Arch. Math. 117 (2021), 579–591

0003-889X/21/050579-13
published online September 21, 2021
https://doi.org/10.1007/s00013-021-01651-8

Archiv der Mathematik

Integral geometry of pairs of planes

Julià Cufí, Eduardo Gallego, and Agustí Reventós

Abstract. We deal with integrals of invariant measures of pairs of planes in the Euclidean space $\mathbb{E}^3$ as considered by Hug and Schneider. In this paper, we express some of these integrals in terms of functions of the visual angle of a convex set. As a consequence of our results, we evaluate the deficit in a Crofton-type inequality due to Blaschke.

Mathematics Subject Classification. Primary 52A15; Secondary 53C65.

Keywords. Invariant measure, Convex set, Visual angle, Constant width.

1. Introduction. The main goal of this paper is to study integrals of invariant measures with respect to Euclidean motions in the Euclidean space $\mathbb{E}^3$, extended to the set of pairs of planes meeting a compact convex set. To carry out this objective, we express these integrals in terms of functions of the dihedral visual angle of the convex set from a line and integrate them with respect to an invariant measure in the space of lines.

The first known formula involving the visual angle of a convex set in the Euclidean plane $\mathbb{E}^2$ is Crofton's formula given in [2]. Other results in this direction were obtained by Hurwitz [8], Masotti [10] and others, in which the use of the Fourier series is the main tool. Recently, the authors [3,4] have dealt with a general type of integral formulas, involving the visual angle, from the point of view of integral geometry.

When trying to generalize these results to higher dimensions, the role played by the Fourier series in the case of the plane has to be replaced by the use of spherical harmonics. In this sense, Theorem 4.1 plays an important role. After stating and proving this result in dimension 3, we realized that Hug and Schneider [7] proved a more general result in any dimension. In fact, the

The authors were partially supported by Grants 2017SGR358, 2017SGR1725 (Generalitat de Catalunya) and PGC2018-095998-B-100 (FEDER/MICINN).

present paper can be considered in some sense as a complement to [7], the novelty being the introduction of the dihedral visual angle.

In Proposition 3.1, we give a characterization of invariant measures in the space of pairs of planes. These will be the kind of measures considered along the paper.

In Section 4, using Hug–Schneider's theorem [7, p. 349], we give an expression for the integral of the sine of the dihedral visual angle of pairs of planes meeting a given compact convex set K in terms of geometrical properties of K, see formula (4.2); we also characterize the compact convex sets of constant width in terms of invariant measures given by Legendre polynomials by means of the following proposition that completes a result in [7].

Proposition 4.3. *Let K be a compact convex set of constant width $\mathcal{W}$ and let $f : [-1,1] \longrightarrow \mathbb{R}$ be an even bounded measurable function. Then*

$$\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = \lambda_0 \pi \mathcal{W}^2, \tag{1.1}$$

where u_i are normal unit vectors to the planes E_i and $\lambda_0 = 2\pi \int_{-1}^{1} f(t)dt$. Moreover, if the above equality holds when $f(t) = P_{2n}(t)$ where P_{2n} is any Legendre polynomial of degree $2n$, $n \neq 0$, then K is of constant width.

In Section 5, we assign to any invariant measure on the space of pairs of planes an appropriate function of the dihedral visual angle of a given convex set. The integral of this function with respect to the measure on the space of lines gives the integral of the above measure extended to those planes meeting the convex set. This result is given in the following theorem.

Theorem 5.2. *Let K be a compact convex set and let $f : [-1,1] \longrightarrow \mathbb{R}$ be an even continuous function. Let H be the $\mathcal{C}^2$ function on $[-\pi,\pi]$ satisfying*

$$H''(x) = f(\cos(x)) \sin^2(x), \quad -\pi \leq x \leq \pi, \quad H(0) = H'(0) = 0.$$

Then

$$\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = \pi H(\pi) F + 2 \int_{G \cap K = \emptyset} H(\omega)\, dG, \tag{1.2}$$

where u_i are normal unit vectors to the planes E_i, $\omega = \omega(G)$ is the visual angle from the line G, and F is the area of the boundary of K.

