Curvature for Polygons

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Abstract. Using a notion of curvature at the vertices of a polygon, we prove an inequality involving the length of the sides of the polygon and the radii of curvature at the vertices. As a consequence, we obtain a discrete version of Ros' inequality.

1. INTRODUCTION. The starting point of this note is the following known inequality. If C is a closed convex curve in $\mathbb{R}^2$ of class $\mathcal{C}^2$ and length L, then

$$\frac{L^2}{4\pi} \leq \frac{1}{2}\int_C \rho(s)\, ds, \tag{1}$$

where $\rho = \rho(s)$ is the radius of curvature of C and ds signifies arclength measure on C. Equality holds if and only if C is a circle. This result can be easily proved using Schwarz's inequality (Proposition 1).

On the other hand, one has the isoperimetric inequality

$$A \leq \frac{L^2}{4\pi}, \tag{2}$$

where A is the area enclosed by C, with equality if and only if C is a circle.

Combining (1) and (2) one gets

$$A \leq \frac{1}{2}\int_C \rho(s)\, ds, \tag{3}$$

which is the two-dimensional Ros' inequality (see [**4**] and [**6**]). The difference $\frac{1}{2}\int_C \rho(s)ds - A$ was studied in [**2**]. The corresponding inequality for the sphere and the hyperbolic plane can be found in [**3**].

Here we consider the inequality (1) for the case where the curve C is a polygon. For this we introduce a notion of radius of curvature at the vertices of a polygon (Definition 1), which is a good approximation of the radius of curvature of a smooth curve. We prove the following inequality:

$$\frac{L^2}{4\pi} \leq \frac{1}{2}\sum_{k=1}^{n} l_k \frac{\rho_k + \rho_{k+1}}{2},$$

where L is the length of the polygon, l_k the length of its sides, and ρ_k the radius of curvature at its vertices. Equality holds if and only if the polygon is umbilical (Theorem 1).

As a consequence, we obtain a discrete version of Ros' inequality (Corollary 1).

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2. ON THE INTEGRAL OF THE RADIUS OF CURVATURE. For the sake of completeness, we give a proof of the well known inequality (1).

Proposition 1. *If C is a closed convex plane curve of class $\mathcal{C}^2$ of length L, then*

$$\frac{L^2}{4\pi} \leq \frac{1}{2} \int_C \rho(s) ds,$$

where $\rho(s)$ is the radius of curvature of C, and ds signifies arclength measure on C. Equality holds if and only if C is a circle.

Proof. Applying Schwarz's inequality we have

$$L = \int_C 1\, ds = \int_C k^{1/2} k^{-1/2}\, ds \leq \left(\int_C k ds \right)^{1/2} \left(\int_C k^{-1} \right)^{1/2}$$
$$= (2\pi)^{1/2} \left(\int_C \rho\, ds \right)^{1/2},$$

where $k = k(s)$ is the curvature of C. Equality holds if and only if $k(s) = \lambda k^{-1}(s)$, for a constant λ. That is, $k(s)$ is constant and C is a circle. ■

3. CURVATURE FOR POLYGONS. Given a plane convex polygon of vertices $P_1, P_2, \ldots, P_n$, we denote by $l_k = |\overrightarrow{P_k P_{k+1}}|$ the length of its sides and by $\alpha_k \pi$ the measure of its external angles (see Figure 1). Of course we have $\sum_{k=1}^{n} \alpha_k = 2$, with $0 < \alpha_k < 1$, and

$$\overrightarrow{P_{k-1} P_k} \cdot \overrightarrow{P_k P_{k+1}} = l_{k-1} \cdot l_k \cdot \cos(\alpha_k \pi).$$

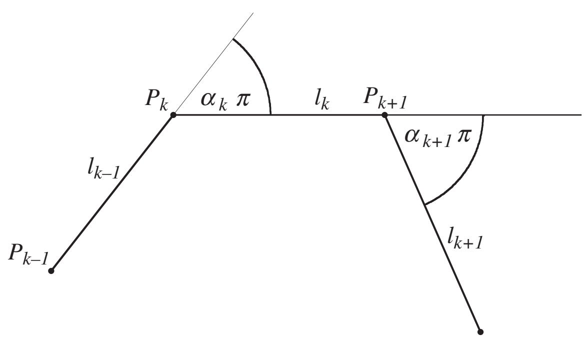

Figure 1.

