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Asymptotic Behaviour of λ-Convex Sets in the Hyperbolic Plane

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Abstract. It is known that the limit Area/Length for a sequence of convex sets expanding over the whole hyperbolic plane is less than or equal to 1, and exactly 1 when the sets considered are convex with respect to horocycles. We consider geodesics and horocycles as particular cases of curves of constant geodesic curvature λ with $0 \leqslant \lambda \leqslant 1$ and we study the above limit Area/Length as a function of the parameter λ.

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1. Introduction

In the study of some problems in geometric probability there is the limit F/L of the quotient between the area F of a convex set and the length L of its boundary. For instance, if σ is the random variable *length of a chord* of a given convex set, the expected value of σ is given by $E(\sigma) = \pi F/L$. In the Euclidean plane it is clear that this quotient, when the convex set becomes 'very large', tends to ∞. But, as it was pointed out by L. A. Santaló and I. Yañez in [4], this is no longer true in the hyperbolic plane.

In fact they proved that for a certain class of convex sets in the hyperbolic plane, concretely the horocyclic convex sets, the limit F/L is 1. In [2], we showed that this limit can attain, in the hyperbolic plane, any value between 0 and 1.

Since horocycles are curves of geodesic curvature ± 1 and geodesics are curves of geodesic curvature 0, both can be considered as particular cases of curves of constant geodesic curvature $\lambda, 0 \leqslant |\lambda| \leqslant 1$.

Thus, if convexity is defined with respect to horocycles this limit is 1 and when convexity is defined with respect to geodesics the limit F/L is less or equal than 1. Hence it is natural to ask the question, first posed to us by A. Borisenko, about the influence of λ upon this limit. In fact, when convexity is defined with respect to λ-geodesics (see 2.5), we shall prove:

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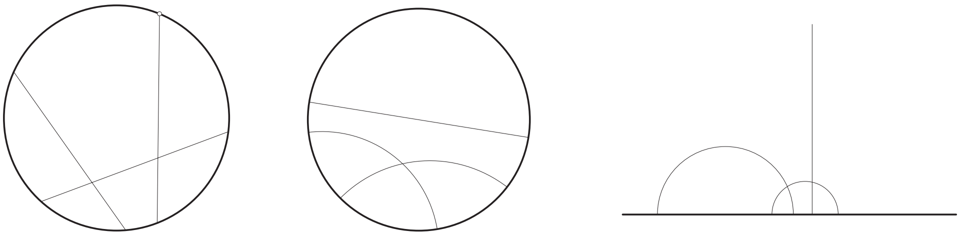

Figure 1. Hyperbolic plane models: *projective, disk and half-plane*.

THEOREM 1. *For each* $\alpha \in [\lambda, 1]$, *there exists a sequence of* λ*-convex polygons* $\{K_n\}$ *expanding over the whole hyperbolic plane such that* $\lim_{n\to\infty} F_n/L_n = \alpha$. *If the sequence is formed by* λ*-convex sets with piecewise* C^2 *boundary, then the* lim sup *and* lim inf *of these ratios lie between* λ *and* 1.

This kind of questions also makes sense in higher dimensions but we restrict our attention here to the two-dimensional case where the Gauss–Bonnet theorem in the hyperbolic plane allows an easier treatment.

2. Convexity in $\mathbb{H}^2$

2.1. THE HYPERBOLIC PLANE

The hyperbolic plane $\mathbb{H}^2$ is the unique complete simply connected Riemannian manifold with constant curvature -1. Its geometry corresponds to the one obtained from the absolute geometry given by the first four Euclid postulates and the Lobatchevsky postulate: through every point P exterior to a line l passes more than one line not intersecting l. It is useful to have different models for this geometry (see Figure 1), we shall describe their points, lines (geodesics) and rigid motions:

Projective model. The set of points is the interior of a conic in the real projective plane and the lines are the restriction of the projective lines to this set. The rigid motions are the projectivities fixing the conic and transforming the interior to itself.

Poincaré disk model. The set of points is the interior of the unit disk and the lines are the arcs of circles orthogonal to the boundary. The rigid motions are the homographies of the complex plane fixing the disk.

Poincaré half-plane model. The set of points is one of the connected components of the complement of a straight line in $\mathbb{R}^2$ and the lines are the arc of circles that meet orthogonally the border. The motions are compositions of inversions with respect to those circles.

