm{supp}(\mu) \subset Q$ and $\mu(S) \sim \mu(Q)$. We obviously have that $T \cap \mathrm{supp}(\mu) \subset Q$. Therefore, let $\mathcal{R}$ denote the family of the maximal descendants of Q in Σ^j that satisfy the following two properties:

- $T \cap \mathrm{supp}(\mu) \subset \bigcup_{R\in\mathcal{R}} R$.
- $R \subset 10T$ for all $R \in \mathcal{R}$.

Our splitting is now

$$\begin{aligned}|\langle f\rangle_T - \langle f\rangle_S| \le{}& \left|\langle f\rangle_T - \sum_{R\in\mathcal{R}} \frac{\mu(T\cap R)}{\mu(T)}\langle f\rangle_R\right| \\ &+ \left|\sum_{R\in\mathcal{R}} \frac{\mu(T\cap R)}{\mu(T)}\langle f\rangle_R - \langle f\rangle_Q\right| + |\langle f\rangle_Q - \langle f\rangle_S| \\ =:{}& \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

By a computation entirely analogous to the ones above, we can see that

$$\mathrm{III} \lesssim \|f\|_{\mathrm{RBMO}^*_{\Sigma^j}} \le \delta(T,S)\,\|f\|_{\mathrm{RBMO}^*_{\Sigma^j}}.$$

For I, we use the doubling property of T in the following way:

$$\begin{aligned}
\mathrm{I} &= \frac{1}{\mu(T)}\left|\int_T f\,d\mu - \sum_{R\in\mathcal{R}}\langle f\rangle_R\int_{T\cap R}d\mu\right| = \frac{1}{\mu(T)}\left|\sum_{R\in\mathcal{R}}\int_{T\cap R}(f-\langle f\rangle_R)\,d\mu\right| \\
&\leq \frac{1}{\mu(T)}\sum_{R\in\mathcal{R}}\int_{T\cap R}|f-\langle f\rangle_R|\,d\mu \leq \frac{1}{\mu(T)}\sum_{R\in\mathcal{R}}\frac{\mu(R)}{\mu(R)}\int_R|f-\langle f\rangle_R|\,d\mu \\
&\leq \frac{\sum_{R\in\mathcal{R}}\mu(R)}{\mu(T)}\|f\|_{\mathrm{RBMO}^*_{\Sigma^j}} \leq \frac{\mu(10T)}{\mu(T)}\|f\|_{\mathrm{RBMO}^*_{\Sigma^j}} \\
&\lesssim \|f\|_{\mathrm{RBMO}^*_{\Sigma^j}} \leq \delta(T,S)\|f\|_{\mathrm{RBMO}^*_{\Sigma^j}},
\end{aligned}$$

since all the cubes $R\in\mathcal{R}$ are contained in $10T$. Finally, we can readily check that

$$\begin{aligned}
\mathrm{II} &\leq \left|\sum_{R\in\mathcal{R}}\frac{\mu(T\cap R)}{\mu(T)}(\langle f\rangle_R-\langle f\rangle_Q)\right| \leq \sup_{R\in\mathcal{R}}|\langle f\rangle_R-\langle f\rangle_Q| \\
&\lesssim \left[\sup_{R\in\mathcal{R}}\delta(R,Q)\right]\|f\|_{\mathrm{RBMO}^*_{\Sigma^j}}.
\end{aligned}$$

Therefore, it is enough to check that for $R\in\mathcal{R}$ we have

$$\delta(R,Q)\lesssim\delta(T,S). \tag{2.3}$$

This can be checked via the following calculation:

$$\begin{aligned}
\delta(R,Q) &= 1+\int_{\alpha B_Q\setminus\alpha B_R}\frac{1}{|y-x_{B_R}|^n}\,d\mu(y) \\
&= 1+\int_{\alpha B_Q\setminus(2S\cup\alpha B_R)}\frac{1}{|y-x_{B_R}|^n}\,d\mu(y) \\
&\quad+\int_{2S\setminus(2T\cup\alpha B_R)}\frac{1}{|y-x_{B_R}|^n}\,d\mu(y) \\
&\quad+\int_{2T\setminus\alpha B_R}\frac{1}{|y-x_{B_R}|^n}\,d\mu(y) =: 1+\mathrm{I}'+\mathrm{II}'+\mathrm{III}'.
\end{aligned}$$

I' and III' are bounded above by an absolute constant since the pairs B_Q and S (on the one hand) and T and B_R (on the other hand) have comparable radii and side length, respectively. For term II' we have

$$\mathrm{II}'\lesssim\int_{4S\setminus 2T}\frac{1}{|y-x_T|^n}\,d\mu(y)\sim\delta(T,S).$$

Therefore, we have (2.3) and the proof is complete. □

Remark 2.4. Contrary to what happens in the Lebesgue measure case (see [**8**]), Theorem B does not imply that the norm in RBMO can be computed as the average of translates and dilates of RBMO_Σ for a given Σ.

