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ON THE INTEGRAL FORMULAS OF CROFTON AND HURWITZ RELATIVE TO THE VISUAL ANGLE OF A CONVEX SET

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Abstract. We provide a unified approach that encompasses some integral formulas for functions of the visual angle of a compact convex set due to Crofton, Hurwitz and Masotti. The basic tool is an integral formula that also allows us to integrate new functions of the visual angle. Also, we establish some upper and lower bounds for the considered integrals, generalizing, in particular, those obtained by Santaló for Masotti's integral.

§1. *Introduction.* In 1868, Crofton showed, in [1], using arguments that nowadays belong to integral geometry, the well-known formula

$$\int_{P\notin K}(\omega-\sin\omega)\,dP=\frac{L^2}{2}-\pi F,$$

where K is a planar compact convex set of area F, L is the length of its boundary and $\omega=\omega(P)$ is the *visual angle* of K from the point P, that is, the angle between the two tangents from P to the boundary of K.

Later, in 1902, Hurwitz, in his celebrated paper [6] considered again the integral of some functions of the visual angle. In particular, he gave a new proof of the Crofton formula using the Fourier series of the radius of curvature of the boundary of K. He also computed

$$\int_{P\notin K}\sin^3\omega\,dP=\frac{3}{4}L^2+\frac{1}{4}\pi^2\gamma_2^2 \tag{1}$$

as well as the integral of a family of special functions that enabled him to show that the quantities $\gamma_k^2=\alpha_k^2+\beta_k^2$, where α_k and β_k are the Fourier coefficients of the radius of curvature, are invariant with respect to rigid motions of K.

In 1955, Masotti [7] considered a Crofton-type formula computing

$$\int_{P\notin K}(\omega^2-\sin^2\omega)\,dP$$

in terms of the area of K, the length of ∂K and the Fourier coefficients of the radius of curvature of ∂K. In 1976, Santaló [9, I.4.5] gave lower and upper bounds for the above integral.

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In this paper, we provide a unified approach that encompasses the previous results and allows us to obtain new integral formulas for functions of the visual angle. The basic tool is an integral formula given by our first result.

THEOREM 3.1. *Let K be a compact convex set with boundary of class C^2 and let L be the length of ∂K. Let $c_k^2 = a_k^2 + b_k^2$, where a_k, b_k are the Fourier coefficients of the support function of K. Then, for every continuous function of the visual angle $f(\omega)$ on $[0, \pi]$ such that $f(\omega) = O(\omega^3)$ as ω tends to zero,*

$$\begin{aligned}\int_{P\notin K} & f(\omega)\, dP \\ &= \left(\int_0^\pi \frac{f(\omega)(1+\cos\omega)^2}{\sin^3\omega}\, d\omega\right)\frac{L^2}{2\pi} + \pi\sum_{k\geqslant 2}\left(\int_0^\pi \frac{f(\omega)h_k(\omega)}{\sin^3\omega}\, d\omega\right)c_k^2,\end{aligned}$$

where h_k, for $k \geqslant 2$, are the universal functions given in (6).

Crofton's formula, the integral of the above mentioned special functions considered by Hurwitz and the Masotti integral formula follow directly from Theorem 3.1. Moreover, we improve the lower bound given by Santaló for Masotti's integral (see Corollary 5.1).

As well the integral, (1) follows from Theorem 3.1. Indeed, a more general result is obtained: we can compute the integral of any power of $\sin\omega$. The corresponding result is the following theorem.

THEOREM 6.1. *Let K be a compact convex set with boundary of class C^2 and length L. Write $c_k^2 = a_k^2 + b_k^2$, where a_k, b_k are the Fourier coefficients of the support function of K. Then*

$$\begin{aligned}\int_{P\notin K} \sin^m\omega\, dP = & \frac{\pi\, m!}{2^{m-1}(m-2)\Gamma((m+1)/2)^2}\frac{L^2}{2\pi} + \frac{\pi^2\, m!}{2^{m-1}(m-2)} \\ & \times \sum_{k\geqslant 2,\ \text{even}} \frac{(-1)^{k/2+1}(k^2-1)}{\Gamma((m+1+k)/2)\Gamma((m+1-k)/2)}c_k^2.\end{aligned}$$

For m odd, the index k in the sum runs only from 2 to $m-1$.

When K is a compact convex set of constant width, one has $c_k = 0$ for k even, so that the integral of $\sin^m\omega$ is, in this case, L^2 multiplied by a factor that depends only on m.

Next, we extend the formulas of Crofton and Masotti by means of the equality

$$\int_{P\notin K} (\omega^m - \sin^m\omega)\, dP = -\pi^m F + M_m\frac{L^2}{2\pi} + \pi\sum_{k\geqslant 2}\beta_k c_k^2, \tag{2}$$

where the quantities M_m and β_k can be explicitly computed (see §7). For instance, in the case $m=3$, we get

$$\begin{aligned}\int_{P\notin K}&(\omega^3-\sin^3\omega)\,dP\\ &=-\pi^3F+\bigg(12\pi\ln(2)-\frac{3\pi}{2}\bigg)\frac{L^2}{2\pi}\\ &\quad+12\pi^2\bigg(\ln(2)-\frac{19}{16}\bigg)c_2^2-6\pi^2\sum_{k\geqslant 3}\bigg(\Psi\bigg(\frac{k+1}{2}\bigg)+\gamma\bigg)c_k^2,\end{aligned}$$

where $\Psi(x)$ is the digamma function $\Psi(x)=(\ln\Gamma(x))'$ and γ is the Euler–Mascheroni constant.

Finally, since the quantities β_k appearing in (2) are not easily handled, we give upper and lower bounds for $\int_{P\notin K}(\omega^m-\sin^m\omega)\,dP$ that generalize those given by Santaló for $m=2$. More precisely, we prove the following theorem.

THEOREM 7.1. *Let K be a compact convex set with boundary of class $\mathcal{C}^2$, area F and length of the boundary L, and let $\omega=\omega(P)$ be the visual angle from the point P. Then*

$$\int_{P\notin K}(\omega^m-\sin^m\omega)\,dP\leqslant-\pi^mF+M_m\frac{L^2}{2\pi},\quad m\geqslant 1,$$

where $M_m=\int_0^\pi((\omega^m-\sin^m\omega)'/(1-\cos\omega))\,d\omega$. Equality holds only for circles.

For the case of constant width, we get the following theorem.

THEOREM 7.2. *Let K be a compact convex set of constant width, with boundary of class $\mathcal{C}^2$, of area F and length of the boundary L, and let $\omega=\omega(P)$ be the visual angle from the point P. Then*

$$\int_{P\notin K}(\omega^m-\sin^m\omega)\,dP\geqslant-\pi^mF+M_m\frac{L^2}{2\pi}-\frac{\pi^{m-1}}{4}\bigg(1-\bigg(\frac{3}{4}\bigg)^m\bigg)\Delta\geqslant 0,$$

where $\Delta=L^2-4\pi F$ is the isoperimetric deficit. The first inequality becomes an equality only for circles.

§2. *Preliminaries.* A set $K\subset\mathbb{R}^2$ is *convex* if it contains the complete segment joining every two points in the set. We shall consider non-empty compact convex sets. The *support function* of K is defined as

$$p_K(u)=\sup\{\langle x,u\rangle : x\in K\}\quad\text{for } u\in\mathbb{R}^2.$$

For a unit vector u, the number $p_K(u)$ is the signed distance of the support line to K with outer normal vector u from the origin. The distance is negative if and only if u points into the open half-plane containing the origin (cf. [**10**, 1.7.1]).

