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# ON NUCLEARITY OF THE $C^*$-ALGEBRA OF AN INVERSE SEMIGROUP

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**Abstract:** We show that the universal groupoid of an inverse semigroup $S$ is topologically (measurewise) amenable if and only if $S$ is hyperfinite and all members of a family of subsemigroups of $S$ indexed by the spectrum of the commutative $C^*$-algebra $C^*(E_S)$ on the idempotents $E_S$ of $S$ are amenable. Thereby we solve some problems raised by A. L. T. Paterson.

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## 1. Introduction

Locally compact (Hausdorff) groups are the central object in abstract harmonic analysis. In the last century a great deal of progress has been made on the analysis of locally compact groups. There are several classes of groups such as Abelian or compact groups for which we know a lot more, but there are many important examples which are not Abelian or compact. Therefore, finding classes of groups which are large enough to include these special cases and restricted enough to satisfy interesting properties is of great importance. One important class of this kind is the class of amenable groups. There are several equivalent definitions of amenability of a locally compact group $G$; for instance $G$ is amenable if and only if the space of bounded continuous functions on $G$ has a state which is invariant under left translations by elements of $G$. The class of amenable groups is fairly large. It contains all Abelian and compact groups and is closed under extensions, taking quotients, or passing to closed subgroups. We refer the interested reader to [**13**] for more details.

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One of the pioneering works on amenable groups in the language of cohomology is that of B. E. Johnson [**9**]. In this important monograph a concept of amenability is defined for Banach algebras and it is shown that amenability of a locally compact group $G$ is equivalent to amenability of its group algebra $L^1(G)$, consisting of all absolutely integrable complex valued functions on $G$ (with respect to a left Haar measure on $G$). Amenability plays a central role in the theory of Banach algebras. It is equivalent to other interesting properties for important subclasses such as $C^*$-algebras or von Neumann algebras. Indeed, the Banach space bidual $A^{**}$ of any $C^*$-algebra $A$ is a von Neumann algebra, called the enveloping von Neumann algebra of $A$ and, by a result of Choi–Effros [**3**], $A$ is nuclear if and only if $A^{**}$ is injective, and this is the case if and only if $A$ is amenable, as shown by Connes [**4**] and Haagerup [**8**]. It is also shown that amenability of a locally compact group $G$, nuclearity of its reduced $C^*$-algebra $C_r^*(G)$, and injectivity of its group von Neumann algebra $VN(G)$ are equivalent [**13**].

Locally compact groups are still the most common theme of research in abstract Harmonic analysis, but there are many examples in which the underlying symmetries are described by more general algebraic structures. Topological semigroups and groupoids are two important examples of such a structure. The main difficulty to deal with harmonic analysis on semigroups is the lack of a Haar measure. One class of (discrete) semigroups for which a good progress on the analysis has been made is the class of inverse semigroups. One of the most important problems for an inverse semigroup $S$, which has been the subject of an intense research, is the problem of finding necessary and sufficient conditions on $S$ such that the semigroup algebra $\ell^1(S)$ is amenable. There is an important negative result by Duncan and Namioka [**6**]. They showed that if $\ell^1(S)$ is amenable, then the set $E_S$ of idempotents of $S$ is finite. As there are amenable inverse semigroups for which $E_S$ is not finite, there is no hope for Johnson's theorem to be true for inverse semigroups. When $E_S$ is finite, it is shown in [**6**] that amenability of $\ell^1(S)$ is equivalent to amenability of all maximal subgroups of $S$. One important recent result in this direction is due to A. L. T. Paterson who showed in his recent monograph that if all the maximal subgroups of $S$ are amenable, then the semigroup von Neumann algebra $VN(S)$ is injective [**14**]. In this result, the finiteness of $E_S$ is not assumed but $S$ is assumed to be countable. The approach of [**14**] is based on the fact that the reduced $C^*$-algebra of $S$ and its universal groupoid $G_S$ are $*$-isomorphic (as it stands in [**14**], this result is proved for the case in which $G_S$ is Hausdorff, and so the above result is valid only if $S$ is E-unitary, but this restriction could be removed [**10**]). The importance of this approach is mainly because we know a lot about amenability of (topological and measured) groupoids [**1**]. In the same monograph, Paterson asks if a similar result could be proved for the reduced $C^*$-algebra $C_r^*(S)$. In particular, he asks if the amenability of all maximal subgroups of $S$ implies nuclearity of $C_r^*(S)$ (something which is true if $E_S$ is finite). This is an interesting problem on its own and could be related to some more central problems in operator algebras. Recently there has been very significant interest in the construction of $C^*$-algebras from inverse semigroups and structural questions for these $C^*$-algebras, including nuclearity, membership to the UCT class, and the possession of a natural Cartan subalgebra (for other related advances, see [**7**] and [**12**]).