It must be pointed out that the two Poincaré models are both conformal to the Euclidean plane, in particular this means that the Euclidean and hyperbolic angles between intersecting curves are identical.

DEFINITION 1. We shall say that two lines are *parallel* if they meet at a point in the border (they meet at infinity), if they don't meet we call them *ultraparallel*.

For practical purposes we shall use polar coordinates. In these coordinates the length element in $\mathbb{H}^2$ is given by

$$ds^2 = dr^2 + \sinh^2 r \, d\theta^2. \tag{1}$$

DEFINITION 2 . Let γ be a regular curve with unit tangent and normal vectors $\mathbf{t}, \mathbf{n}$ such that $\{\mathbf{t}, \mathbf{n}\}$ is compatible with a given orientation. Then $\nabla_{\mathbf{t}}\mathbf{t} = \kappa_g \mathbf{n}$ and κ_g is the (signed) *geodesic curvature* of γ.

2.2. HYPERBOLIC TRIGONOMETRY

We shall need the hyperbolic trigonometric formulas for triangles. If a, b, c are the sides of a triangle and α, β, γ are the opposite angles, then

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}, \tag{2}$$

$$\cosh a = \cosh b \cdot \cosh c - \sinh b \cdot \sinh c \cdot \cos \alpha, \tag{3}$$

$$\cos \alpha = -\cos \beta \cdot \cos \gamma + \sin \beta \cdot \sin \gamma \cdot \cosh a. \tag{4}$$

They are, respectively, the law of sines and the first and the second law of cosines.

As an application we give an expression for the area of an isosceles triangle that we shall use later. This expression is, obviously, independent of the model.

Let $\triangle OAB$ be an hyperbolic isosceles triangle with $d(O, A) = d(O, B) = R$, $d(A, B) = d$ and $\angle AOB = \alpha$. Let P be the midpoint of AB. By the law of sines applied to $\triangle OPA$, we have

$$\frac{\sinh R}{\sin \frac{1}{2}\pi} = \frac{\sinh \frac{1}{2}d}{\sin \frac{1}{2}\alpha}.$$

Therefore

$$d = 2 \operatorname{arcsinh} (\sinh R \cdot \sin \tfrac{1}{2}\alpha) \tag{5}$$

By the second law of cosines (4) applied to the same $\triangle OPA$, we have

$$\cosh R = \cot \tfrac{1}{2}\alpha \cdot \cot \gamma$$

where $\gamma = \angle OAP$. Since the area F of $\triangle OAB$ is given by

$$F = \pi - (\alpha + 2\gamma)$$

we have

$$F = \pi - \left(\alpha + 2 \arctan \frac{1}{\tan \frac{1}{2}\alpha \cdot \cosh R} \right) \tag{6}$$

2.3. CONVEX SETS

A set $K \subset \mathbb{H}^2$ is said to be convex when the segment joining a pair of points in K is contained in K. In the projective model, convex sets are seen as convex sets in Euclidean plane. A closed convex curve is a curve such that the region it encloses is convex.

DEFINITION 3 . A closed convex set with nonempty interior will be called a *convex domain*.

As in the Euclidean case we have

LEMMA 1. *A compact domain K with C^2 boundary is convex if and only if its geodesic curvature does not change the sign.*

2.4. HOROCYCLIC CONVEX SETS

Horocycles are curves orthogonal to a bundle of parallel lines. They can be considered as circles centered at infinity. In the half-plane model horocycles are the circles tangent to $y = 0$ and the curves $y = \text{ct}$. Given two points in $\mathbb{H}^2$ there are two and only two horocycle arcs joining them and the geodesic line passing through them lies between the horocyclic arcs. Horocycles have geodesic curvature ± 1.

DEFINITION 4 . A subset $K \subset \mathbb{H}^2$ is said to be *h-convex* or *convex with respect to horocycles* if for each pair of points belonging to K, the entire segments of the two horocycles joining them also belong to K.

It is clear that every h-convex set is convex but, as can be seen taking a convex polygon, not every convex set is h-convex.

