RUBRIQUE des C.R.A.S.P.: *Géométrie Différentielle*

COURBURE ET CHAMPS DE PLANS

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Résumé. Soit M une variété riemanniene orientée munie de deux champs de plans $\mathcal{F}$ et $\mathcal{H}$ orientés, orthogonaux et complémentaires l'un de l'autre.

Suivant Albert ([1]) on obtient quelques formules intégrales donnant des relations entre la géométrie de la variété d'une part, et celle des champs de plans (courbure, intégrabilité et deuxième forme fondamentale), d'autre part.

On generalise un resultat de [3] a codimension arbitraire et sans hypothèse d'intégrabilité.

CURVATURE AND PLANE FIELDS

Abstract. Let M be an oriented Riemannian manifold equipped with two oriented , complementary and orthogonal distributions of planes $\mathcal{F}$ and $\mathcal{H}$

Following Albert ([1]) we obtain some change integral formulas in the sense that on one side of them there is a term depending on the geometry of the manifold (curvature), and on the other side there are terms depending on the geometry of the plane fields (curvature, integrability and second fundamental form).

We generalize a result from [3] to arbitrary codimension and avoiding the integrability condition.

I PRELIMINARIES AND NOTATION. Let (M,g) be a Riemannian manifold of dimension n, $\mathcal{F}$ be a distribution of p-planes and $\mathcal{H}$ be the orthogonal distribution with rank $q=n-p$.

We will denote by $\{e_1,\dots,e_n\}$ a local orthonormal frame field adapted to $\mathcal{F}$, that is, with $e_A\in\Gamma(\mathcal{F})$ for $1\le A\le p$ and $e_\alpha\in\Gamma(\mathcal{H})$ for $p+1\le\alpha\le n$. Let $\{\theta^1,\dots,\theta^n\}$ be the associated dual basis.

The connection and curvature forms of the Levi-Civita connection associated with g will be denoted by ω^i_j and Ω^i_j respectively. They satisfy the structural equations:

$$
\begin{aligned}
d\theta^i &= -\sum_{k=1}^{n}\omega^i_k\wedge\theta^k\\
\Omega^i_j &= d\omega^i_j+\sum_{k=1}^{n}\omega^i_k\wedge\omega^k_j
\end{aligned}
\tag{1}
$$

By ∇' and R' we mean the covariant derivation symbol and the Riemann tensor of g (cf.[5]). Then, for every $X,Y,Z\in\Gamma(\mathcal{F})$ we define

$$\nabla_X Y = h\nabla'_X Y$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$\alpha(X,Y) = v\nabla'_X Y$$

where h is the orthogonal projection of TM to $\mathcal{F}$.

It is easy to prove the following "Gauss equation" for a plane field

(2) $R'(W,Z,X,Y) = R(W,Z,X,Y) + g(\alpha(X,Z),\alpha(Y,W)) - g(\alpha(Y,Z),\alpha(X,W))$

If $k'_{ij} = R'(e_i,e_j,e_i,e_j)$ and $k_{ij} = R(e_i,e_j,e_i,e_j)$ we define the sectional curvatures by

$$\tau'_M = \sum_{i,j=1}^{n} k'_{ij}, \quad \tau'_{\mathcal{F}} = \sum_{A,B=1}^{p} k'_{AB}, \quad \tau_{\mathcal{F}} = \sum_{A,B=1}^{p} k_{AB}$$

We have

$$\tau'_M = \tau'_{\mathcal{F}} + 2\tau'_m + \tau'_{\mathcal{H}}$$

where τ'_m is called the mixed scalar curvature.

The integrability and fundamental tensors of $\mathcal{F}$ are defined as:

$$A_{\mathcal{F}}(X,Y) = \frac{1}{2} v(\nabla'_X Y - \nabla'_Y X)$$
$$S_{\mathcal{F}}(X,Y) = \frac{1}{2} v(\nabla'_X Y + \nabla'_Y X)$$

where $X,Y \in \Gamma(\mathcal{F})$ and v is the orthogonal projection of TM to $\mathcal{H}$. $A_{\mathcal{H}}$ and $S_{\mathcal{H}}$ are defined in a similar way.