The present paper was motivated by an example suggested by David Cowan which negatively answers the above question (see Section 3). In our search for necessary and sufficient conditions for nuclearity of $C_r^*(S)$, we realized that a larger class of subsemigroups of $S$ should be considered. The maximal subgroups of $S$ are indexed by elements of $E_S$ which is a commutative inverse semigroup. Motivated by [**10**], we considered a class of subsemigroups of $S$ indexed by the spectrum of the commutative $C^*$-algebra $C^*(E_S)$ which includes all maximal subgroups. It is shown that amenability of all these subgroups plus an extra measure theoretic property of $S$ is equivalent to the nuclearity of $C_r^*(S)$. The latter property is called hyperfiniteness and is motivated by the celebrated theorem of Connes–Feldman–Weiss [**5**] (see also [**1**]). The proof uses the fact that $C_r^*(S)$ and $C_r^*(G_S)$ are $*$-isomorphic for any inverse semigroup $S$ [**10**]. It is quite natural to solve the amenability problem for an inverse semigroup by passing to its universal groupoid. This is mainly because there is an extensive literature on amenability of ($r$-discrete) groupoids [**1**]. We freely use such concepts as quasi-invariant measures, principal groupoid, discrete (hyperfinite) equivalence relations, ample groupoids, for which we refer the reader to [**17**] and [**1**].

## 2. Universal groupoid

In this section we briefly review the basic properties of the universal groupoid of an inverse semigroup. Our main references are [**14**] and [**10**]. Recall that a (discrete) semigroup $S$ is called an *inverse semigroup* if for each $s \in S$, there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. The set of idempotents in $S$ is denoted by $E_S$. This is a commutative subsemigroup of $S$ which has a natural ordering defined by $e \leq f$ if and only if $ef = e$. For each $e \in E_S$, the set

$$S_e^e = \{s \in S : ss^* = s^*s = e\}$$

is a subgroup of $S$. Indeed it is easy to see that these are exactly the maximal subgroups of $S$ (see Subsection 4.5 of [**14**]). The *left regular representation* of $S$ in $\ell^2(S)$ is naturally defined by $\pi_2(s)(\delta_t) = \delta_{st}$ if $tt^* \leq s^*s$, and $\pi_2(s)(\delta_t) = 0$ otherwise. This lifts to a (faithful) $*$-representation of $\ell^1(S)$ in $\ell^2(S)$. The $C^*$-algebra and $W^*$-algebra generated by $\pi_2(S)$, which are the norm and weak operator closures of $\pi_2(\ell^1(S))$ in $\mathfrak{B}(\ell^2(S))$, are called the reduced $C^*$-algebra and the von Neumann algebra of $S$ and denoted by $C_r^*(S)$ and $VN(S)$, respectively. Similarly the full $C^*$-algebra $C^*(S)$ could be defined by considering all representations of $S$ (see [**14**] for details).

Now we briefly discuss the construction of the universal groupoid $G_S$ of an inverse semigroup $S$. For each unital, discrete, commutative $*$-semigroup $T$, the set $T^*$ of semicharacters of $T$ with the topology of pointwise convergence is a unital, locally compact, Hausdorff $*$-semigroup in which the subset $\hat{T}$ of bounded semicharacters form a compact unital subsemigroup [**2**]. If $T$ is not unital, then $\hat{T}$ is not necessarily compact. In each case, elements of $\hat{T}$ take value in the closed unit disk $\mathbb{D}$ (in the unit circle $\mathbb{T}$, when $S$ is a group). When $T$ is an idempotent semigroup, elements of $T^* = \hat{T}$ take only values 0 and 1. When $S$ is an inverse semigroup, $E_S$ is a commutative inverse semigroup, which is unital exactly when it has a maximum element in its natural order. Consider the commutative Banach $*$-algebra $\ell^1(E_S)$ and let $X$ be the space of all multiplicative bounded linear functionals on $\ell^1(E_S)$ with the relative weak$^*$-topology, hence $X$ is the spectrum of the abelian $C^*$-algebra $C^*(E_S)$. We may then regard elements of $E_S$ as continuous functions on $X$. For each $s \in S$, and $x \in X$ with $s^*s(x) = 1$, we define $s.x \in X$ by $s.x(e) = x(s^*es)$ ($e \in E_S$). This defines an action of $S$ on $\hat{E}_S$ which keeps (the dense subset) $E_S$ invariant (see [**18**] for more details on the space of semicharacters $\hat{E}_S$, its topology, and the above action). Indeed, one can easily check that $s.e = ses^*$, for each $s \in S$, $e \in E_S$. There is an alternative way of thinking about the action of $S$ on $X$. For each $s \in S$, consider the subset

$$X_s := \{x \in X : x(s^*s) = 1,\, s.x = x\}$$

of $X$. It is clear that $X_s$ is compact and open in $X$. Now each $s \in S$ could be considered as a surjective homeomorphism $s\colon X_s \to X_{s^*}$, and $S$ could be considered as a semigroup of partial homeomorphisms on $X$, acting relatively free on $X$ [**17**, Definition I.2.11].