2.5. λ-CONVEX SETS

Given a geodesic line l in the Euclidean plane, the set of equidistant points to l are two parallel lines symmetric with respect to l. In the hyperbolic plane this is no longer true. The set of equidistant points to l are two curves called *equidistants*. If we consider the half-plane model $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, the equidistants to the hyperbolic line $x = 0$ are Euclidean lines passing through $(0, 0)$. Indeed any

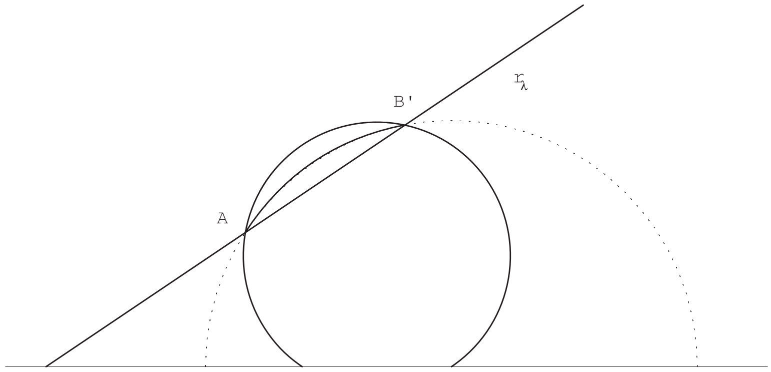

Figure 2. λ-geodesic segments with end points A and B'.

geodesic arc with center in $(0, 0)$ going from $x = 0$ to $y = mx$ has the same length because they are (Euclidean) homothetic and every homothety with center in the axis $y = 0$ is the composition of two inversions with respect to circumferences centered in $y = 0$ which are hyperbolic isometries. In fact, if $m = \tan\theta$ the length of these geodesic arcs is equal to $\arctan e^{\theta/2}$. In this model equidistant lines are, in general, arcs of Euclidean circles meeting the infinity at two points.

DEFINITION 5 . A λ-*geodesic* is an equidistant line that meets the infinity line with angle α such that $|\cos\alpha| = \lambda$.

Remark. When $\lambda = 0$ they are geodesics and when $\lambda = 1$ they are horocycles.

LEMMA 2 . *Given two points in* $\mathbb{H}^2$ *and* $0 < \lambda \leqslant 1$ *there are two and only two* λ-*geodesic lines through them. They are included in the region bounded by the two horocycles determined by the given points and are symmetric with respect the geodesic passing through these points*

Proof. Let A, B be points on $\mathbb{H}^2$ and r_λ the λ-geodesic of type $y = mx + b$ passing through A. If B' is a point in r_λ such that the geodesic segment AB' has the length of AB (see Figure 2), the λ-geodesics through A and B' correspond, via a rigid motion, to the λ-geodesics through A and B.

Now consider the inversion of r_λ with respect to the geodesic determined by A and B'. In this way we obtain the unique Euclidean circumference C_λ in the real plane intersecting $y = 0$ with the same angle as r_λ and passing through A and B (it passes through the center of the geodesic between A and B'). □

Remark. Since the two Poincaré models are conformal, the λ-geodesic lines in the Poincaré disk are also the arcs of Euclidean circles intersecting the border with an angle α such that $|\cos\alpha| = \lambda$.

Consider now the equidistant $\gamma(t) = (t, \tan\alpha \cdot t)$, $t > 0$ in the half-plane. The tangent and normal unit vectors are $\mathbf{t} = \sin\alpha \cdot t(1, \tan\alpha)$ and $\mathbf{n} = \sin\alpha \cdot t(-\tan\alpha, 1)$, hence $\nabla_{\mathbf{t}}\mathbf{t} = \cos\alpha \cdot \mathbf{n}$. Therefore, if $\cos\alpha = \lambda$, we have

LEMMA 3. *The geodesic curvature of a λ-geodesic is $\pm\lambda$.*

Remark. λ-geodesics through a point P with direction v in P are solutions of the second order differential equations $\kappa_g = \pm\lambda$. Then, with these initial conditions, there are two λ-geodesic lines, one with positive geodesic curvature and the other one with negative geodesic curvature.

DEFINITION 6. A subset $K \subset \mathbb{H}^2$ is said to be λ-*convex* if for each pair of points belonging to K, the entire segments of the λ-convex lines joining them also belong to K.

Remark. It follows from Lemma 2 that every λ-convex set is convex.