2.4. Relationship between the different BMO-type spaces. We have considered three different kinds of BMO spaces associated with μ. If we fix a sequence of nested partitions Σ that generates an admissible filtration, we have

$$\mathrm{RBMO} \subsetneq \mathrm{RBMO}^*_\Sigma \subsetneq \mathrm{RBMO}_\Sigma.$$

This means that operators that map $L^\infty(\mu)$ to RBMO also map $L^\infty(\mu)$ to RBMO^*_Σ. On the other hand, Theorem B implies

$$L^\infty(\mu) \subsetneq \mathrm{RBMO} = \bigcap_{j=1}^{N} \mathrm{RBMO}^*_{\Sigma^j} \subsetneq \bigcap_{j=1}^{N} \mathrm{RBMO}_{\Sigma^j}, \tag{2.4}$$

where Σ^j are the families given by Theorem A. As we said before, it was shown in [**5**] that

$$[\mathrm{RBMO}_{\Sigma^j}, L^1(\mu)]_{\frac{1}{p}} = L^p(\mu), \quad 1 < p < \infty,$$

for each j, and we obviously have

$$[L^\infty(\mu), L^1(\mu)]_{\frac{1}{p}} = L^p(\mu), \quad 1 < p < \infty.$$

Therefore, (2.4) implies that each space $\mathrm{RBMO}^*_{\Sigma^j}$ interpolates as well:

$$[\mathrm{RBMO}^*_{\Sigma^j}, L^1(\mu)]_{\frac{1}{p}} = L^p(\mu), \quad 1 < p < \infty.$$

This means that even though they are not martingale BMO spaces, our new spaces RBMO^*_Σ are suitable counterparts of the usual dyadic BMO in the nonhomogeneous setting, because they satisfy the following three properties:

- They serve as endpoints for boundedness of operators (and particularly, Calderón–Zygmund ones).
- They interpolate with the L^p scale in the usual sense.
- The intersection of finitely many of them yields a norm comparable to that of the natural BMO space in this setting, which is RBMO.

Appendix A. Usual dyadic cubes in the nondoubling setting

The usual one third trick can be applied in the nondoubling setting so long as the cubes involved are are all doubling. This can be applied to the RBMO norm. We shall use the following version of the trick, that is also used in its more standard version in Section 1:

Lemma A.1. *There exist 3^d dyadic systems $\mathcal{D}^1, \mathcal{D}^2, \dots, \mathcal{D}^{3^d}$ on $\mathbb{R}^d$ such that, for all pairs of cubes Q_1 and Q_2, there exists $1 \leq k \leq 3^d$ and cubes $T_1, T_2 \in \mathcal{D}^k$ such that*

$$Q_1 \subset T_1 \subset 6Q_1 \text{ and } Q_2 \subset T_2 \subset 6Q_2.$$

Lemma A.1 is essentially known. Its proof is a minimal variation of Lemma 2.5 of [**9**] (the only difference with the result there is the fact that the cubes Q_1 and Q_2 in the statement of Lemma A.1 need not be dyadic).

Remark A.2. Following the arguments in [**2**] (see also [**3**]) one can see that the optimal number of dyadic systems such that Lemma A.1 holds is $2d+1$.