Now let $\Sigma = \{(s, x) \in S \times X : s^*s(x) = 1\}$ with the relative product topology, and define an equivalence relation on $\Sigma$ by $(s, x) \sim (t, y)$ if and only if $x = y$ and there is $e \in E_S$ such that $e(x) = 1$ and $se = te$. Let $G_S$

be the quotient of $\Sigma$ with respect to this equivalence relation and $\pi\colon \Sigma \to G_S$ be the quotient map. We write $[s,x]$ for the equivalence class $\pi((s,x))$ of $(s,x)$. Then $G_S$ is a locally compact groupoid with $G_S^{(0)} = X$ under the quotient topology and the following operations:

$$s[s,x]=x, \quad r[s,x]=s.x, \quad [s,x]^{-1}=[s^*,s.x], \quad \text{and} \quad [s,t.y][t,y]=[st,y],$$

for $s,t \in S$ and $x,y \in X$. The groupoid $G_S$ is called the universal groupoid of $S$. It is an $r$-discrete groupoid which is not Hausdorff in general (it is Hausdorff if $S$ is E-unitary, see [**14**], or see [**18**, Theorem 5.17] for a complete characterization). Each $e \in E_S$ defines an element $\varepsilon_e \in X$ by $f(\varepsilon_e) = 1$ if $e \leq f$ and $f(\varepsilon_e) = 0$ otherwise, where $f \in E_S$. The fact that $\varepsilon_e \in X$ follows from the observation that $\bar{e} := \{f \in E_S : f \geq e\}$ forms a filter in the semilattice $E_S$ [**14**, p. 174]. Also each $s \in S$ defines an element $\varepsilon_s = [s, \varepsilon_{s^*s}] \in G_S$. This provides an embedding of $S$ in $G_S$ which sends $E_S$ onto a dense subset of $X$. We usually identify $E_S$ with its image. For each $x \in X = C^*(E_S)\hat{}$,

$$S_x := \{s \in S : x(s^*s) = 1,\, s.x = x\}$$

is a subsemigroup of $S$. Indeed if $s,t \in S_x$, then

$$x((st)^*(st)) = x(t^*s^*st) = t.x(s^*s) = x(s^*s) = 1.$$

The maximal group homomorphic image of $S_x$ is the discrete group $G(S_x) = S_x/\sim$, where for $s,t \in S_x$, we define $s \sim t$ if and only if there is $e \in E_S$ such that $e(x) = 1$ and $se = te$. When $x = e \in E_S$, then $S_e = \{s \in S : s^*s \geq e\}$ and $G(S_e) = S_e^e$. In general, it is easy to see that the group $G(S_x)$ is isomorphic to the isotropy group $(G_S)_x^x$. Moreover, the principal groupoid $\Gamma_S := \{(r(\gamma), s(\gamma)) : \gamma \in G_S\}$ of $G_S$ [**17**, p. 39] is an $r$-discrete groupoid whose ample semigroup $\Gamma_S^a$ is the ample inverse semigroup generated by $S$ [**17**, Proposition I.2.13].

**Lemma 2.1.** *For each $x \in X$, the maximal group homomorphic image of the subsemigroup*

$$S_x = \{s \in S : x(s^*s) = 1,\, s.x = x\}$$

*is isomorphic to the isotropy group*

$$(G_S)_x^x = \{[s,x] : s \in S,\, x(s^*s) = 1\}.$$

**Lemma 2.2.** *$X = C^*(E_S)\hat{}$ with the weak$^*$ topology is homeomorphic to $\hat{E}_S$ with the topology of pointwise convergence.*

## 3. Amenability of the universal groupoid

Our main reference for amenability of groupoids is [**1**] to which we refer the reader for the definition of (locally compact and measured) groupoids and different types of amenability defined for them (in particular, see Definitions 2.2.8 and 3.3.1 in [**1**], respectively for topological and measurewise amenability). As in the previous section, throughout this section $S$ is an arbitrary inverse semigroup with universal groupoid $G_S$. For a general (measured or locally compact) groupoid $G$, the $C^*$-algebras $C^*_r(G)$ and $C^*(G)$ could be defined as in the group case. Also for each quasi-invariant measure $\mu$ on the set $X = G^{(0)}$ of the unit elements, the von Neumann algebra $VN(G, \mu)$ is defined similar to the group case (see [**1**] for details). Note that in [**1**] an alternative von Neumann algebra $W^*(G, \mu)$ is also defined, which is naturally isomorphic to $VN(G, \mu)$ (see the remarks in the beginning of Subsection 6.2 in [**1**]).