If K and K' are λ-convex sets then $K \cap K'$ is also a λ-convex set. Thus

LEMMA 4. *λ-convexity is stable under intersection.*

A λ-geodesic line l divides the plane in two regions having l as common boundary. Only one of these regions is λ-convex, it is the one containing the geodesic lines passing through every pair of points in l. It will be called the *λ-convex region* determined by l and denoted by K_l.

Remark. Definition 6 is equivalent to the following one: a closed set K is λ-convex if for every point of the boundary there exists a λ-geodesic line l through it such that the λ-convex region determined by l contains K. This λ-convex line is called *supporting* λ-geodesic of K. Thus, in higher dimensions a closed set K is λ-convex if it is supported by umbilical hypersurfaces with principal curvature λ.

DEFINITION 7. A λ-polygon is the region obtained by intersection of a finite number of λ-convex regions K_{l_i} determined by λ-geodesic lines l_i.

EXAMPLE 1. Let K be a regular convex polygon with vertices a_i and edges $e_i = a_i a_{i+1}$ on a circle C. For every pair (a_i, a_{i+1}) consider the λ-geodesic segment joining a_i and a_{i+1} not in K, let l_i be the λ-geodesic line containing this segment. The intersection $\cap K_{l_i}$ defines a regular λ-convex polygon denoted by K_λ. Let us prove that $K \subset K_\lambda$. Consider the Poincaré disk model, the λ-geodesic curves l_i are seen as Euclidean circles and they intersect C in a_i and a_{i+1}, since different circles intersect in at most two points we conclude that K_{l_i} contains the rest of the vertices. Therefore $K \subset K_\lambda$.

Remark. The same construction can be done with an arbitrary convex polygon K but the associated polygon K_λ not necessarily includes K. This is so because the λ-geodesic line joining two consecutive vertices can intersect the other sides of the original polygon K.

In the smooth case we can characterize convexity in terms of the curvature of the boundary.

PROPOSITION 1 . *Let K be a compact domain bounded by a curve of class C^2. Then K is λ-convex if and only if the geodesic curvature of the boundary satisfies $\kappa_g \geqslant \lambda$ (or $\kappa_g \leqslant -\lambda$ if we consider the opposite orientation).*

Proof. Let $p \in \partial K$ such that $\kappa_g(p) < \lambda$. Let us consider geodesic normal coordinates (x, y) in p such that $\partial/\partial x$ is tangent to ∂K in p and $\partial/\partial y$ is the interior normal in p. With respect to these coordinates, in a neighbourhood of p, the boundary is the graph of

$$y = \tfrac{1}{2}\kappa_g(p)x^2 + \mathrm{o}(x^2)$$

and the geodesic and the λ-geodesic curves with direction $\partial/\partial x$ in p are the graph of

$$y \equiv 0, \quad y = \pm\tfrac{1}{2}\lambda x^2 + \mathrm{o}(x^2).$$

Let $q = (x_0, y_0)$ with $x_0 > 0$ be a point in the interior λ-geodesic and near p. The λ-geodesics joining q with p are contained in K. One of them arrives to p tangential to ∂K. Take $\varepsilon > 0$ small enough such that the point $q_\varepsilon = (x_0, y_0 - \varepsilon)$ lies in K. By continuity there is a λ-geodesic joining q_ε and p that crosses the boundary. This contradicts the λ-convexity of K. Hence $\kappa_g \geqslant \lambda$.

Conversely, if K is not λ-convex there are two points $x, y \in \partial K$ such that the λ-geodesic between them is not contained in K. By hypothesis $\kappa_g \geqslant \lambda \geqslant 0$, and this implies that K is convex, therefore the geodesic segment r between x and y is contained in K. Let $0 \leqslant \mu < \lambda$ be the supremum of all nonnegative numbers such that the μ-geodesic between x and y is contained in K. If this μ-geodesic touches ∂K in a point we should have $\kappa_g \leqslant \mu < \lambda$ in this point, a contradiction. This implies, because μ is the supremum, that the μ-geodesic is tangent to ∂K at x or at y. But then $\kappa_g \leqslant \mu < \lambda$ at x or at y, a contradiction. □

Remark. In fact, if K is a λ-convex domain such that ∂K is *piecewise* C^2, i.e. a finite union of C^2 arcs, on every regular point $p \in \partial K$ the geodesic curvature satisfies also $\kappa_g \geqslant \lambda$.