Definition 1. Given a plane convex polygon of vertices $P_1, P_2, \ldots, P_n$, and sides of lengths $l_1, l_2, \ldots, l_n$, we define the radius of curvature at the vertex P_k by

$$\rho_k = \frac{l_{k-1} + l_k}{2\alpha_k \pi}.$$

In particular, the curvature at the vertex P_k is given by

$$\kappa_k = \frac{1}{\rho_k} = \frac{2\alpha_k \pi}{l_{k-1} + l_k},$$

an expression that essentially agrees with the classical definition of curvature as the ratio of the angle to the length. Note also that $l_0 = l_n$.

Note. This notion of curvature was also considered in [**1**] and [**5**]. Another natural definition of radius of curvature of a polygon is the following. If P_{i-1}, P_i, P_{i+1} are consecutive vertices of a polygon, the radius of curvature R_i at P_i is the radius of the circumscribed circle around the triangle $P_{i-1}P_iP_{i+1}$ (see [**7**] and Figure 2).

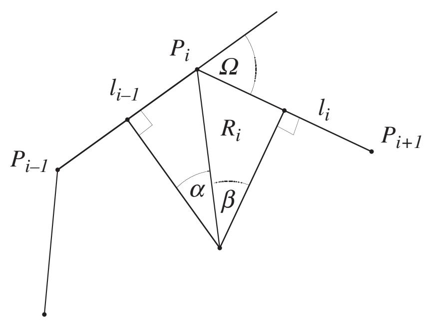

Figure 2.

The relation between R_i and ρ_i is (with the notation of the figure)

$$\rho_i = R_i \frac{\sin\alpha + \sin\beta}{\alpha + \beta}.$$

In particular, since $\Omega = \alpha + \beta$, R_i tends to ρ_i when the external angles of the polygon tend to zero.

In order to justify Definition 1, we remark that if we have a sequence of polygons approximating a smooth curve then both sequences R_i and ρ_i approximate the radius of curvature function.

More precisely, let $\gamma : [0, L] \longrightarrow \mathbb{R}^2$ be the arclength parametrization of a smooth closed curve C and consider the dyadic polygons whose consecutive vertices are

$$P_k^{(n)} = \gamma(s_k^{(n)}), \quad \text{where} \quad s_k^{(n)} = k\frac{L}{2^n} \in [0, L], \quad \text{and} \quad k = 1, 2, 3, \ldots, 2^n.$$

Then the radius of curvature at the vertex $P_k^{(n)}$ of the nth dyadic polygon and the radius of curvature of C at this point, have the same limit when $n \to \infty$, for $k = 1, 2, 3, \ldots, 2^n$.

4. A DISCRETE VERSION OF ROS' INEQUALITY. Now we give a discrete version of inequality (1). For this, we shall need the following result.

Lemma 1. *Let $a_1, \ldots, a_n \in \mathbb{R}^+$ and let $f : (\mathbb{R}^+)^n \longrightarrow \mathbb{R}$ be the function given by*

$$f(x_1, \ldots, x_n) = \frac{a_1^2}{x_1} + \cdots + \frac{a_n^2}{x_n}.$$

If $x_1 + \cdots + x_n = 2$*, then*

$$f(x_1, \ldots, x_n) \geq \frac{1}{2}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Proof. Applying Schwarz's inequality we have

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} x_i^{1/2}\frac{a_i}{x_i^{1/2}} \leq \left(\sum_{i=1}^{n} x_i\right)^{1/2}\left(\sum_{i=1}^{n} \frac{a_i^2}{x_i}\right)^{1/2} = \sqrt{2}\left(\sum_{i=1}^{n} \frac{a_i^2}{x_i}\right)^{1/2}. \qquad \blacksquare$$

Definition 2. A convex polygon is called umbilical if the radius of curvature at its vertices is constant.