**Proposition 3.1.** *The $C^*$-algebras $C^*(S)$ and $C^*_r(S)$ are $*$-isomorphic to $C^*(G_S)$ and $C^*_r(G_S)$, respectively. If $S$ is countable, the von Neumann algebras $VN(S)$ and $VN(G_S, \mu_0)$ are $*$-isomorphic, where*

$$\mu_0 = \sum_{i=1}^{\infty} 2^{-n} \delta_{e_n}.$$

*Proof:* The first statement is proved in [**14**, Theorems 4.4.1 and 4.4.2] for the countable, E-unitary case and in [**10**] for the general case. The second statement is included in the proof of Theorem 4.5.2 in [**14**]. □

For a groupoid $G$, the isotropy is the family of groups $G^x_x := \{\gamma \in G : s(\gamma) = r(\gamma) = x\}$ indexed by $x \in G^{0)}$.

**Lemma 3.2.** *$G_S$ has discrete isotropy everywhere. Also it has a continuous Haar system. When $S$ is countable, $G_S$ has a (discrete) quasi-invariant measure $\mu$.*

*Proof:* This follows from the fact that $G_S$ is étale. Alternatively, consider the quotient map $\pi\colon S \times X \to G_S$. Then, for $x \in X$, the set $\pi^{-1}((G_S)^x_x) = \{s \in S : s^*s(x) = 1,\ s.x = x\} \times \{x\}$ is discrete in $S \times X$, so $(G_S)^x_x$ is discrete in $G_S$. The Haar system of $G_S$ consists of counting measures which is clearly a continuous system. The (discrete) quasi-invariant measure $\mu$ on $G_S$ is constructed in the proof of Theorem 4.4.2 in [**14**]. □

The next two lemmas can be found in [**1**, Corollary 5.3.33 and Theorem 5.3.42]:

**Lemma 3.3.** *If $(G, \lambda, \mu)$ is a measured groupoid and $(\Gamma, \alpha, \mu)$ is its associated principal groupoid, then $G$ is amenable if and only if $\Gamma$ is amenable and, for $\mu$-a.a. $x \in G^{(0)}$, the isotropy groups $G_x^x$ are amenable.*

**Lemma 3.4.** *A discrete equivalence relation $(\Gamma, \alpha, \mu)$ is amenable if and only if it is hyperfinite.*

The following follows from Lemma 3.2 and [**17**, Proposition I.2.13].

**Lemma 3.5.** *Let $\Gamma_S$ be the principal groupoid of $G_S$. Then $\Gamma_S$ is an $r$-discrete ample groupoid with $\Gamma_S^{(0)} = X$ (identified with the diagonal subspace of $X \times X$). Moreover, the elements of $\Gamma_S^a$ are exactly finite disjoint unions of the sets of the form*

$$s \bullet K := \{(x, s.x) \in X \times X : x(s^*s) = 1,\, x \in K\},$$

*where $s \in S$ and $K$ is a compact open subset of $X$.*

**Lemma 3.6.** *Let $\mu$ be a probability measure on $X$. Then the following are equivalent:*

(i) *$\mu$ is quasi-invariant for $(G_S, \lambda)$.*
(ii) *$\mu$ is quasi-invariant for $(\Gamma_S, \alpha)$.*
(iii) *For each $s \in S$, $s.\mu \sim \mu$ on $X_s$.*

*Proof:* The equivalence of (i) and (ii) is true for any $r$-discrete groupoid and its principal groupoid [**1**, p. 72]. Now $\mu$ is quasi-invariant for $(\Gamma_S, \alpha)$ if and only if it is quasi-invariant under $\Gamma_S^a$ [**17**, I.3.22]. By Lemma 3.5, this means that if each of the measures $\int_X \alpha_x \, d\mu(x)$ and $\int_X \alpha^x \, d\mu(x)$ vanishes on a set of the form $s \bullet K$, where $s \in S$ and $K \subseteq X$ is compact, then so does the other. Switching the role of source and range maps for the sets of the form $s \bullet K$ amounts to switching the role of $x$ and $y := s.x$. Now $x$ satisfies $x(s^*s) = 1$ and $s.x = x$ if and only if $y$ satisfies $y(ss^*) = 1$ and $s^*.y = y$. Therefore, (ii) means that, for each $s$, $K$ as above, $\mu(K \cap X_s) = 0$ if and only if $\mu(s.K \cap X_{s^*}) = 0$, where $s.K := \{s.x : x \in K\}$. But $\mu(s.K \cap X_{s^*}) = \mu(s.(K \cap X_s)) = s.\mu(K \cap X_s)$, so (ii) is equivalent to (iii). □