EXAMPLE 2. The region bounded by a circle of radius r is h-convex. A simple computation in polar coordinates shows that $\kappa_g = \coth r$, and this value is greater than one. Note that in the limit case we obtain the horocycles whose geodesic curvature is one.

Remark. Observe that there are no λ-convex sets K bounded by a C^2-smooth curve with $0 < \kappa_g < \lambda_0 \leqslant 1$: at the intersection points with the circumdisc of radius R we obtain $\kappa_g \geqslant \coth R > 1$

2.6. SOME ISOPERIMETRIC INEQUALITIES

In the next sections we shall use inequalities involving the area and the perimeter of a compact convex domain. Let K be a compact convex domain in $\mathbb{H}^2$. Then it is known (see, for example [3]) that if L denotes the perimeter of ∂K and F the enclosed area we have

$$L^2 - 4\pi F - F^2 \geqslant 0 \tag{7}$$

and equality holds if and only if K is a geodesic circle. This is the *isoperimetric inequality* in the hyperbolic plane.

In the next sections we shall make an extensive use of some relations derived from the Gauss–Bonnet formula. If we assume that ∂K is piecewise C^2, the geodesic curvature is well defined except in a finite number of points. Therefore

$$\sum_i \int_{\partial K_i} \kappa_p = 2\pi - \sum_i \alpha_i + F, \tag{8}$$

where α_i are the exterior angles in the singular points and ∂K_i are the arcs of ∂K. From this we have the following lemma.

LEMMA 5 . *Let K be a compact convex domain in $\mathbb{H}^2$ with ∂K piecewise C^2. Then we have*

(a)

$$F \geqslant \sum_i \alpha_i - 2\pi. \tag{9}$$

(b) *If K is h-convex*

$$L \leqslant 2\pi - \sum_i \alpha_i + F. \tag{10}$$

(c) *If K is λ-convex*

$$\lambda L \leqslant 2\pi - \sum_i \alpha_i + F. \tag{11}$$

Remark. When K is a λ-polygon we have the equality

$$\lambda L = F + \sum_{i=1}^{n} \beta_i - (n-2)\pi, \quad \lambda \in [0, 1], \tag{12}$$

where n is the number of vertices and β_i are the interior angles.

3. A Problem in Hyperbolic Geometric Probability

Given a rectifiable curve C in the Euclidean plane and a line l, denote by $n(C \cap l)$ the number of intersection points, counted with their multiplicities. If dl denotes a measure of lines invariant under rigid motions the classical Cauchy–Crofton formula states that

$$\int_{l \cap C \neq \emptyset} n(l \cap C)\,\mathrm{d}l = \delta L,$$

where L is the length of C. We choose the unique measure dl such that $\delta = 2$. Therefore, since $n(l \cap C) = 2$ for the boundary C of a convex domain, and l not tangent to C, the measure of lines that intersect a convex domain is equal to its perimeter. Note that the set of lines, tangent to C, forms a set of measure zero.

Given a convex domain K let $\sigma(l)$ be the length of the chord $l \cap K$. It is easy to see that the expected value of the random variable σ is

$$E(\sigma) = \frac{\pi F}{L}.$$

These results, that are easily proved in the Euclidean case, remain true in the hyperbolic plane (cf. [3]).

DEFINITION 8. We say that a sequence of compact convex domains $\{K_n\}$ *expands over the whole plane* if $K_n \subset K_m$ when $n < m$ and for every point P there exists a K_N such that $P \in K_N$.

In the Euclidean plane it can be proved that $F/L \geqslant r_i/2$ where r_i is the radius of the greatest circumference contained in K (this easily follows from the expression $F = \frac{1}{2}\int p\,\mathrm{d}s$ where p is the distance between the origin of the circumference and the support lines of the convex domain). Then, if $\{K_n\}$ is a sequence of compact convex domains expanding over the whole plane, the mean value $E(\sigma)$ tends to infinity.