We shall denote by $p(\varphi)$ the 2π-periodic function obtained by evaluating $p_K(u)$ on $u = (\cos\varphi, \sin\varphi)$. Note that ∂K is the envelope of the one parametric family of lines given by

$$x\cos\varphi + y\sin\varphi = p(\varphi).$$

If the support function $p(\varphi)$ is differentiable, we can parametrize the boundary ∂K by

$$\gamma(\varphi) = p(\varphi)N(\varphi) + p'(\varphi)N'(\varphi),$$

where $N(\varphi) = (\cos\varphi, \sin\varphi)$. When p is a $\mathcal{C}^2$ function, the radius of curvature $\rho(\varphi)$ of ∂K at the point $\gamma(\varphi)$ is given by $p(\varphi) + p''(\varphi)$. Then, convexity is equivalent to $p(\varphi) + p''(\varphi) \geqslant 0$. From now on, we will assume that p is of class C^2.

It can be seen (cf. [9, I.1.2]) that the length L of ∂K and the area F of K are given in terms of the support function, respectively, by

$$L = \int_0^{2\pi} p\, d\varphi \quad \text{and} \quad F = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\varphi. \tag{3}$$

We will consider $\omega = \omega(P)$ the *visual angle* of ∂K from an exterior point P, that is, the angle between the tangents from P to ∂K. For a function $f(\omega)$ of the visual angle ω, we will deal with the integral of $f(\omega)$ with respect to the area measure dP.

Sets of constant width form a special class of convex sets; they are those whose orthogonal projection on any direction have the same length w. In terms of the support function p of K, constant width means that $p(\varphi)+p(\varphi+\pi) = w$. Expanding p in Fourier series

$$p(\varphi) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\varphi + b_n \sin n\varphi), \tag{4}$$

it follows that

$$p(\varphi) + p(\varphi+\pi) = 2\sum_{n=0}^{\infty}(a_{2n}\cos 2n\varphi + b_{2n}\sin 2n\varphi),$$

and therefore constant width is equivalent to $a_n = b_n = 0$ for all even $n > 0$.

Given a compact convex set K with support function $p(\varphi)$, the *Steiner point* of K is defined by the vector-valued integral

$$s(K) = \frac{1}{\pi}\int_0^{2\pi} p(\varphi)N(\varphi)\, d\varphi.$$

This functional on the space of convex sets is additive with respect to the Minkowski sum. The Steiner point is rigid motion equivariant; this means that $s(gK) = gs(K)$ for every rigid motion g. We remark that $s(K)$ can be considered, in the $\mathcal{C}^2$ case, as the centroid with respect to the curvature measure in the boundary ∂K; also it is known that $s(K)$ lies in the interior of K (cf. [5, p. 56]). In terms of the Fourier coefficients of $p(\varphi)$ given in (4), the Steiner point is

$$s(K) = (a_1, b_1).$$

The relation between the support function $p(\varphi)$ of a convex set K and the support function $q(\varphi)$ of the same convex set but with respect to a new reference with origin at the point (a, b) and axes parallel to the previous x and y axes, is given by

$$q(\varphi) = p(\varphi) - a\cos\varphi - b\sin\varphi.$$

Hence, taking the Steiner point as a new origin,

$$q(\varphi) = a_0 + \sum_{n\geqslant 2}(a_n \cos n\varphi + b_n \sin n\varphi).$$

The associated *pedal curve* to K will be the curve that in polar coordinates with respect to the Steiner point as origin is given by $r = p(\varphi)$. In fact, it is the geometrical locus of the orthogonal projection of the center on the tangents to the curve. The area A enclosed by the pedal curve is

$$A = \frac{1}{2}\int_0^{2\pi} p(\varphi)^2\,d\varphi.$$

§3. *First integral formula.* Let $f(\omega)$ be a function of the visual angle $\omega = \omega(P)$ of a given compact convex set K from a point P outside K. In this section, we will give a formula to compute the integral of $f(\omega)$ with respect to the area measure dP, $\int_{P\notin K} f(\omega)\,dP$, in terms of the area of K, the length of the boundary of K, and the Fourier coefficients of the support function of K.

For each point $P \notin K$, let φ be the angle at the origin formed by the normal to one of the tangents from P to ∂K with the x axis; the pair (φ, ω) can be considered as a system of coordinates of $\mathbb{R}^2\backslash K$.

Denoting by A, A_1 the contact points of the tangents from P to ∂K, and by $p = p(\varphi)$ the support function of K with respect to an origin O inside K (see Figure 1), we have

$$A = (p\cos\varphi - p'\sin\varphi,\, p\sin\varphi + p'\cos\varphi)$$

and hence, denoting

$$\varphi_1 = \pi + \varphi - \omega$$

we get

$$A_1 = (-p_1\cos(\varphi-\omega) + p_1'\sin(\varphi-\omega),\, -p_1\sin(\varphi-\omega) - p_1'\cos(\varphi-\omega)),$$

where $p_1(\varphi) = p(\varphi_1)$, $p_1'(\varphi) = p'(\varphi_1)$.

The intersection point $P = (X, Y)$ of the tangent lines to ∂K at points A and A_1 is given by

$$X = -\frac{1}{\sin\omega}(p\sin(\varphi-\omega) + p_1\sin\varphi),$$
$$Y = \frac{1}{\sin\omega}(p\cos(\varphi-\omega) + p_1\cos\varphi).$$

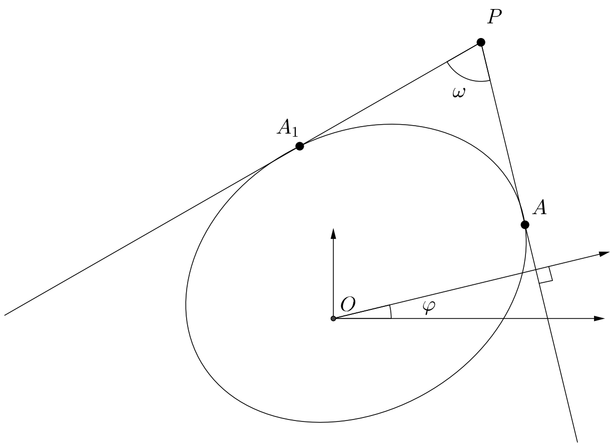

Figure 1: The visual angle ω.

From this, it is easy to see that the distances $T = PA$ and $T_1 = PA_1$ are given by the positive quantities

$$\begin{aligned} T &= \frac{1}{\sin\omega}(p\cos\omega - p'\sin\omega + p_1), \\ T_1 &= \frac{1}{\sin\omega}(p_1\cos\omega + p_1'\sin\omega + p), \end{aligned} \tag{5}$$

due to the fact that the origin is inside K.

The area element dP of $\mathbb{R}^2 \backslash K$ is

$$dP = dX \wedge dY = \left(\frac{\partial X}{\partial\omega}\frac{\partial Y}{\partial\varphi} - \frac{\partial X}{\partial\varphi}\frac{\partial Y}{\partial\omega}\right) d\varphi \wedge d\omega.$$

A straightforward computation shows that

$$dP = \frac{TT_1}{\sin\omega}\, d\varphi \wedge d\omega.$$

This expression of the area element, introduced by Crofton in [1], appears also in [8] and [9, I.2.2].