A probability measure on $X$ is called $\theta$-invariant if it satisfies any of the above conditions. Next, let us put

$$[s] = \{x \in X : x(s^*s) = 1\} \quad (s \in S).$$

**Definition 3.7.** A subset $S_0$ of $S$ is called $\theta$-invariant if it has the following properties:

(i) For each $x \in X$, there is $s \in S_0$ such that $x \in X_s$.
(ii) For each $s \in S_0$ and $x \in [s]$, there exists $t \in S_0$ with $x \in X_{ts}$.

(iii) For each $s, t \in S_0$, there is some $u \in S_0$ such that $[t] \cap [st] \subseteq [u]$ and $st.x = u.x$, for each $x \in [t] \cap [st]$.

Note that any subsemigroup $S_0$ of $S$ with $E_{S_0} = E_S$ is $\theta$-invariant (for (i), take any $e \in \mathrm{supp}(x)$ and choose $s \in S_0$ with $s^*s = e$, and for (ii) and (iii), take $t = s^*$ and $u = st$, respectively) but the converse is not true in general.

**Lemma 3.8.** *The subgroupoids of $\Gamma_S$ with the same unit space $X$ are exactly the subsets of the form*

$$\Gamma_0 := \{(x, s.x) \in X \times X : s \in S_0,\, x \in X,\, x(s^*s) = 1\},$$

*where $S_0 \subseteq S$ is $\theta$-invariant.*

*Proof:* First assume that $\Gamma_0$ is a subgroupoid of $\Gamma_S$ with $\Gamma_0^{(0)} = X$ (recall that the left hand side is indeed the diagonal of $X \times X$, naturally identified with the right hand side). It is clear that $\Gamma_0$ is of the above form for some subset $S_0$ of $S$. We need to show that $S_0$ is $\theta$-invariant. Given $x \in X$, then $(x, x) \in \Gamma_0^{(0)}$, so there is $s \in S_0$ such that $x \in [s]$ and $(x, x) = (x, s.x)$. Hence $x \in X_s$. Next, let $s \in S_0$ and $x \in [s]$. Then $(x, s.x) \in \Gamma_0$, so $(s.x, x) = (x, s.x)^{-1} \in \Gamma_0$, and so there is $t \in S_0$ such that $s.x(t^*t) = 1$ and $t.(s.x) = ts.x = x$. Therefore, $x \in X_{ts}$. Finally, let $s, t \in S_0$ and $[s] \cap [st] \neq \emptyset$. Choose any $x \in [s] \cap [st]$. Then $(x, t.x), (t.x, s.(t.x)) \in \Gamma_0$, hence $(x, st.x) \in \Gamma_0$, that is, $st.x = u.x$ for some $u \in S_0$ with $x(u^*u) = 1$.

Conversely, assume that $S_0$ is $\theta$-invariant. For each $x \in X$, there is $s \in S_0$ such that $x \in X_s$. Therefore, $(x, x) \in \Gamma_0$, so $\Gamma_0^{(0)} = X$. Next, let $(x, s.x) \in \Gamma_0$. Then $x \in [s]$. Take $t \in S_0$ such that $x \in X_{ts}$. Then $t.(s.x) = ts.x = x$ and $s.x(t^*t) = x((ts)^*(ts)) = 1$, so $(s.x, x) = (s.x, t.(s.x)) \in \Gamma_0$. Finally, let $(x, t.x), (y, s.y) \in S_0$ with $y = t.x$. Then $x \in [t] \cap [st]$. Choose $u \in S_0$ as in (iii) and then $(x, s.y) = (x, st.x) = (x, u.x) \in \Gamma_0$. $\square$

Before we define a hyperfinite inverse semigroup, one should note that there is a slight error in the definition of hyperfinite groupoid in [**1**, Definition 5.3.41]. The correct statement (in the notations of [**1**]) is that $(R, \mu)$ is hyperfinite if there is an increasing sequence of bounded Borel subequivalence relations $R_n$ such that $\cup_n R_n$ is co-null with respect to $\mu \circ \lambda$.

**Definition 3.9.** $S$ is called hyperfinite if there exists a sequence $\{S_n\}$ of $\theta$-invariant subsets of $S$ such that

(i) For each $x \in X$, $\sup_{n \geq 1} |\{s.x : s \in S_n,\, x(s^*s) = 1\}| < \infty$.