Then we relate this result to Blaschke's work [1]. If K is a convex set of mean curvature M and area of its boundary F, it is known the following Crofton–Herglotz formula

$$\int_{G \cap K = \emptyset} (\omega^2 - \sin^2 \omega)\, dG = 2M^2 - \frac{\pi^3 F}{2},$$

where $\omega = \omega(G)$ is the dihedral visual angle of K from the line G. This equality reveals the significance of the function of the visual angle $\omega^2 - \sin^2 \omega$. One can ask what role the function $\omega - \sin \omega$ plays; this function, interpreting ω as

the visual angle in the plane, is significant thanks to Crofton's formula. In dimension 3, the inequality

$$\int_{G\cap K=\emptyset} (\omega - \sin\omega)\, dG \geq \frac{\pi}{4}(M^2 - 2\pi F)$$

was established in [1]. Here we provide a simple formulation of the deficit in this inequality by means of the following result.

Theorem 5.3. *Let K be a compact convex set with support function p, area of its boundary F, and mean curvature M. Let L_u be the length of the boundary of the projection of K on* $\operatorname{span}\{u\}^\perp$ *and let $\omega = \omega(G)$ be the visual angle of K from the line G. Then*

$$\text{(i)} \int_{u\in S^2} L_u^2 du = \pi M^2 + 4\pi \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2} \|\pi_{2n}(p)\|^2,$$

$$\text{(ii)} \int_{G\cap K=\emptyset} (\omega - \sin\omega) dG = \frac{\pi}{4}(M^2 - 2\pi F) + \pi \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2} \|\pi_{2n}(p)\|^2,$$

with $\pi_{2n}(p)$ the projection of the support function p of K on the vector space of spherical harmonics of degree $2n$.

Moreover, equality holds both in (5.5) and (5.6) if and only if K is of constant width.

In Section 6, we give a formulation of Theorem 5.2 in terms of the Fourier series of the function of the visual angle assigned to an invariant measure. As a consequence, one obtains that the integral of any invariant measure in the space of pairs of planes extended to those meeting a compact convex set K is an infinite linear combination of integrals of even powers of the sine of the visual angle of K. From this, we exhibit in Proposition 6.3 a simple family of polynomial functions that are in some sense a basis for the integrals considered in Theorem 4.1. In fact, every invariant integral can be written as an infinite linear combination of integrals with respect to the invariant measures given by those polynomial functions.

2. Preliminaries.

2.1. Support function. The *support function* of a compact convex set K in the Euclidean space $\mathbb{E}^3$ is defined as $p_K(u) = \sup\{\langle x, u\rangle : x \in K\}$ for u belonging to the unit sphere S^2. If the origin O of $\mathbb{E}^3$ is an interior point of K, then the number $p_K(u)$ is the distance from the origin to the support plane of K in the direction given by u. The *width* w of K in a direction $u \in S^2$ is $w(u) = p_K(u) + p_K(-u)$.

From now on, we will write $p(u) = p_K(u)$ and will assume that $p(u)$ is of class C^2; in this case, we shall say that the boundary of K, ∂K, is of class $\mathcal{C}^2$.

2.2. Spherical harmonics. Let us recall that a *spherical harmonic* of order n on the unit sphere S^2 is the restriction to S^2 of an harmonic homogeneous polynomial of degree n. It is known that every continuous function on S^2

can be uniformly approximated by finite sums of spherical harmonics (see for instance [6]).

More precisely, the function $p(u)$ can be written in terms of spherical harmonics as

$$p(u) = \sum_{n=0}^{\infty} \pi_n(p)(u), \tag{2.1}$$

where $\pi_n(p)$ is the projection of the support function p on the vector space of spherical harmonics of degree n. An orthogonal basis of this space is given in terms of the longitude θ and the colatitude φ in S^2 by

$$\{\cos(j\theta)(\sin\varphi)^j\, P_n^{(j)}(\cos\varphi), \quad \sin(j\theta)(\sin\varphi)^j\, P_n^{(j)}(\cos\varphi) : \quad 0 \le j \le n\}$$

where $P_n^{(j)}$ denotes the jth derivative of the nth Legendre polynomial P_n (cf. [6]).