Hence, the integral on $\mathbb{R}^2 \backslash K$ of a suitable function of the visual angle $f(\omega)$ is given by

$$\int_{P\notin K} f(\omega)\, dP = \int_0^\pi \int_0^{2\pi} \frac{f(\omega)}{\sin\omega} TT_1\, d\varphi\, d\omega = \int_0^\pi \frac{f(\omega)}{\sin\omega}\left(\int_0^{2\pi} TT_1\, d\varphi\right) d\omega.$$

Now we will write the product TT_1 in terms of the Fourier coefficients of $p(\varphi)$ given in (4) and the Fourier coefficients of $p_1(\varphi)$ given by

$$p_1(\varphi) = a_0 + \sum_{k>0}(A_k\cos k\varphi + B_k\sin k\varphi),$$

which are related to the coefficients of $p(\varphi)$ by

$$A_k = (-1)^{k+1}(-a_k \cos k\omega + b_k \sin k\omega),$$
$$B_k = (-1)^{k+1}(-a_k \sin k\omega - b_k \cos k\omega).$$

Substituting these Fourier series in (5), a straightforward but long calculation gives

$$\int_0^{2\pi} T T_1 \, d\varphi = \frac{1}{\sin^2 \omega}\left(\frac{L^2}{2\pi}(1+\cos\omega)^2 + \pi \sum_{k>0} c_k^2 h_k(\omega)\right),$$

where $c_k^2 = a_k^2 + b_k^2$ and

$$\begin{aligned} h_k(\omega) = 2\cos\omega + (-1)^{k+1}\,&(-\cos k\omega(1+\cos^2\omega) \\ &- 2k \sin k\omega \sin\omega \cos\omega + k^2 \cos k\omega \sin^2\omega). \end{aligned} \tag{6}$$

These functions can also be written as

$$\begin{aligned} h_k(\omega) = \frac{(-1)^k}{4}&[(k+1)^2 \cos((k-2)\omega) \\ &+ (k-1)^2 \cos((k+2)\omega) - 2(k^2-3)\cos(k\omega)] + 2\cos\omega. \end{aligned}$$

Notice that $h_1 \equiv 0$, $h_k(0) = 2(1+(-1)^k)$ and $h_k(\omega) = O((\omega-\pi)^4)$, as ω tends to π. Hence we have obtained the following result.

THEOREM 3.1. *Let K be a compact convex set with boundary of class C^2 and let L be the length of ∂K. Let $c_k^2 = a_k^2 + b_k^2$, where a_k, b_k are the Fourier coefficients of the support function of K. Then, for every continuous function of the visual angle $f(\omega)$ on $[0,\pi]$ such that $f(\omega) = O(\omega^3)$, as ω tends to zero,*

$$\begin{aligned} &\int_{P\notin K} f(\omega)\, dP \\ &\quad = \left(\int_0^\pi \frac{f(\omega)(1+\cos\omega)^2}{\sin^3\omega}\, d\omega\right)\frac{L^2}{2\pi} + \pi \sum_{k\geqslant 2}\left(\int_0^\pi \frac{f(\omega) h_k(\omega)}{\sin^3\omega}\, d\omega\right) c_k^2, \end{aligned} \tag{7}$$

where h_k, for $k \geqslant 2$, are the universal functions given in (6).

As a first example, we can easily compute the integral in (1) to get

$$\begin{aligned} \int_{P\notin K} \sin^3\omega \, dP &= \frac{L^2}{2\pi}\int_0^\pi (1+\cos\omega)^2\, d\omega + \pi \sum_{k\geqslant 2} c_k^2 \int_0^\pi h_k(\omega)\, d\omega \\ &= \frac{3}{4}L^2 + \frac{9}{4}\pi^2 c_2^2. \end{aligned} \tag{8}$$

Notice that $\gamma_2^2 = 9c_k^2$ (see the footnote on page 884). This formula shows that the quantity c_2^2 is invariant with respect to Euclidean motions of K.

§4. *The area of level sets and second integral formula.* As an application of Theorem 3.1, we will now compute the area $F(\omega)$ enclosed by the locus C_ω of the points from which the convex set K is viewed under the same angle ω. The corresponding formula (11) was first given by Hurwitz in [6, p. 383] and we will use it later to obtain another version of formula (7).

Applying formula (7) with f the characteristic function of the domain enclosed by the level set C_ω gives

$$\begin{aligned} F(\omega) &= F + \left(\int_\omega^\pi \frac{(1+\cos\tau)^2}{\sin^3\tau}\,d\tau\right)\frac{L^2}{2\pi} + \pi\sum_{k\geqslant 2}\left(\int_\omega^\pi \frac{h_k(\tau)}{\sin^3\tau}\,d\tau\right)c_k^2 \\ &= F + \frac{L^2}{2\pi}\left[-\frac{1}{2\sin^2(\tau/2)}\right]_\omega^\pi + \pi\sum_{k\geqslant 2}\int_\omega^\pi\left(\frac{h_k(\tau)}{\sin^3\tau}\,d\tau\right)c_k^2 \\ &= F + \frac{L^2}{4\pi}\cot^2(\omega/2) + \pi\sum_{k\geqslant 2}\int_\omega^\pi\left(\frac{h_k(\tau)}{\sin^3\tau}\,d\tau\right)c_k^2. \end{aligned} \tag{9}$$

Using (3) and the Fourier expansions of p and p', one gets (see, for instance, [2] or [5, 4.2.2])

$$F = \frac{L^2}{4\pi} - \frac{\pi}{2}\sum_{k\geqslant 2}(k^2-1)c_k^2$$

and equation (9) can be written as

$$F(\omega) = \frac{L^2}{4\pi}\frac{1}{\sin^2(\omega/2)} - \frac{\pi}{2}\sum_{k\geqslant 2}(k^2-1)c_k^2 + \pi\sum_{k\geqslant 2}\int_\omega^\pi\left(\frac{h_k(\tau)}{\sin^3\tau}\,d\tau\right)c_k^2.$$

Equivalently,

$$\begin{aligned} F(\omega)\sin^2\omega = \frac{L^2}{2\pi}(1+\cos\omega) - \frac{\pi}{2}\sin^2\omega\sum_{k\geqslant 2}(k^2-1)c_k^2 \\ + \pi\sin^2\omega\sum_{k\geqslant 2}\int_\omega^\pi\left(\frac{h_k(\tau)}{\sin^3\tau}\,d\tau\right)c_k^2. \end{aligned}$$

Introducing the functions

$$g_k(\omega) = 1 + \frac{(-1)^k}{2}((k+1)\cos(k-1)\omega - (k-1)\cos(k+1)\omega), \tag{10}$$

already considered by Hurwitz, and using that

$$\frac{h_k(\tau)}{\sin^3\tau} = -\left(\frac{g_k(\tau)}{\sin^2\tau}\right)'$$

we get the following proposition.

PROPOSITION 4.1. *Under the same hypothesis as that of Theorem 3.1, the area $F(\omega)$ enclosed by the locus C_ω of the points from which the compact convex set K is viewed under the same angle ω is given by*

$$F(\omega)\sin^2\omega = \frac{L^2}{2\pi}(1+\cos\omega) + \pi\sum_{k\geqslant 2} c_k^2 g_k(\omega), \tag{11}$$

where the functions $g_k(\omega)$ are defined in (10).

Notice that the asymptotic behavior of $F(\omega)$ for ω near zero is given by

$$\lim_{\omega\to 0}(F(\omega)\sin^2\omega) = \frac{L^2}{\pi} + 2\pi\sum_{k\geqslant 2,\ \text{even}} c_k^2, \tag{12}$$

an equality that appears in [**6**, p. 383]. In the special case of a compact set of constant width, it is

$$\lim_{\omega\to 0}(F(\omega)\sin^2\omega) = \frac{L^2}{\pi}.$$

As $dP = P(\varphi,\omega)\,d\varphi\wedge d\omega$, we can write

$$F(\omega) = F + \int_0^{2\pi}\int_\omega^\pi P(\varphi,\tau)\,d\tau\,d\varphi,$$

so that $F'(\omega) = -\int_0^{2\pi} P(\varphi,\omega)\,d\varphi$. Therefore

$$\int_{P\notin K} f(\omega)\,dP = -\int_0^\pi f(\omega)F'(\omega)\,d\omega.$$

Integrating by parts and using the functions g_k given in (10), we obtain the following proposition.