If $A_{\mathcal{F}}(e_A,e_B) = \sum_{\alpha=p+1}^{n} a^{\alpha}_{AB} \cdot e_\alpha$ and $S^\alpha(v,w) = g(S_{\mathcal{F}}(v,w),e_\alpha)$ we put:

$$\|A_{\mathcal{F}}\|^2 = \sum_{\substack{A<B\\ \alpha}} (a^{\alpha}_{AB})^2$$

$$\sigma_{2,\mathcal{F}} = \frac{1}{2} \sum_{\alpha} \{(trS^\alpha)^2 - tr(S^\alpha)^2\}$$

Notice that these definitions do not depend on the orthonormal adapted basis elected.

From (2) and the fact that $\alpha = S_{\mathcal{F}} + A_{\mathcal{F}}$ we obtain

$$\tau'_{\mathcal{F}} = \tau_{\mathcal{F}} - 2 \cdot (\sigma_{2,\mathcal{F}} + \|A_{\mathcal{F}}\|^2) \tag{3}$$

II A BASIC MORPHISM. On an oriented Riemannian manifold (M,g) it is equivalent to giving a reduction $P_{\mathcal{F}}$ of the bundle of orthonormal direct frames P to the structural group $SO(p) \times SO(q)$ than an oriented distribution of p-planes $\mathcal{F}$.

Following Albert ([1]), let W^* be the Weil algebra associated to the affine group $ASO(n) = \mathbf{R}^n \times SO(n)$ with Lie algebra $\mathfrak{a}so(n) = \mathbf{R}^n \oplus \mathfrak{m} \oplus (\mathfrak{a}so(p) \oplus \mathfrak{a}so(q))$. Considering the identity map $id : \mathfrak{a}so(n) \to \mathfrak{a}so(n)$ either as an element of $\Lambda^1 \mathfrak{a}so(n)^* \otimes \mathfrak{a}so(n)$ or as an element of $S^1 \mathfrak{a}so(n) \otimes \mathfrak{a}so(n)$ we have the decomposition $i = \theta_0 + \sigma_0 + \pi_0$ or $I = \Theta_0 + \Sigma_0 + \Pi_0$.

Let W^*_{pq} be the $SO(p)\otimes SO(q)$ differential graded algebra (cf.[4]) ($SO(p)\otimes SO(q)$ –DG– algebra for short) obtained by restricting to $SO(p)\otimes SO(q)$ the action of $ASO(n)$ over W^*. Let J be the ideal of W^*_{pq} generated by the components, over an arbitrary basis of $\mathbf{R}^n$, of Θ_0 and $d\Theta_0$, and $\hat{W}^*_{pq}$ be the quotient $SO(p)\otimes SO(q)$ –DG– algebra $\hat{W}^*_{pq}/J$.

Then, associated to each p-plane field F on M, one obtains the morphism of $SO(p)\otimes SO(q)$ –DG– algebras

$$\varphi_{\mathcal{F}} : \hat{W}^*_{pq} \longrightarrow A^*(P_{\mathcal{F}})$$

and restricting to the basic elements (i.e. $i_X\omega = 0 = \mathcal{L}_X\omega$)

$$B\varphi_{\mathcal{F}} : B\hat{W}^*_{pq} \longrightarrow A^*(M)$$

When the plane field is n-dimensional we have $B\varphi_M : B\hat{W}^*_{n0} \to A^*(M)$ and the commutative diagram

$$\begin{tikzcd} B\hat{W}^*_{n0} \arrow[rr, "j"] \arrow[dr, "B\varphi_M"'] & & B\hat{W}^*_{pq} \arrow[dl, "B\varphi_{\mathcal{F}}"] \\ & A^*(M) & \end{tikzcd} \tag{4}$$