In the hyperbolic plane this is no longer true. Indeed if $\{K_n\}$ is formed by h-convex domains bounded by piecewise C^2 curves it is known that

$$\lim \frac{F_n}{L_n} = 1. \tag{13}$$

Nevertheless in [2] it was proved

THEOREM 2 . *For every nonnegative* $\alpha \leqslant 1$ *there exists a sequence* $\{K_n\}$ *of compact convex domains expanding over the whole hyperbolic plane* $\mathbb{H}^2$ *such that*

$$\lim \frac{F_n}{L_n} = \alpha. \tag{14}$$

Roughly speaking we can say that the perimeter becomes much larger than the area. This is a consequence of the so called 'edge effect' in the hyperbolic plane.

Now we shall see, using an approach different from that in [2], how these examples can be constructed. Let K_n be a regular polygon formed by $3\cdot 2^{n-1}$ isosceles triangles inscribed in a circle of radius R_n. Then if d_n is the length of the basis of one of these triangles and h_n is its area, then $F_n/L_n = h_n/d_n$. If $\alpha_n = 2\pi/(3\cdot 2^{n-1})$ is the central angle, by (5)

$$d_n = 2 \text{ arcsinh } (\sinh R_n \cdot \sin(\tfrac{1}{2}\alpha_n))$$

and using (6)

$$h_n = \pi - \left(\alpha_n + 2\arctan \frac{1}{\tan\frac{1}{2}\alpha_n \cdot \cosh R_n}\right).$$

But

$$\lim(\tan\tfrac{1}{2}\alpha_n \cdot \cosh R_n) = \lim \frac{\alpha_n}{2}\frac{2}{\mu\alpha_n} = \frac{1}{\mu},$$

hence

$$\lim h_n = \pi - 2\cdot \arctan \mu. \tag{15}$$

In an analogous way

$$\lim d_n = 2 \text{ arcsinh } \frac{1}{\mu}. \tag{16}$$

Taking $R_n = n$ we have that $\lim h_n/d_n = 0$. And taking $R_n = \log(4/\mu\alpha_n)$ with $\mu > 0$ we have that

$$\lim \frac{F_n}{L_n} = \lim \frac{h_n}{d_n} = \frac{\pi - 2\arctan \mu}{2 \text{ arcsinh } \frac{1}{\mu}}$$

which attains, depending on the parameter μ, all values between 0 and 1.

4. Asympotic Behaviour of λ-convex Sets in $\mathbb{H}^2$

We have seen that the quotient F_n/L_n tends to 1 in the horocyclical case and that this limit can take any value less or equal than 1 in the general convex case. Since horocycles and geodesics can be considered as the extremal case of λ-geodesics ($\lambda = 1$ and $\lambda = 0$ respectively) it is natural to ask what is the asymptotic behaviour of F/L in the λ-convex case. As a consequence of Lemma 5 we have

PROPOSITION 2. *Let K_n be a family of compact λ-convex domains with piecewise C^2 boundary that expands over the whole hyperbolic plane. Then*

$$\lambda \leqslant \liminf \frac{F_n}{L_n} \leqslant \limsup \frac{F_n}{L_n} \leqslant 1. \tag{17}$$

Proof. Taking (7) into account and dividing by L^2 we have that $F/L < 1$. On the other hand, changing in (11) the exterior angles α_i by the interior ones β_i and dividing by L we have

$$\frac{F}{L} \geqslant \lambda + \frac{(n-2)\pi - \sum_{i=1}^{n} \beta_i}{L} \geqslant \lambda - \frac{2\pi}{L}.$$

Hence the proposition follows. □

We shall study if there are sequences $\{K_n\}$ of λ-convex sets expanding over the whole hyperbolic plane such that the limit F_n/L_n is some fixed value between λ and 1.

4.1. λ-LENGTH

As it was said in Lemma 2, given two points A and B there are exactly two λ-geodesic lines from A to B and the geodesic segment AB lies between them. The symmetry with respect to these geodesics permutes the λ-geodesic lines so that they have, between A and B, the same length l.

Now we are going to compute this length $l = l(A, B)$ as function of the hyperbolic distance $d = d(A, B)$.

By transitivity, we can assume $A = i$, $B = bi$ with $b > 1$. Recall that $d = \log b$. The parametric equations of the Euclidean circles through A and B that meet $y = 0$ with angle α are given by

$$\begin{aligned} x &= \pm\frac{1}{2\lambda}\sqrt{(1+b)^2 - (b-1)^2\lambda^2} + \frac{1+b}{2\lambda}\cos\theta, \\ y &= \frac{1+b}{2} + \frac{1+b}{2\lambda}\sin\theta, \end{aligned}$$

where $\lambda = |\cos\alpha|$.