Of course all regular polygons are umbilical. Note that the radius of curvature of an umbilical polygon must be equal to $L/2\pi$, where L is the length of the polygon. This fact is easily demonstrated by simply adding the equalities

$$l_{k-1} + l_k = 2\alpha_k \pi\rho, \quad \text{for } k = 1, \ldots, n,$$

where ρ is the constant radius of curvature.

Theorem 1. *Let* L *be the length of a convex polygon with sides* l_k *and radii of curvature* ρ_k*. Then we have*

$$\frac{L^2}{4\pi} \leq \frac{1}{2}\sum_{k=1}^{n} l_k \frac{\rho_k + \rho_{k+1}}{2}.$$

Equality holds if and only if the polygon is umbilical.

Proof. By definition of ρ_k, the second term of this inequality is

$$\frac{1}{2}\sum_{k=1}^{n} l_k \frac{\rho_k + \rho_{k+1}}{2} = \frac{1}{8\pi}\sum_{k=1}^{n} l_k \left(\frac{l_{k-1} + l_k}{\alpha_k} + \frac{l_k + l_{k+1}}{\alpha_{k+1}}\right)$$
$$= \frac{1}{8\pi}\sum_{k=1}^{n} \frac{(l_k + l_{k+1})^2}{\alpha_{k+1}}.$$

Since $\alpha_1 + \cdots + \alpha_n = 2$, we can apply Lemma 1 and obtain

$$\frac{1}{8\pi}\sum_{k=1}^{n} \frac{(l_k + l_{k+1})^2}{\alpha_{k+1}} \geq \frac{1}{8\pi}\frac{1}{2}(2L)^2.$$

Hence

$$\frac{1}{2}\sum_{k=1}^{n} l_k \frac{\rho_k + \rho_{k+1}}{2} \geq \frac{1}{4\pi}L^2$$

and the inequality of the theorem is proved.

By the proof of Lemma 1, equality is attained when

$$\alpha_{k+1} = \frac{2(l_k + l_{k+1})}{\sum_{i=1}^{n}(l_i + l_{i+1})} = \frac{l_k + l_{k+1}}{L} = \frac{2\alpha_{k+1}\pi\rho_k}{L}.$$

Hence

$$\rho_k = \frac{L}{2\pi}, \quad k = 1, \dots, n$$

and the polygon is umbilical. ∎

We get now the corresponding discrete version of Ros' inequality.

Corollary 1. *If A is the area of a convex polygon with sides l_k and radii of curvature ρ_k, then we have*

$$A \leq \frac{1}{2}\sum_{k=1}^{n} l_k \frac{\rho_k + \rho_{k+1}}{2}. \tag{4}$$

Proof. The proof is a direct consequence of Theorem 1 and the isoperimetric inequality $L^2 - 4\pi A \geq 0$. ∎

As a concluding remark, we note that Ros' inequality (3) for regular curves is the limit of inequality (4) for $n \to \infty$.

If $l_k^{(n)}$ denotes the length of the side $P_k^{(n)} P_{k+1}^{(n)}$ and $\tilde{\rho}_k{}^{(n)}$ the arithmetic mean

$$\tilde{\rho}_k{}^{(n)} = \frac{\rho_k^{(n)} + \rho_{k+1}^{(n)}}{2}$$

where $\rho_k^{(n)}$ is the discrete radius of curvature at the vertex $P_k^{(n)}$, then by considering dyadic polynomials as at the end of Section 3 we claim that

$$\int_C \rho(s)\,ds = \lim_{n\to\infty}\left(\sum_{k=1}^{2^n} l_k^{(n)}\tilde{\rho}_k{}^{(n)}\right). \tag{5}$$

Indeed, since by the definition of the Riemann integral

$$\int_C \rho(s)\,ds = \lim_{n\to\infty}\left(\sum_{k=1}^{2^n}\frac{L}{2^n}\,\rho(s_k^{(n)})\right),$$

the assertion follows easily.

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