PROPOSITION 4.2. *With the same hypothesis as that of Theorem 3.1,*

$$\int_{P\notin K} f(\omega)\,dP = -[f(\omega)F(\omega)]_{0_+}^{\pi_-} + \frac{L^2}{2\pi}M(f) + \pi\sum_{k\geqslant 2}\beta_k(f)c_k^2,$$

where

$$M(f) = \int_0^\pi \frac{f'(\omega)}{1-\cos\omega}\,d\omega \quad \text{and} \quad \beta_k(f) = \int_0^\pi \frac{f'(\omega)g_k(\omega)}{\sin^2\omega}\,d\omega. \tag{13}$$

Notice that $M(f)$ and $\beta_k(f)$ depend only on the function f and not on the shape of the convex set K.

As an application of Proposition 4.2, we can easily prove Crofton's formula

$$\int_{P\notin K}(\omega-\sin\omega)\,dP = -\pi F + \frac{L^2}{2}. \tag{14}$$

Indeed, $M(\omega-\sin\omega)=\pi$ and $\beta_k(\omega-\sin\omega)=\int_0^\pi g_k(x)/(1+\cos(x))\,dx=0$, as can be easily seen by integrating by parts and using elementary trigonometric identities. Since

$$-\lim_{\omega\to\pi} f(\omega)F(\omega)+\lim_{\omega\to 0} f(\omega)F(\omega)=-\pi F,$$

the formula follows.

We will now find another expression for the universal factors $g_k(\omega)/\sin^2(\omega)$ appearing in the integral defining the coefficients $\beta_k(f)$.

LEMMA 4.3. *The following identities hold.*

$$\frac{g_k(\omega)}{\sin^2(\omega)}=\frac{1}{1-\cos\omega}+2\sum_{j=1,\,\text{odd}}^{k-1} j\cos(j\omega)\quad \textit{for } k \textit{ even},$$
$$\frac{g_k(\omega)}{\sin^2(\omega)}=-2\sum_{j=2,\,\text{even}}^{k-1} j\cos(j\omega)\quad \textit{for } k \textit{ odd}.$$

Proof. From the expression of the conjugate Dirichlet kernel,

$$\sum_{j=1,\,\text{odd}}^{k-1}\sin(j\omega)=\frac{1-\cos(k\omega)}{2\sin\omega}\quad \text{for } k \text{ even}, \tag{15}$$

and

$$\sum_{j=1,\,\text{even}}^{k-1}\sin(j\omega)=\frac{\cos(\omega)-\cos(k\omega)}{2\sin\omega}\quad \text{for } k \text{ odd}.$$

Differentiating these formulas proves the lemma. □

From this Lemma and Proposition 4.2, we get the following proposition.

PROPOSITION 4.4. *With the same hypothesis as that of Theorem 3.1,*

$$\begin{aligned}\int_{P\notin K} f(\omega)\,dP &= -[f(\omega)F(\omega)]_0^\pi+\frac{L^2}{2\pi}M(f)\\ &\quad+\pi\sum_{k\geqslant 2,\,\text{even}}\left(M(f)+2\sum_{j=1,\,\text{odd}}^{k-1}\int_0^\pi f'(\omega)\,j\cos(j\omega)\,d\omega\right)c_k^2\\ &\quad+\pi\sum_{k\geqslant 3,\,\text{odd}}\left(-2\sum_{j=2,\,\text{even}}^{k-1}\int_0^\pi f'(\omega)\,j\cos(j\omega)\,d\omega\right)c_k^2, \end{aligned}\tag{16}$$

where $M(f)$ is given in (13).

This is a useful formula because it does not involve auxiliary functions and the coefficients of the c_k^2 do not depend on the convex set.

§5. *Some applications of the integral formulas.*

5.1. *Hurwitz functions.* In formula (16), the individual Fourier coefficients, a_k, b_k, of the support function of K do not appear. Rather, it is the $c_k^2 = a_k^2 + b_k^2$ that seems to be of interest. So it will be important to see how these quantities depend on the geometry of K. In fact, Hurwitz, in [**6**, p. 392], found a formula relating the c_k^2 with the length of ∂K and the integral outside K of an elementary function of the visual angle. By a direct application of formula (16), we can prove the following theorem.

THEOREM 5.1 (Hurwitz, [**6**]). *Let K be a compact convex set with boundary of class C^2 and length L. Let $c_k^2 = a_k^2 + b_k^2$, where a_k, b_k are the Fourier coefficients of the support function of K. For the functions $f_m(\omega)$ given by*

$$f_m(\omega) = -2\sin\omega + \frac{m+1}{m-1}\sin((m-1)\omega) - \frac{m-1}{m+1}\sin((m+1)\omega), \quad (17)$$

we have[1]

$$\int_{P\notin K} f_m(\omega)\, dP = L^2 + (-1)^m \pi^2 (m^2-1) c_m^2 \quad m \geqslant 2. \quad (18)$$

Proof. In order to apply formula (16), we need to compute $[f_m(\omega)F(\omega)]_0^\pi$, $M(f_m)$ and the integrals $\int_0^\pi f_m'(\omega)\cos(j\omega)\, d\omega$, for j an integer.

First, we have $[f_m(\omega)F(\omega)]_0^\pi = 0$ since, by (12),

$$\lim_{\omega\to 0} f_m(\omega)F(\omega) = c \lim_{\omega\to 0} \frac{f_m(\omega)}{\sin^2\omega} = 0.$$

For $M(f_m)$, we need the equalities

$$f_m'(\omega) = 2(1-\cos\omega)\left(1 + 2\sum_{j=1}^{m-1} j\cos(j\omega) + (m-1)\cos(m\omega)\right) \quad m \geqslant 2,$$

which are obtained by direct computation using $2\cos(\omega)\cos(j\omega) = \cos(j+1)\omega + \cos(j-1)\omega$. Then

$$\begin{aligned} M(f_m) &= \int_0^\pi \frac{f_m'(\omega)}{1-\cos\omega}\, d\omega \\ &= 2\int_0^\pi \left(1 + 2\sum_{j=1}^{m-1} j\cos(j\omega) + (m-1)\cos(m\omega)\right) d\omega = 2\pi. \end{aligned}$$

[1] There is a misprint with the sign in Hurwitz's paper. Moreover, the c_k coefficients appearing in this formula are different from those in Hurwitz's paper because the latter correspond to the Fourier series of the curvature radius function. In fact, $\alpha_k = (1-k^2)a_k$, $\beta_k = (1-k^2)b_k$, where α_k, β_k are the Fourier coefficients of the radius of curvature.

Finally,

$$
\begin{aligned}
\int_0^\pi f_m'(\sin(j\omega))'\,d\omega &= -\int_0^\pi f_m''\sin(j\omega)\,d\omega \\
&= -\int_0^\pi (2\sin\omega + 2(m^2-1)\cos(m\omega)\sin\omega)\sin(j\omega)\,d\omega \\
&= -\pi\delta_{1,j} - 2(m^2-1)\int_0^\pi \sin\omega\cos(m\omega)\sin(j\omega)\,d\omega \\
&= -\pi\delta_{1,j} - 2(m^2-1)\frac{\pi}{4}(\delta_{j,m+1}-\delta_{j,m-1}),
\end{aligned}
$$

where $\delta_{ij}=0$ for $i\neq j$ and $\delta_{ii}=1$.