III RESULTS. On $\hat{W}^*_{pq}$ we consider the element

$$\alpha = \sum_{i<j} (-1)^{i+j+1}\Omega^i_{0j}\wedge\theta^1_0\wedge\cdots\wedge\hat{\theta}^i_0\wedge\cdots\wedge\hat{\theta}^j_0\wedge\cdots\wedge\theta^n_0$$

(where $\Omega_0 = \Theta_0 + \Sigma_0 + \Pi_0$). Then, if ν is the volume element of (M, g), we have the following

Theorem.

a) $\alpha \in B\hat{W}^*_{n0}$
b) $B\varphi_M(\alpha) = \frac{1}{2}\tau'_M\cdot\nu$
c) $B\varphi_{\mathcal{F}}\circ j(\alpha) = \{\frac{1}{2}(\tau_{\mathcal{F}}+\tau_{\mathcal{H}}) + \sigma_{2,\mathcal{F}} + \sigma_{2,\mathcal{H}} + \|A_{\mathcal{F}}\|^2 + \|A_{\mathcal{H}}\|^2\}\cdot\nu + d\zeta$

The proof of a) and b) are straightforward computations. To prove c) we use the decomposition of $id : \mathfrak{a}so(n) \to aso(n)$ given in II, the fundamental equations (1) and formula (3).

As a consequence of the commutativity of diagram (4) we have

Corollary 1.

a)

$$\tau'_M\cdot\nu = \{\tau_{\mathcal{F}} + \tau_{\mathcal{H}} + 2(\sigma_{2,\mathcal{F}} + \sigma_{2,\mathcal{H}}) + 2(\|A_{\mathcal{F}}\|^2 + \|A_{\mathcal{H}}\|^2)\}\cdot\nu + d\zeta'$$

b) *In addition, if M is closed*

$$\int_M \tau'_M = \int_M \tau_{\mathcal{F}} + \int_M \tau_{\mathcal{H}} + 2\int_M(\sigma_{2,\mathcal{F}} + \sigma_{2,\mathcal{H}}) + 2\int_M(\|A_{\mathcal{F}}\|^2 + \|A_{\mathcal{H}}\|^2)$$

NOTE 1. In this way we have obtained a change formula, because at left hand side of b) there is a term depending only on the geometry of the manifold and on the other side there are terms depending on the geometry of the plane fields: integrability ($\|A\|^2$), curvature (τ) and second fundamental form (σ_2).

NOTE 2. This point of view unifies several results obtained by some authors using different techniques (cf.[2], [3], [6], [7]). For instance, from (3) and the above theorem we have the following corollary, from which Ranjan ([6]) and Rocamora ([7]) formulas can be derived.

Corollary 2.

a)
$$\frac{1}{2}\tau'_m \cdot \nu = (\sigma_{2,\mathcal{F}} + \sigma_{2,\mathcal{H}} + \|A_{\mathcal{F}}\|^2 + \|A_{\mathcal{H}}\|^2) \cdot \nu + d\zeta''$$

b) *In addition, if M is closed*

$$\frac{1}{2}\int_M \tau'_m = \int_M (\sigma_{2,\mathcal{F}} + \sigma_{2,\mathcal{H}}) + \int_M (\|A_{\mathcal{F}}\|^2 + \|A_{\mathcal{H}}\|^2)$$

In the case of a manifold of constant sectional curvature we generalize a result from [3] to arbitrary codimension and avoiding the integrability condition.

Corollary 3. *If M is closed with constant sectional curvature k*

$$\frac{1}{2}pqk \cdot vol(M) = \int_M (\sigma_{2,\mathcal{F}} + \|A_{\mathcal{F}}\|^2) + \int_M (\sigma_{2,\mathcal{H}} + \|A_{\mathcal{H}}\|^2)$$

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