Hence the arclength is given by

$$\mathrm{d}s = \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{y^2} = \frac{\mathrm{d}\theta}{|\lambda + \sin\theta|}$$

and so

$$l = \int_{\arcsin \frac{1-b}{1+b}\lambda}^{\arcsin \frac{b-1}{b+1}\lambda} \frac{\mathrm{d}\theta}{\lambda + \sin\theta}$$

because $\lambda + \sin\theta > 0$ in this interval.

The evaluation of this integral gives us

$$l = \frac{1}{\sqrt{1-\lambda^2}} \log |\Phi(b, \lambda)|,$$

where $\Phi(b, \lambda)$ verifies

$$\frac{\Phi(b, \lambda) - 1}{\Phi(b, \lambda) + 1} = \frac{(b-1)\sqrt{1-\lambda^2}}{\sqrt{(1+b)^2 - (b-1)^2\lambda^2}}.$$

Since $\tanh(\frac{1}{2}\log x) = (x-1)/(x+1)$ we have

$$l = \frac{1}{\sqrt{1-\lambda^2}} \left(2 \ \mathrm{arctanh} \ \frac{(b-1)\sqrt{1-\lambda^2}}{\sqrt{(b+1)^2 - (b-1)^2\lambda^2}} \right), \tag{18}$$

which we express in terms of $d = d(A, B)$ in the following:

PROPOSITION 3. *The λ-length l between two points at hyperbolic distance d is given by*

$$\begin{aligned} l &= \frac{1}{\sqrt{1-\lambda^2}} \left(2 \ \mathrm{arctanh} \ \frac{(e^d-1)\sqrt{1-\lambda^2}}{\sqrt{(e^d+1)^2 - (e^d-1)^2\lambda^2}} \right) \\ &= \frac{2}{\sqrt{1-\lambda^2}} \ \mathrm{arcsinh} \ \left(\sqrt{1-\lambda^2} \sinh \tfrac{1}{2} d \right). \end{aligned}$$

Remark. Observe that if $\lambda = 0$ we obtain $l = d$ and when $\lambda = 1$ (h-convex case) $l = 2\sinh \frac{1}{2} d$.

The angle β between the geodesic and the λ-geodesic in A is equal to the angle θ corresponding to the polar coordinate of A. Hence

$$\tan\beta = \tan\theta = \frac{\lambda(b-1)}{\sqrt{(b+1)^2 - (b-1)^2\lambda^2}}. \tag{19}$$

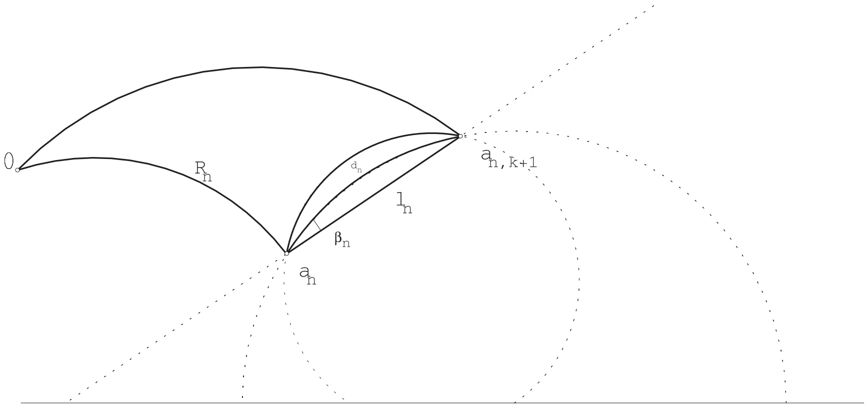

Figure 3. A triangle forming the λ-polygon K_n.