If $\mathcal{D}$ is a dyadic system, we can define the dyadic version of RBMO by

$$\|f\|_{\mathrm{RBMO}_{\alpha,\beta,\mathcal{D}}} := \sup_{\substack{Q\in\mathcal{D} \\ Q\ (\alpha,\beta)\text{-doubling}}} \frac{1}{\mu(Q)} \int_Q |f - \langle f\rangle_Q|\, d\mu + \sup_{\substack{Q,R\in\mathcal{D} \\ Q,R\ (\alpha,\beta)\text{-doubling}}} \left| \frac{\langle f\rangle_Q - \langle f\rangle_R}{\delta(Q,R)} \right|.$$

Immediately, we get the following:

Corollary A.3. *Fix $\alpha \geq 2$ and $\beta > (6\alpha)^d$. Let $\mathcal{D}^j$, $1 \leq j \leq 2d+1$, be the dyadic systems given by Lemma A.1. Then we have*

$$\|f\|_{\mathrm{RBMO}} \sim \sup_{1\leq j\leq 3^d} \|f\|_{\mathrm{RBMO}_{\alpha,\beta,\mathcal{D}^j}}.$$

Proof: First, it is immediate that

$$\|f\|_{\mathrm{RBMO}_{\alpha,\beta,\mathcal{D}^j}} \leq \|f\|_{\mathrm{RBMO}}$$

for all j. For the reverse inclusion, we may assume that the cubes in the definition of the RBMO norm are $(6\alpha,\beta)$-doubling. Then, given such a cube Q we may use Lemma A.1 to find an index k and a cube $T \in \mathcal{D}^k$ such that $Q \subset T \subset 6Q$. We know that

$$\mu(\alpha T) \leq \mu(6\alpha Q) \leq \beta\mu(Q) \leq \beta\mu(T),$$

so T is (α,β)-doubling. Therefore, we can estimate

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q |f-\langle f\rangle_Q|\,d\mu &\leq \frac{1}{\mu(Q)}\int_Q |f-\langle f\rangle_T|\,d\mu + |\langle f\rangle_T-\langle f\rangle_Q| \\ &\leq 2\frac{1}{\mu(Q)}\int_Q |f-\langle f\rangle_T|\,d\mu \\ &\leq 2\frac{\mu(T)}{\mu(Q)}\frac{1}{\mu(T)}\int_T |f-\langle f\rangle_T|\,d\mu \\ &\leq 2\frac{\mu(6\alpha Q)}{\mu(Q)}\frac{1}{\mu(T)}\int_T |f-\langle f\rangle_T|\,d\mu \\ &\lesssim \|f\|_{\mathrm{RBMO}_{\alpha,\beta,\mathcal{D}^k}}.\end{aligned}$$

Finally, given $(6\alpha,\beta)$-doubling cubes Q and R, we apply Lemma A.1 with $Q_1=Q$ and $Q_2=R$. Then, by Lemma 2.1 and the computation above we find that

$$\begin{aligned}|\langle f\rangle_Q-\langle f\rangle_R| &\leq |\langle f\rangle_Q-\langle f\rangle_{T_1}|+|\langle f\rangle_{T_2}-\langle f\rangle_R|+|\langle f\rangle_{T_1}-\langle f\rangle_{T_2}| \\ &\leq \frac{1}{\mu(Q)}\int_Q |f-\langle f\rangle_{T_1}|\,d\mu \\ &\quad+\frac{1}{\mu(R)}\int_R |f-\langle f\rangle_{T_2}|\,d\mu+|\langle f\rangle_{T_1}-\langle f\rangle_{T_2}| \\ &\lesssim \|f\|_{\mathrm{RBMO}_{\alpha,\beta,\mathcal{D}^k}}+\delta(T_1,T_2)\|f\|_{\mathrm{RBMO}_{\alpha,\beta,\mathcal{D}^k}} \\ &\lesssim \delta(Q,R)\|f\|_{\mathrm{RBMO}_{\alpha,\beta,\mathcal{D}^k}}.\end{aligned}$$

This shows that

$$\|f\|_{\mathrm{RBMO}} \lesssim \sup_{1\leq j\leq 3^3}\|f\|_{\mathrm{RBMO}_{\alpha,\beta,\mathcal{D}^j}}. \qquad \square$$

Corollary A.3 is more similar to Mei's statement in [**11**] and simpler, because it only relies on usual dyadic cubes. However, the spaces $\mathrm{RBMO}_{\mathcal{D}^j}$ are not martingale BMO spaces in general (in fact, the quantity $\|f\|_{\mathrm{RBMO}_{\alpha,\beta,\mathcal{D}}}$ need not be a norm modulo constants). It is far from clear whether they interpolate with the L^p scale or not in case μ is not doubling. This indicates that our approach via David–Mattila construction yields a more useful result.

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Primera versió rebuda el 28 de juny de 2018,
darrera versió rebuda el 14 de gener de 2019.