Hence, if k is even,

$$
\begin{aligned}
2\sum_{j=1,\text{ odd}}^{k-1}\int_0^\pi f_k'(\sin(j\omega))'\,d\omega &= -2\pi - (m^2-1)\pi(-\delta_{k-1,m-1}) \\
&= -2\pi + (k^2-1)\pi\delta_{k,m}
\end{aligned}
$$

since

$$
\delta_{j+2,m+1} - \delta_{j,k-1} = 0, \quad j = 1,\ldots,k-1.
$$

And, if k is odd,

$$
2\sum_{j=2,\text{ even}}^{k-1}\int_0^\pi f_m'(\sin(j\omega))'\,d\omega = -(m^2-1)\pi(-\delta_{k-1,m-1}) = (m^2-1)\pi\delta_{k,m}.
$$

Substituting in (16) gives

$$
\begin{aligned}
\int_{P\notin K} f_m(\omega)\,dP &= L^2 + \pi\sum_{k\geqslant 2,\text{ even}} c_k^2(2\pi - 2\pi + (k^2-1)\pi\delta_{k,m}) \\
&\quad + \pi\sum_{k\geqslant 3,\text{ odd}} c_k^2(-(k^2-1)\pi\delta_{k,m}) \\
&= L^2 + (-1)^m\pi^2 c_m^2(m^2-1).
\end{aligned}
$$

The proof is now complete. □

The theorem shows that the quantities c_m^2 are invariant with respect to Euclidean motions of K.

Notice that the functions f_m are related to the functions g_m introduced in (10) by

$$
g_m(\omega) = 1 + \frac{(-1)^m}{2}(f_m'(\omega) + 2\cos(\omega)).
$$

5.2. *Masotti integral formula.* In [7], Masotti gives without proof a Crofton-type formula evaluating $\int_{P\notin K}(\omega^2 - \sin^2\omega)\,dP$. Here, we will derive Masotti's formula from (16).

Consider the function $f(\omega) = \omega^2 - \sin^2\omega$, which clearly satisfies the hypothesis of Theorem 3.1, and let us compute $[f(\omega)F(\omega)]_0^\pi$, $M(f)$ and the integrals $\int_0^\pi f'(\omega)\cos(j\omega)\,d\omega$, for j an integer.

We have

$$\lim_{\omega\to\pi} f(\omega)F(\omega) = \pi^2 F$$

and

$$\lim_{\omega\to 0} f(\omega)F(\omega) = \lim_{\omega\to 0}\frac{\omega^2-\sin^2\omega}{\sin^2\omega}\left(\frac{L^2}{2\pi}(1+\cos\omega)+\pi\sum_{k\geqslant 2}c_k^2 g_k(\omega)\right) = 0,$$

since the term inside the parentheses is bounded. Hence, $[f(\omega)F(\omega)]_0^\pi = \pi^2 F$.

On the other hand,

$$\begin{aligned} M(f) &= \int_0^\pi \frac{f'(\omega)}{1-\cos\omega}\,d\omega = \int_0^\pi \frac{2\omega-\sin(2\omega)}{1-\cos\omega}\,d\omega \\ &= [\sin^2(\omega/2) - 3\cos^2(\omega/2) - 2\omega\cot(\omega/2)]_0^\pi = 8. \end{aligned}$$

Moreover, for $j \neq 2$,

$$\begin{aligned} &\int_0^\pi f'(\omega)\cos(j\omega)\,d\omega \\ &\quad = \int_0^\pi (2\omega - \sin(2\omega))\cos(j\omega)\,d\omega \\ &\quad = \left[\frac{2}{j^2}(\cos(j\omega) + j\omega\sin(j\omega)) - \frac{\cos((j-2)\omega)}{2(j-2)} + \frac{\cos((j+2)\omega)}{2(j+2)}\right]_0^\pi \\ &\quad = \frac{8(1-(-1)^j)}{j^2(j^2-4)}, \end{aligned}$$

and

$$\int_0^\pi (2\omega - \sin(2\omega))\cos(2\omega)\,d\omega = 0.$$

It follows that

$$\sum_{j=1,\,\mathrm{odd}}^{k-1}\int_0^\pi f'(\omega)\cos(j\omega)\,d\omega = \sum_{j=1,\,\mathrm{odd}} \frac{16}{j(j^2-4)} = \frac{4k^2}{1-k^2}.$$

Summing up, we obtain the following theorem.

THEOREM 5.2 (Masotti, [7]). *Let K be a compact convex set of area F with boundary of class C^2 and length L. Let $c_k^2 = a_k^2 + b_k^2$, where a_k, b_k are the Fourier coefficients of the support function of K. Then*

$$\int_{P\notin K}(\omega^2 - \sin^2\omega)\,dP = -\pi^2 F + \frac{4L^2}{\pi} + 8\pi\sum_{k\geqslant 2,\,\mathrm{even}}\left(\frac{1}{1-k^2}\right)c_k^2. \quad (19)$$

Moreover, the equality

$$\int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP = -\pi^2 F + \frac{4L^2}{\pi}$$

holds if and only if the compact convex set K has constant width.

In addition to the obvious inequality

$$\int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP \leqslant -\pi^2 F + \frac{4L^2}{\pi}$$

that follows from (19), Santaló states, in [**9**, I.4.5], the lower bound

$$\int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP \geqslant (16-\pi^2)F, \tag{20}$$

with equality only for circles. We will now improve this last inequality.

THEOREM 5.3. *Under the same hypothesis as that in Theorem 5.2,*

$$-\pi^2 F + \frac{4L^2}{\pi} - \frac{4}{3}\left(H - \frac{L^2}{\pi}\right) \leqslant \int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP \leqslant -\pi^2 F + \frac{4L^2}{\pi}, \tag{21}$$

where $H = \lim_{\omega\to 0} F(\omega)\sin^2\omega$ is given in (12). Equality in the left-hand side holds if and only if ∂K is a circle or a curve parallel to an astroid.

Proof. For the left-hand side, just write

$$\int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP \geqslant -\pi^2 F + \frac{4L^2}{\pi} - \frac{8\pi}{3} \sum_{k\geqslant 2,\ \text{even}} c_k^2$$
$$= -\pi^2 F + \frac{4L^2}{\pi} - \frac{4}{3}\left(H - \frac{L^2}{\pi}\right).$$

Equality in the left-hand side holds if and only if the support function of K with respect to the Steiner point is of the form $p(\varphi) = a_0 + a_2\cos(2\varphi) + b_2\sin(2\varphi)$. This means that ∂K is a circle or a curve parallel to an astroid (see, for instance, [**2**]). □

In terms of the area A of the pedal curve of K with respect to the Steiner point, we get the following corollary.

COROLLARY 5.1. *Under the same hypothesis as that in Theorem 5.2,*

$$\int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP \geqslant (16-\pi^2)F + \frac{32}{3}(A - F).$$

Equality holds if and only if ∂K is a circle or a curve parallel to an astroid.

Proof. This is a simple consequence of (21) and the two inequalities $H - L^2/\pi \leqslant A - F$ and $\Delta \geqslant 3\pi(A-F)$ (see [**2**]). □

The above argument shows that (21) improves Santaló's inequality (20).