It can be seen that $\pi_0(p) = \mathcal{W}/2 = M/4\pi$ where $\mathcal{W} = 1/4\pi \int_{S^2} w(u)\, du$ is the *mean width* of K, and M is the *mean curvature* of K. Moreover, $\pi_1(p) = \langle s(K), \cdot \rangle$ where $s(K)$ denotes the *Steiner point* of K (cf. [6, p. 182]). It is clear that $\pi_0(p)$ is invariant under Euclidean motions and that $\pi_1(p)$ is not. It is known that $\pi_n(p)$ is invariant under translations for every $n \neq 1$ (cf. [12, p. 5]).

One can easily check that K has constant width if and only if $\pi_n(p) = 0$ for $n \neq 0$ even.

2.3. Measures in the space of planes. The space of affine planes $\mathcal{A}_{3,2}$ in $\mathbb{E}^3$ is a homogeneous space of the group of isometries of $\mathbb{E}^3$. It can also be considered as a line bundle $\pi : \mathcal{A}_{3,2} \longrightarrow \mathrm{Gr}(3,2)$ where $\mathrm{Gr}(3,2)$ is the Grassmannian of planes through the origin in $\mathbb{E}^3$ and $\pi(E)$ is the plane parallel to E through the origin. The fiber on $E_0 \in \mathrm{Gr}(3,2)$ is identified with $E_0^{\perp}$. Each plane $E \in \mathcal{A}_{3,2}$ is then uniquely determined by the pair $(\pi(E), E \cap \pi(E)^{\perp})$. Every pair $(E_0, p) \in \mathrm{Gr}(3,2) \times \mathbb{R}^3$ determines an element $E_0 + p \in \mathcal{A}_{3,2}$.

We shall also consider the space of affine lines $\mathcal{A}_{3,1}$ in $\mathbb{E}^3$; it is a vector bundle $\pi : \mathcal{A}_{3,1} \longrightarrow \mathrm{Gr}(3,1)$ where $\mathrm{Gr}(3,1)$ is the Grassmannian of lines through the origin and every affine line $G \subset \mathbb{E}^3$ can be identified with $(\pi(G), G \cap \pi(G)^{\perp})$.

Both the isometry group of $\mathbb{E}^3$ and the isotropy group of a fixed plane $E \in \mathcal{A}_{3,2}$ are unimodular groups; so the Haar measure of the group of isometries is projected into an isometry-invariant measure m on $\mathcal{A}_{3,2}$.

For a measurable set $B \subset \mathcal{A}_{3,2}$, we consider

$$m(B) = \int_{\mathcal{A}_{3,2}} \chi_B(E) dE := \int_{\mathrm{Gr}(3,2)} \left(\int_{E_0^{\perp}} \chi_B(E_0 + p) dp \right) d\nu$$

where χ_B is the characteristic function of B, dp denotes the ordinary Lebesgue measure on $E_0^{\perp}$, and $d\nu$ a normalized isometry-invariant measure on $\mathrm{Gr}(3,2)$ such that $\nu(\mathrm{Gr}(3,2)) = 2\pi$.

More generally, if $f : \mathcal{A}_{3,2} \to \mathbb{R}$ and $\bar{f} : \mathrm{Gr}(3,2) \times \mathbb{R}^3 \to \mathbb{R}$ are related by $\bar{f}(E_0, p) = f(E_0 + p)$, we have

$$\int_{\mathcal{A}_{3,2}} f(E)\, dE := \int_{\mathrm{Gr}(3,2)} \left(\int_{E_0^\perp} \bar{f}(E_0, p) dp \right) d\nu.$$

Notice that the only measures on $\mathcal{A}_{3,2}$ invariant under isometries are those of the form $f(E)dE$ with f a constant function.

In a similar way, one has a normalized isometry-invariant measure on $\mathcal{A}_{3,1}$ that will be denoted by dG. For more details, see [9].