(ii) For each $\theta$-invariant measure $\mu$ on $X$,

$$\int_X |\{s.x : s \notin \cup_n S_n,\, x(s^*s) = 1\}|\, d\mu(x) = 0.$$

**Proposition 3.10.** *The following are equivalent:*

(i) *$S$ is hyperfinite.*

(ii) *$\Gamma_S$ is measurewise amenable.*

*Proof:* If $S$ is hyperfinite, let $\{S_n\}$ be the corresponding sequence of $\theta$-invariant subsets of $S$ and put

$$\Gamma_n := \{(x, s.x) \in X \times X : s \in S_n,\, x \in X,\, x(s^*s) = 1\} \quad (n \geq 1).$$

By Lemma 3.8, $\Gamma_n$ is a subgroupoid of $\Gamma_S$ with the same unit space. Now let $\alpha$ be the Haar system of $\Gamma_S$ so that $\alpha \circ \mu$ is the pseudo-image of $\lambda \circ \mu$. Then conditions (i) and (ii) of Definition 3.9 translate into $\sup_{n\geq 1} \alpha_x((\Gamma_n)_x) < \infty$ $(x \in X)$ and $\alpha \circ \mu((\Gamma_S \backslash (\cup_n \Gamma_n))) = 0$. This means that $\Gamma_S$ is hyperfinite as a discrete equivalence relation on $X$ [**1**, Definition 5.3.41] and so $(\Gamma_S, \alpha, \mu)$ is amenable by the Connes–Feldman–Weiss Theorem [**5**], [**1**, Theorem 5.3.42]. This being true for each quasi-invariant measure $\mu$, $\Gamma_S$ is measurewise amenable.

Conversely if $\Gamma_S$ is measurewise amenable, then for each $\theta$-invariant measure $\mu$, $(\Gamma_S, \alpha, \mu)$ is amenable and so hyperfinite [**1**, Theorem 5.3.42]. Again Lemma 3.8 shows that $S$ is hyperfinite. □

**Example 3.11.** The bicyclic semigroup $\mathcal{C}$ is hyperfinite.

*Proof:* Recall that the bicyclic semigroup is the finitely presented monoid with two generators $p$ and $q$ and one relation $pq = 1$. It is easy to see that each discrete hyperfinite equivalence relation $\Gamma$ has finite orbits. Now one can readily check that $E_{\mathcal{C}} = (\mathbb{N}, \max)$ and $X = \hat{E_{\mathcal{C}}} = (\mathbb{N} \cup \{\infty\}, \max)$. Hence $(x, y) \in \Gamma_{\mathcal{C}}$ if and only if $x, y \in \mathbb{N}$ or $x = y = \infty$. Therefore, $\Gamma_{\mathcal{C}}$ has two orbits: $(\Gamma_{\mathcal{C}})_\infty = \{(\infty, \infty)\}$ and $(\Gamma_{\mathcal{C}})_x = \mathbb{N} \times \mathbb{N}$ $(x \in \mathbb{N})$. In particular, $\Gamma_{\mathcal{C}}$ is hyperfinite. This shows that $\mathcal{C}$ is hyperfinite. □

Now we are ready to prove our main result. Recall that a mean on a discrete semigroup $S$ is a state on $\ell^\infty(S)$, which is called left (right, bi-) invariant if it is invariant under the left (right, left, and right) translation action(s) of $S$ on $\ell^\infty(S)$. The semigroup $S$ is left (right) amenable if there exists a left (right) invariant mean on $S$, and amenable if there is a bi-invariant mean. We show that the amenability of the universal groupoid of an inverse semigroup $S$ is equivalent to hyperfiniteness of $S$ plus (left) amenability of a class of subsemigroups of $S$ indexed by the compact space $X = \hat{E}_S$. When $S$ is a (discrete) group, this family is a singleton consisting of $S$ itself, and $S$ is automatically hyperfinite, as we can take $S_n = S$, $n \geq 1$ and the supremum in Definition 3.9(i) is always 1.

**Theorem 3.12.** *Let $S$ be an inverse semigroup and $G_S$ be its universal groupoid. Then the following are equivalent:*

(i) *$G_S$ is measurewise amenable.*
(ii) *$G_S$ is topologically amenable.*
(iii) *$S$ is hyperfinite and for each $x \in X = \hat{E}_S$, the subsemigroup $S_x$ is (left) amenable.*

*Proof:* The equivalence of (i) and (ii) follows from [**1**, Theorem 3.3.7 and Remark 3.3.9] and the fact that $G_S$ is $r$-discrete (étale) [**10**]. The equivalence of (i) and (iii) follows from Lemmas 3.3, 2.1, and Proposition 3.10. $\square$

It is asked in [**14**] if amenability of all maximal subgroups $S_e^e$, $e \in E_S$, implies the nuclearity of $C_r^*(S)$. The following example (suggested to the authors by David Cowan) answers this question negatively.