§6. *Integral powers of the sine of the visual angle.* In (8), we have seen that

$$\int_{P\notin K} \sin^3 \omega \, dP = \frac{3}{4}L^2 + \frac{9}{4}\pi^2 c_2^2.$$

In this section, we compute the integral of $\sin^m(\omega)$ for integer values of m greater than three. We have that $[\sin^m(\omega)F(\omega)]_0^\pi = 0$, so

$$\int_{P\notin K} \sin^m(\omega)\, dP = M(\sin^m(\omega))\frac{L^2}{2\pi} + \pi \sum_{k\geqslant 2} \beta_k(\sin^m \omega) c_k^2,$$

where M and β_k are as given in (13).

As, for j even, $\cos(\omega)\cos(j\omega)$ is an odd function with respect $\pi/2$, it can be seen from (16) that $\beta_k(\sin^m \omega) = 0$ for every k odd.

When K is a convex set of constant width, $c_k = 0$ for every even value of k, and we get

$$\int_{P\notin K} \sin^m(\omega)\, dP = M(\sin^m(\omega))\frac{L^2}{2\pi}.$$

Since $M(\sin^m \omega)$ does not depend on the convex set K, we can compute this constant by applying the above formula to the unit circle centered at the origin. If r is the distance to the origin of the point P, we have $\sin(\omega) = 2\sin(\omega/2)\cos(\omega/2) = 2\sqrt{r^2-1}/r^2$, and so

$$\begin{aligned}\int_{P\notin K} \sin^m(\omega)\, dP &= \int_0^{2\pi}\int_1^\infty \sin^m(\omega) r\, dr\, d\theta = 2\pi\, 2^m \int_1^\infty \left(\frac{\sqrt{r^2-1}}{r^2}\right)^m r\, dr \\ &= 2^m \pi B\left(\frac{m}{2}+1, \frac{m}{2}-1\right),\end{aligned}$$

where $B(x, y)$ is the beta function. Hence

$$M(\sin^m \omega) = 2^{m-1} B\left(\frac{m}{2}+1, \frac{m}{2}-1\right) = 2^{m-1}\frac{\Gamma(m/2+1)\Gamma(m/2-1)}{(m-1)!}.$$

Using the relation

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

we obtain

$$\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m}{2}-1\right) = \frac{\pi m!(m-1)!}{2^{2m-2}(m-2)\Gamma((m+1)/2)^2},$$

and hence

$$M(\sin^m(\omega)) = \frac{\pi\, m!}{2^{m-1}(m-2)\Gamma((m+1)/2)^2}. \tag{22}$$

Notice that $M(\sin^m \omega)$ decreases with m, its maximum value $3\pi/2$ is attained for $m = 3$ and it behaves as $1/\sqrt{m}$ when m tends to infinity.

We have proved the following proposition.

PROPOSITION 6.1. *Let K be a compact convex of constant width with boundary of class C^2 and length L. Then*

$$\int_{P\notin K} \sin^m \omega \, dP = \frac{\pi\, m!}{2^{m-1}(m-2)\Gamma((m+1)/2)^2} \frac{L^2}{2\pi}.$$

For general convex sets, we have the following theorem.

THEOREM 6.1. *Let K be a compact convex set with boundary of class C^2 and length L. Write $c_k^2 = a_k^2 + b_k^2$, where a_k, b_k are the Fourier coefficients of the support function of K. Then*

$$\int_{P\notin K} \sin^m \omega \, dP = M(\sin^m \omega)\frac{L^2}{2\pi} + \frac{m!\pi^2}{2^{m-1}(m-2)} \times \sum_{k\geqslant 2,\text{ even}} \frac{(-1)^{k/2+1}(k^2-1)}{\Gamma((m+1+k)/2)\Gamma((m+1-k)/2)} c_k^2, \quad (23)$$

where $M(\sin^m(\omega))$ is given in (22). For m odd, the index k in the sum runs only from 2 to $m-1$.

Proof. From (16), it is clear that we need to compute

$$M(\sin^m \omega) + 2 \sum_{j=1,\text{ odd}}^{k-1} \int_0^\pi (\sin^m \omega)' j \cos(j\omega)\, d\omega \quad \text{for } k \text{ even}.$$

Using (15) and integrating by parts gives

$$\sum_{j=1,\text{ odd}}^{k-1} \int_0^\pi (\sin^m \omega)'(\sin(j\omega))'\, d\omega = -\int_0^\pi (m(m-1)\sin^{m-2}\omega - m^2 \sin^m \omega)\frac{1-\cos(k\omega)}{2\sin\omega}\, d\omega.$$

Denoting $I_{m,k} = \int_0^\pi \sin^m \omega \cos(k\omega)\, d\omega$,

$$\sum_{j=1,\text{ odd}}^{k-1} \int_0^\pi (\sin^m \omega)'(\sin(j\omega))'\, d\omega = -\frac{m(m-1)}{2} I_{m-3,0} + \frac{m^2}{2} I_{m-1,0} + \frac{m(m-1)}{2} I_{m-3,k} - \frac{m^2}{2} I_{m-1,k}. \quad (24)$$

By induction on m and using known relations of the gamma function, it can be seen that

$$I_{m,k} = (-1)^{k/2} \frac{2^{-m} m! \pi}{\Gamma(1+(m-k)/2)\Gamma(1+(m+k)/2)},$$

(see, for instance, [**4**, p. 372]). Performing the operation on the right-hand side of (24) with these values of $I_{m,k}$, we obtain

$$\sum_{j=1,\text{ odd}}^{k-1} \int_0^{\pi} (\sin^m \omega)'(\sin(j\omega))'\, d\omega$$
$$= -\frac{1}{2}M(\sin^m \omega) + \frac{m!\pi}{2^m(m-2)} \frac{(-1)^{k/2+1}(k^2-1)}{\Gamma((m+1+k)/2)\Gamma((m+1-k)/2)}.$$

From this, formula (23) follows.

When m is odd and $k > m$, we have that $m+1-k$ is an even non-positive integer and hence $\Gamma((m+1-k)/2) = \infty$. From this remark, the last assertion of the theorem is proved. □

For instance, for the special cases $m = 3$, 4 and 5, we obtain the equalities

$$\int_{P\notin K} \sin^3 \omega\, dP = \frac{3}{4}L^2 + \frac{9}{4}\pi^2 c_2^2,$$
$$\int_{P\notin K} \sin^4 \omega\, dP = \frac{4}{3\pi}L^2 + \pi \sum_{k=2,\text{even}}^{\infty} \frac{24}{9-k^2} c_k^2,$$
$$\int_{P\notin K} \sin^5 \omega\, dP = \frac{5}{16}L^2 + \frac{5\pi^2}{4}c_2^2 - \frac{25\pi^2}{16}c_4^2.$$

To finish, we make the following remarks

(a) If $m = 2r$, the coefficient of c_k^2 in (23) for $k \leqslant r$ is positive if and only if $k/2$ is odd. For $k > r$, this coefficient is positive if and only if r is odd.

(b) If $m = 2r-1$, the coefficient of c_k^2 vanishes for $k > m$.

In [**6**, p. 392], Hurwitz computed the integral of $\sin^3(\omega)$ and the integrals of the functions $f_m(\omega)$ given in (17) without any relationship between them. We will show now that the integrals of the powers of the sine of the visual angle are a linear combination of the integrals of the functions f_m.

PROPOSITION 6.2. *For a compact convex set K with boundary of class C^2 and $m \geqslant 3$,*

$$\int_{P\notin K} \sin^m(\omega)\, dP = \frac{m!}{2^{m-1}(m-2)} \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{\Gamma((m+1)/2+p)\Gamma((m+1)/2-p)} \cdot \int_{P\notin K} f_{2p}(\omega)\, dP,$$

where the functions $f_{2p}(\omega)$ are given in (17).