4.2. A FAMILY OF λ-CONVEX SETS

Let $\{K_n\}$ be a family of λ-convex sets defined as in Example 1 from regular polygons with central angle $\alpha_n = 2\pi/(3 \cdot 2^{n-1})$ and radius $R_n = \log(4/\mu\alpha_n)$. Let $(a_{n,k})$ be the vertices of K_n, and let f_n be the area of the figure formed by the points $O, a_{n,k}, a_{n,k+1}$, the geodesic segments $Oa_{n,k}$, $Oa_{n,k+1}$ and the λ-geodesic segment between $a_{n,k}$ and $a_{n,k+1}$ in ∂K_n (see Figure 3). We shall denote by l_n the length of this λ-segment. Let h_n be the area of the hyperbolic triangle $Oa_{n,k}a_{n,k+1}$, and β_n be the angle between the geodesic and the λ-geodesic in each vertex.

We are interested in the quotient F_n/L_n but

$$\frac{F_n}{L_n} = \frac{f_n}{l_n}.$$

By the Gauss–Bonnet theorem, we have

$$f_n = h_n + \lambda l_n - 2\beta_n \tag{20}$$

and, hence,

$$\frac{f_n}{l_n} = \lambda + \frac{h_n - 2\beta_n}{l_n}. \tag{21}$$

Remark. Since $\operatorname{arcsinh} y = \log(y + \sqrt{1 + y^2})$ the formula (16) can be written as

$$\lim e^{d_n} = \left(\frac{1}{\mu} + \frac{1}{\mu}\sqrt{1+\mu^2}\right)^2$$

or, equivalently,

$$\lim e^{d_n} = \frac{\nu + 1}{\nu - 1}, \qquad \text{where } \nu^2 = 1 + \mu^2.$$

This notation will be useful in the next lemmas.

LEMMA 6. *The relation, in the limit, between the parameter* $\nu = \sqrt{1+\mu^2}$ *and the length* l_n *of the above family of* λ*-polygons is given by*

$$\lim l_n = \frac{1}{\sqrt{1-\lambda^2}}\left(2\ \mathrm{arctanh}\sqrt{\frac{1-\lambda^2}{\nu^2-\lambda^2}}\right). \tag{22}$$

Proof. It is a direct substitution of

$$\begin{aligned}\lim(e^{d_n}+1) &= \frac{\nu+1}{\nu-1}+1 = \frac{2\nu}{\nu-1},\\ \lim(e^{d_n}-1) &= \frac{\nu+1}{\nu-1}-1 = \frac{2}{\nu-1}\end{aligned}$$

in the formula of Theorem 3. □

Substituting the expression of l given by Formula 18 in Formula 19 we obtain

LEMMA 7. *Let* P *and* Q *be two points on a* λ*-geodesic* r *and let* l *be the length of* r *between* P *and* Q*. Let* s *be the geodesic between* P *and* Q *and let* β *be the angle between* r *and* s *in* P*. Then*

$$\beta = \arctan\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\tanh\left(\frac{l}{2}\sqrt{1-\lambda^2}\right)\right). \tag{23}$$

This expression can also be written as

$$\beta = \arctan\left(\frac{\lambda}{\sqrt{\coth^2\frac{1}{2}d-\lambda^2}}\right),$$

where d is the hyperbolic distance between P and Q.

Substituting now the expression of $\lim l_n$ obtained in Lemma 6 in formula (23) we have

LEMMA 8. *With the same notation:*

$$\lim\beta_n = \arctan\frac{\lambda}{\sqrt{\nu^2-\lambda^2}}.$$

Substituting the values obtained in the above lemmas in formula (21) one obtains

LEMMA 9. *The relation, in the limit, between area and length of the family of* λ*-convex polygons considered in this section is given by* $\lim F_n/L_n = \lambda+\varphi(\lambda,\mu)$, *where*

$$\varphi(\lambda,\mu) = \frac{\pi-2\left(\arctan\mu+\arctan\frac{\lambda}{\sqrt{\mu^2+1-\lambda^2}}\right)}{\frac{1}{\sqrt{1-\lambda^2}}\left(2\ \mathrm{arctanh}\sqrt{\frac{1-\lambda^2}{\mu^2+1-\lambda^2}}\right)}.$$

Since for each λ the function $\varphi(\lambda, \mu)$ is continuous with respect to μ and

$$\lim_{\mu \to 0} \varphi(\lambda, \mu) = 0$$

and

$$\lim_{\mu \to \infty} \varphi(\lambda, \mu) = 1 - \lambda,$$

the proof of Theorem 1 is complete.

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