**Example 3.13.** All the maximal subgroups of the free inverse semigroup $S$ on two generators are amenable, but $C_r^*(S)$ is not nuclear.

*Proof:* Let $S$ be the free inverse semigroup on two generators. Then $G_S^{(0)} = E_S \cup \{1\}$, where 1 is the characteristic function of $E$ (acting as a multiplicative linear functional on $\ell^1(S)$). It is clear that for each $e \in E_S$, $S_e^e$ is the trivial group [**16**, Proposition VIII.1.14]. Also clearly $S_1^1 = \mathbb{F}_2$, the free group on two generators. Now if $C_r^*(S)$ is nuclear, then so is $C_r^*(G_S)$. Hence by [**1**, Corollary 6.2.14] $G_S$ is measurewise amenable. Therefore, the above theorem implies that all the isotropy groups of $G_S$ must be amenable. But the isotropy group at $x = 1$ is $\mathbb{F}_2$ which is non amenable. Note that $VN(S)$ is injective in this case [**14**, Theorem 4.5.2]. $\square$

**Corollary 3.14.** *Let $S$ be a hyperfinite inverse semigroup. Consider the following statements:*

(i) *The subsemigroup $S_x$ is (left) amenable for each $x \in \hat{E}_S$.*
(ii) *$VN((G_S)_x^x)$ is injective for each $x \in G_S^{(0)}$.*
(iii) *$C_r^*((G_S)_x^x)$ is nuclear for each $x \in G_S^{(0)}$.*
(iv) *$C_r^*(S)$ is nuclear.*
(v) *$VN(S)$ is injective.*

*Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v).*

*Proof:* (i) implies (ii) by the above theorem and [**14**, Theorem 4.5.2]. The proof of the converse is included in the proof of the above theorem. The equivalence of (i), (ii), and (iii) is well-known, as each $(G_S)_x^x$ is a discrete group and the (left) amenability of each $S_x$ is equivalent to the amenability of its maximal group homomorphic image $(G_S)_x^x$. If $C_r^*(S)$ is nuclear, then its enveloping von Neumann algebra $VN(S)$ is injective, so (iv) implies (v). If (i) holds, then by the above theorem $G_S$ is measurewise amenable, so by [**1**, Corollary 6.2.14] $C_r^*(G_S)$ is nuclear, and so by Proposition 3.1, $C_r^*(S)$ is nuclear. Finally, if $C_r^*(S)$ is nuclear, then so is $C_r^*(G_S)$. Also by Lemma 3.2, $G_S$ has a continuous Haar system and discrete isotropy, and thus $G_S$ is measurewise amenable [**1**, Corollary 6.2.14]. Hence (i) holds by the above theorem. □

**Example 3.15.** The Cuntz–Renault semigroup $S_n$ satisfies the equivalent conditions of the above theorem.

*Proof:* It is well known that $S_n$ has amenable universal groupoid [**17**]. Indeed its universal groupoid is a graph groupoid and these are amenable in general [**15**]. An alternative way of seeing this is to observe that $C_r^*(S_n)$ is nuclear and use the above corollary. It is well known that $C_r^*(S_n) \simeq \mathcal{A}_n \times \mathbb{C}$, where $\mathcal{A}_n$ is the Cuntz–Toeplitz algebra (an extension of the Cuntz algebra $\mathcal{O}_n$ by the algebra of compact operators $\mathcal{K}(\ell^2)$) [**14**, p. 209]. □

**Corollary 3.16.** *If $S$ is hyperfinite and every subsemigroup $S_x$, $x \in \hat{E}_S$, is amenable, then $C^*(S)$ and $C_r^*(S)$ are $*$-isomorphic.*

*Proof:* If the hypothesis holds, then by the above theorem $G_S$ is measurewise amenable, so by Lemma 3.2 and [**1**, Proposition 6.1.8], $C^*(G_S)$ and $C_r^*(G_S)$ are $*$-isomorphic. Therefore, by Proposition 3.1, $C^*(S)$ and $C_r^*(S)$ are $*$-isomorphic. □

More generally assume that $S$ acts on a $C^*$-algebra $A$ by endomorphisms (see for instance [**11**, Definition 5.1 and Example 5.2(b)]). Then we have

**Corollary 3.17.** *If $S$ is hyperfinite and all subsemigroups $S_x$, $x \in \hat{E}_S$, are amenable and $\alpha$ is an action of $S$ on a $C^*$-algebra $A$ by endomorphisms, then the full and reduced crossed products $A \rtimes_\alpha S$ and $A \rtimes_{\alpha,r} S$ are $*$-isomorphic. In particular, $A \rtimes_\alpha E_S$ and $A \rtimes_{\alpha,r} E_S$ are always $*$-isomorphic.*