Proof. Substituting in (23) the value of c_k^2 given by (18) gives

$$\begin{aligned}\int_{P\notin K}&\sin^m\omega\,dP\\ &=\frac{m!}{2^m(m-2)}\\ &\quad\times\left(\frac{1}{\Gamma((m+1)/2)^2}-2\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{\Gamma((m+1)/2+p)\Gamma((m+1)/2-p)}\right)L^2\\ &\quad+\frac{m!}{2^{m-1}(m-2)}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{\Gamma((m+1)/2+p)\Gamma((m+1)/2-p)}\\ &\quad\cdot\int_{P\notin K}f_{2p}(\omega)\,dP.\end{aligned}\tag{25}$$

Using the standard notation for hypergeometric series,

$$\begin{aligned}2\sum_{p=1}^{\infty}&\frac{(-1)^{p+1}}{\Gamma((m+1)/2+p)\Gamma((m+1)/2-p)}\\ &=2\frac{{}_2F_1((3-m)/2,1;(3+m)/2;1)}{\Gamma((m+1)/2+1)\Gamma((m+1)/2-1)},\end{aligned}$$

and by the Gauss summation formula (see [3, Vol. III, p. 147]) we obtain

$$\begin{aligned}2&\frac{{}_2F_1((3-m)/2,1;(3+m)/2;1)}{\Gamma((m+1)/2+1)\Gamma((m+1)/2-1)}\\ &=\frac{2}{\Gamma((m+1)/2+1)\Gamma((m+1)/2-1)}\cdot\frac{\Gamma(m-1)\Gamma((m+3)/2)}{\Gamma(m)\Gamma((m+1)/2)}\\ &=\frac{1}{\Gamma((m+1)/2)^2}.\end{aligned}$$

Hence the coefficient of L^2 in (25) vanishes. □

§7. *Extension of the Crofton and Masotti formulas and related inequalities.* In this section, we consider the integral

$$\int_{P\notin K}(\omega^m-\sin^m\omega)\,dP,$$

where ω is the visual angle of the convex set K from the point P. For $m=1$ and $m=2$, these are the integrals appearing in Crofton's formula (14) and in the Masotti integral formula (19), respectively.

For the general case, we have, by Proposition 4.2,

$$\int_{P\notin K}(\omega^m-\sin^m\omega)\,dP=-\pi^m F+M_m\frac{L^2}{2\pi}+\pi\sum_{k\geqslant 2}\beta_k c_k^2,\tag{26}$$

where $M_m = M(\omega^m - \sin^m \omega)$ and $\beta_k = \beta_k(\omega^m - \sin^m \omega)$ are given in (13). The quantities M_m can be explicitly computed. In fact, $M(\sin^m \omega)$ is given in (22) and

$$M(\omega^m) = 2m(m-1)\pi^{m-2}\left(\frac{1}{m-2} + \sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(m-2+2k)(2k)!}\right),$$

where B_{2k} are the Bernoulli numbers (see [4, p. 189]). As the parenthesized expression tends to zero as $1/m^2$ when m tends to infinity, we see that $M(\omega^m)$ behaves like e^m and therefore M_m grows exponentially with m.

For β_k, recall that, by Proposition 4.4,

$$\beta_k = \begin{cases} M_m + 2 \displaystyle\sum_{j=1,\,\mathrm{odd}}^{k-1} \int_0^{\pi} (\omega^m - \sin^m \omega)' j \cos(j\omega)\, d\omega & \text{for } k \text{ even}, \\ -2 \displaystyle\sum_{j=1,\,\mathrm{even}}^{k-1} \int_0^{\pi} (\omega^m)' j \cos(j\omega)\, d\omega & \text{for } k \text{ odd}. \end{cases} \tag{27}$$

Although the integrals appearing in the expression of the β_k can be explicitly computed, they are not easily handled.

For instance, in the case $m = 3$, it can be seen that

$$\begin{aligned} &\int_{P\notin K} (\omega^3 - \sin^3 \omega)\, dP \\ &\quad = -\pi^3 F + \left(12\pi \ln(2) - \frac{3\pi}{2}\right)\frac{L^2}{2\pi} + 12\pi^2\left(\ln(2) - \frac{19}{16}\right)c_2^2 \\ &\qquad - 6\pi^2 \sum_{k\geqslant 3}\left(\Psi\left(\frac{k+1}{2}\right) + \gamma\right)c_k^2, \end{aligned}$$

where $\Psi(x)$ is the digamma function, $\Psi(x) = (\ln \Gamma(x))'$, and γ is the Euler–Mascheroni constant.

7.1. *Upper bounds.* We now obtain an upper bound for $\int_{P\notin K}(\omega^m - \sin^m \omega)\, dP$. For $m = 3$, since $\Psi(x) > 0$ for $x \geqslant 2$,

$$\int_{P\notin K} (\omega^3 - \sin^3 \omega)\, dP \leqslant -\pi^3 F + \left(12\pi \ln(2) - \frac{3\pi}{2}\right)\frac{L^2}{2\pi}. \tag{28}$$

For the general case, we obtain the following result.

THEOREM 7.1. *Let K be a compact convex set with boundary of class $\mathcal{C}^2$, area F and length of the boundary L, and let $\omega = \omega(P)$ be the visual angle from the point P. Then*

$$\int_{P\notin K} (\omega^m - \sin^m \omega)\, dP \leqslant -\pi^m F + M_m \frac{L^2}{2\pi} \quad \text{for } m \geqslant 1,$$

where $M_m = \int_0^{\pi} ((\omega^m - \sin^m \omega)'/(1 - \cos\omega))\, d\omega$. Equality holds only for circles.

Remark 7.1. Since $M_1 = \pi$, $M_2 = 8$ and $M_3 = (12\ln(2) - 3\pi/2)$, this result agrees with Crofton's formula (14) for $m = 1$, and it is a generalization of the upper bounds for Masotti's integral given in (21) for $m = 2$ and of the upper bound given in (28) for $m = 3$.

Proof. We shall see that $\beta_k \leqslant 0$, for $k \geqslant 2$. Writing $V_{r,j} = \int_0^\pi \omega^r \cos(j\omega)\, d\omega$, the following recurrence formula can be checked:

$$V_{r,j} = \frac{r}{j^2}((-1)^j \pi^{r-1} - (r-1)V_{r-2,j}).$$

This gives $V_{r,j} \geqslant 0$ for j even and $V_{r,j} \leqslant 0$ for j odd. As, for k odd, we have $\beta_k = -2m \sum_{j=2,\text{ even}}^{k-1} j V_{m-1,j}$, it follows that $\beta_k \leqslant 0$ for k odd.