*Proof:* By [**11**, Theorem 6.5] there is a natural action of $G_S$ on $A \rtimes_\alpha E_S$, and $A \rtimes_\alpha S$ and $A \rtimes_{\alpha,r} S$ are $*$-isomorphic, respectively with $(A \rtimes_\alpha E_S) \rtimes$

$G_S$ and $(A \rtimes_\alpha E_S) \rtimes_r G_S$. It is easy to see that $A \rtimes_\alpha E_S$ could be regarded as a continuous $G$-bundle of $C^*$-algebras in the sense of [**1**] indexed by $X = G_S^{(0)}$. Therefore, by [**1**, Proposition 6.1.10] $(A \rtimes_\alpha E_S) \rtimes G_S$ and $(A \rtimes_{\alpha,r} E_S) \rtimes_r G_S$ are $*$-isomorphic. □

## References

[1] C. Anantharaman-Delaroche and J. Renault, *"Amenable Groupoids"*, With a foreword by G. Skandalis and Appendix B by E. Germain, Monographies de L'Enseignement Mathématique **36**, L'Enseignement Mathématique, Geneva, 2000.

[2] C. Berg, J. P. R. Christensen, and P. Ressel, *"Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions"*, Graduate Texts in Mathematics **100**, Springer-Verlag, New York, 1984. DOI: 10.1007/978-1-4612-1128-0.

[3] M.-D. Choi and E. G. Effros, Nuclear $C^*$-algebras and injectivity: the general case, *Indiana Univ. Math. J.* **26(3)** (1977), 443–446. DOI: 10.1512/iumj.1977.26.26034.

[4] A. Connes, On the cohomology of operator algebras, *J. Functional Analysis* **28(2)** (1978), 248–253. DOI: 10.1016/0022-1236(78)90088-5.

[5] A. Connes, J. Feldman, and B. Weiss, An amenable equivalence relation is generated by a single transformation, *Ergodic Theory Dynam. Systems* **1(4)** (1981), 431–450 (1982). DOI: 10.1017/s014338570000136x.

[6] J. Duncan and I. Namioka, Amenability of inverse semigroups and their semigroup algebras, *Proc. Roy. Soc. Edinburgh Sect. A* **80(3–4)** (1978), 309–321. DOI: 10.1017/S0308210500010313.

[7] R. Exel and C. Starling, Amenable actions of inverse semigroups, *Ergodic Theory Dynam. Systems* **37(2)** (2017), 481–489. DOI: 10.1017/etds.2015.60.

[8] U. Haagerup, All nuclear $C^*$-algebras are amenable, *Invent. Math.* **74(2)** (1983), 305–319. DOI: 10.1007/BF01394319.

[9] B. E. Johnson, *"Cohomology in Banach Algebras"*, Memoirs of the American Mathematical Society **127**, American Mathematical Society, Providence, R.I., 1972.

[10] M. Khoshkam and G. Skandalis, Regular representation of groupoid $C^*$-algebras and applications to inverse semigroups, *J. Reine Angew. Math.* **2002(546)** (2002), 47–72. DOI: 10.1515/crll.2002.045.

[11] M. Khoshkam and G. Skandalis, Crossed products of $C^*$-algebras by groupoids and inverse semigroups, *J. Operator Theory* **51(2)** (2004), 255–279.

[12] D. Milan, $C^*$-algebras of inverse semigroups: amenability and weak containment, *J. Operator Theory* **63(2)** (2010), 317–332.

[13] A. L. T. Paterson, *"Amenability"*, Mathematical Surveys and Monographs **29**, American Mathematical Society, Providence, RI, 1988. DOI: 10.1090/surv/029.

[14] A. L. T. Paterson, *"Groupoids, Inverse Semigroups, and their Operator Algebras"*, Progress in Mathematics **170**, Birkhäuser Boston, Inc., Boston, MA, 1999. DOI: 10.1007/978-1-4612-1774-9.

[15] A. L. T. Paterson, Graph inverse semigroups, groupoids and their $C^*$-algebras, *J. Operator Theory* **48(3)**, suppl. (2002), 645–662.

[16] M. Petrich, *"Inverse Semigroups"*, Pure and Applied Mathematics (New York), A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984.

[17] J. RENAULT, *"A Groupoid Approach to $C^*$-Algebras"*, Lecture Notes in Mathematics **793**, Springer, Berlin, 1980. DOI: 10.1007/BFb0091072.

[18] B. STEINBERG, A groupoid approach to discrete inverse semigroup algebras, *Adv. Math.* **223(2)** (2010), 689–727. DOI: 10.1016/j.aim.2009.09.001.

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