For k even, by integrating by parts, we get

$$\beta_k - \beta_{k+2} = 2\int_0^\pi (\omega^m - \sin^m \omega)'' \sin((k+1)\omega)\, d\omega.$$

It can be seen that, for $m \geqslant 4$, the function $\psi(\omega) := (\omega^m - \sin^m \omega)''$ is non-negative and increasing on the interval $[0, \pi]$. Then, partitioning $[0, \pi]$ by $t_j = j\pi/(k+1)$ with $j = 0, \ldots, k+1$, we get

$$\begin{aligned}
&\int_0^\pi \psi(\omega)\sin(k+1)\omega\, d\omega \\
&\quad = \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} \psi(\omega)\sin(k+1)\omega\, d\omega > \sum_{j=1}^{k}\int_{t_j}^{t_{j+1}} \psi(\omega)\sin(k+1)\omega\, d\omega \\
&\quad = \sum_{j=2,\text{ even}}^{k}\left(\int_{t_j}^{t_{j+1}} \psi(\omega)\sin(k+1)\omega\, d\omega - \int_{t_{j-1}}^{t_j} \psi(\omega)|\sin(k+1)\omega|\, d\omega\right) \\
&\quad = \sum_{j=2,\text{ even}}^{k}\int_{t_j}^{t_{j+1}}\left(\psi(\omega) - \psi\left(\omega - \frac{\pi}{k+1}\right)\right)\sin(k+1)\omega\, d\omega > 0,
\end{aligned}$$

so that $\beta_k - \beta_{k+2} > 0$ for all $k \geqslant 2$. Thus, in order to see that $\beta_k \leqslant 0$, it is enough to show that $\beta_2 \leqslant 0$, with

$$\beta_2 = \int_0^\pi (\omega^m - \sin^m \omega)'\left(\frac{1}{1-\cos\omega} + 2\cos\omega\right) d\omega.$$

For simplicity in the exposition, we write $h_m(\omega) = \omega^{m-1} - \sin^{m-1}\omega\cos\omega$ and $g(\omega) = 1/(1-\cos\omega) + 2\cos\omega$. The function g has exactly one root in the interval $[0, \pi]$, $\zeta = \arccos((1-\sqrt{3})/2)$.

Notice that

$$h_m(\omega)g(\omega) = \omega^{m-1}g(\omega) - \sin^{m-1}\omega\cos\omega g(\omega) \leqslant \omega^{m-1}g(\omega) + 1$$

for $0 \leqslant \omega \leqslant \pi$. Then, in order to see that $\beta_2 < 0$, it suffices to prove that $\int_0^\pi \omega^{m-1} g(\omega)\, d\omega \leqslant -\pi$. First, we see that the sequence $\int_0^\pi \omega^{m-1} g(\omega)\, d\omega$ decreases with m, that is,

$$\int_0^\pi (\omega^m - \omega^{m-1}) g(\omega)\, d\omega = \int_0^\pi \omega^{m-1}(\omega - 1) g(\omega)\, d\omega \leqslant 0.$$

The integrand is positive if $\omega \in [1, \zeta]$ and negative otherwise. Therefore, if we see that

$$\int_1^\zeta \omega^m (\omega - 1) g(\omega)\, d\omega < \int_\zeta^\pi \omega^{m-1}(\omega - 1)|g(\omega)|\, d\omega,$$

the proof is complete.

On the one hand, $\int_1^\zeta \omega^{m-1}(\omega - 1) g(\omega)\, d\omega < \zeta^{m-1}(\zeta - 1) \int_1^\zeta g(\omega)\, d\omega$. On the other hand,

$$\begin{aligned}\int_\zeta^\pi \omega^{m-1}(\omega - 1)|g(\omega)| &> \int_{\zeta+\varepsilon}^\pi \omega^{m-1}(\omega - 1)|g(\omega)|\, d\omega \\ &> (\zeta + \epsilon)^{m-1}(\zeta + \epsilon - 1) \int_{\zeta+\epsilon}^\pi |g(\omega)|\, d\omega\end{aligned}$$

for $0 < \epsilon < \pi - \zeta$. Considering $\epsilon = 1$ and using the fact that the function $2\sin\omega - \cot(\omega/2)$ is an antiderivative of g, a simple computation shows that

$$\zeta^{m-1}(\zeta - 1) \int_1^\zeta g(\omega)\, d\omega < (\zeta + 1)^{m-1} \zeta \int_{\zeta+1}^\pi |g(\omega)|\, d\omega$$

and the sequence $\int_0^\pi \omega^{m-1} g(\omega)\, d\omega$ decreases with m. Since $\int_0^\pi \omega^2 g(\omega)\, d\omega < -\pi$, we have proved that $\beta_k \leqslant \beta_2 < 0$. This gives us the upper bound

$$\int_{P \notin K} (\omega^m - \sin^m \omega)\, dP \leqslant -\pi^m F + M_m \frac{L^2}{2\pi} \quad \text{for } m \geqslant 4.$$

As the cases $m = 1, 2, 3$ are already known, this finishes the proof of the inequality in the theorem. Finally, since the β_k are not zero, equality holds only when $c_k = 0$ for $k \geqslant 2$, that is, when ∂K is a circle. □

7.2. *Lower bounds.* For the case of constant width, we have the following theorem.

THEOREM 7.2. *Let K be a compact convex set of constant width, with boundary of class $\mathcal{C}^2$, of area F and length of the boundary L, and let $\omega = \omega(P)$ be the visual angle from the point P. Then*

$$\int_{P \notin K} (\omega^m - \sin^m \omega)\, dP \geqslant -\pi^m F + M_m \frac{L^2}{2\pi} - \frac{\pi^{m-1}}{4}\left(1 - \left(\frac{3}{4}\right)^m\right)\Delta \geqslant 0,$$

where $\Delta = L^2 - 4\pi F$ is the isoperimetric deficit. The first inequality becomes an equality only for circles.

Proof. In the constant width case, we have $c_k = 0$ for k even and the only contribution in β_k comes from ω^m when k is odd. Therefore, since

$$\int_{P \notin K} (\omega^m - \sin^m \omega)\, dP = -\pi^m F + M_m \frac{L^2}{2\pi} + \pi \sum_{k>2,\text{ odd}} \beta_k c_k^2,$$

an upper bound for the positive quantity $K_m = -\pi \sum_{k>2,\text{ odd}} \beta_k c_k^2$ will give a lower bound for $\int_{P \notin K} (\omega^m - \sin^m \omega)\, dP$. Using (27),

$$K_m = 2\pi m \sum_{k>2,\text{ odd}} \left(\sum_{j=2,\text{ even}}^{k-1} j \int_0^\pi \omega^{m-1} \cos(j\omega)\, d\omega \right) c_k^2.$$

The following estimate holds.

$$\begin{aligned}\int_0^\pi \omega^{m-1} \cos(j\omega)\, d\omega &< \int_{\pi-\pi/2j}^{\pi} \omega^{m-1} \cos(j\omega)\, d\omega \\ &\leqslant \int_{\pi-\pi/2j}^{\pi} \omega^{m-1}\, d\omega = \frac{\pi^m}{m}\left(1 - \left(\frac{3}{4}\right)^m\right).\end{aligned}$$

Moreover, $\sum_{j=2,\text{ even}} j = \frac{1}{4}(k^2 - 1)$, so that

$$K_m \leqslant \frac{\pi^{m+1}}{2}\left(1 - \left(\frac{3}{4}\right)^m\right) \sum_{k>2,\text{ odd}} (k^2 - 1) c_k^2. \tag{29}$$

As $\Delta = 2\pi^2 \sum_{k \geqslant 2} (k^2 - 1) c_k^2$ (cf. [2]),

$$K_m \leqslant \frac{\pi^{m-1}}{4}\left(1 - \left(\frac{3}{4}\right)^m\right)\Delta,$$

which gives the desired lower bound.

From formula (26) applied to a circle, it follows that $M_m \geqslant \pi^m/2$ and so

$$\begin{aligned}\int_{P \notin K} (\omega^m - \sin^m \omega)\, dP &\geqslant -\pi^m F + M_m \frac{L^2}{2\pi} - \frac{\pi^{m-1}}{4}\left(1 - \left(\frac{3}{4}\right)^m\right)\Delta \\ &\geqslant \frac{\pi^{m-1}}{4}\left(\frac{3}{4}\right)^m \Delta \geqslant 0.\end{aligned}$$

Finally, if some $c_k \neq 0$, we have strict inequality in (29), and hence the first inequality becomes an equality only for circles. $\square$

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