3. Invariant measures in the space of ordered pairs of planes. We consider measures in the space $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ of pairs of planes in $\mathbb{E}^3$ of the form $m_{\tilde{f}} := \tilde{f}(E_1, E_2)dE_1 dE_2$. We want to study which functions $\tilde{f}$ give an isometry-invariant measure, that is, a measure $m_{\tilde{f}}$ satisfying $m_{\tilde{f}}(B) = m_{\tilde{f}}(gB)$ for every Euclidean motion g. For instance, it is known that for a given compact convex set K, one has $\int_{E \cap K \neq \emptyset} dE = M$. So when $\tilde{f}(E_1, E_2) = 1$, we have

$$\int_{K \cap E_i \neq \emptyset} dE_1 dE_2 = M^2 = 4\pi^2 \mathcal{W}^2, \tag{3.1}$$

where M and $\mathcal{W}$ are the mean curvature and the mean width of K, respectively.

Proposition 3.1. *The measure given by $\tilde{f}(E_1, E_2)dE_1\, dE_2$ in $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ is invariant under isometries of $\mathbb{E}^3$ if and only if $\tilde{f}(E_1, E_2) = f(\langle u_1, u_2 \rangle)$ where $\pi(E_i)^\perp = \mathrm{span}\{u_i\}$, $i = 1, 2$, and $f : [-1, 1] \to \mathbb{R}$ is an even measurable function.*

Proof. Suppose that $\tilde{f}(E_1, E_2)dE_1\, dE_2$ is invariant. Using the representation of an element $E \in \mathcal{A}_{3,2}$ as a pair $(\pi(E), p)$ where $p = E \cap \pi(E)^\perp$, we can write

$$\tilde{f}(E_1, E_2) = F(\pi(E_1), p_1; \pi(E_2), p_2)$$

for some $F : (\mathrm{Gr}(3,2) \times \mathbb{E}^3)^2 \to \mathbb{R}$. For any translation τ, it is

$$\begin{aligned} \tilde{f}(E_1 + \tau, E_2 + \tau) &= F(\pi(E_1), p_1 + \langle \tau, u_1 \rangle u_1; \pi(E_2), p_2 + \langle \tau, u_2 \rangle u_2) \\ &= F(\pi(E_1), p_1; \pi(E_2), p_2), \end{aligned}$$

the last equality holds due to the invariance of $\tilde{f}(E_1, E_2)dE_1\, dE_2$. Now, as it can be easily checked, for each pair $p_i' \in \pi(E_i)^\perp, i = 1, 2$, there is a translation τ such that $p_i' = p_i + \langle \tau, u_i \rangle u_i$, and so F is independent of p_1 and p_2 and we can write $\tilde{f}(E_1, E_2) = H(\pi(E_1), \pi(E_2))$ for some function H on $\mathrm{Gr}(3,2) \times \mathrm{Gr}(3,2)$.

Given $t \in [-1, 1]$, consider $(V, W) \in \mathrm{Gr}(3,2)^2$ such that $V = \mathrm{span}\{v\}^\perp$, $W = \mathrm{span}\{w\}^\perp$, and $t = \langle v, w \rangle$ with v, w unit vectors. The function $f(t) = H(V, W)$ is well defined since for any rotation θ, we have that $H(\theta V, \theta W) = H(V, W)$ and it is even. So it is proved that there exists a measurable and even function $f : [-1, 1] \to \mathbb{R}$ such that

$$\tilde{f}(E_1, E_2) = f(\langle u_1, u_2 \rangle).$$

If $\tilde{f}$ is as above, it is clear that $\tilde{f}(E_1, E_2)dE_1 dE_2$ gives rise to an isometry-invariant measure. □

4. Integral of functions of pairs of planes meeting a convex set. Let K be a compact convex set in the Euclidean space $\mathbb{E}^3$. According to equality (3.1), it is a natural question to evaluate

$$\int_{E_i\cap K\neq\emptyset} \tilde{f}(E_1,E_2)dE_1dE_2,$$

where $\tilde{f}(E_1,E_2)dE_1dE_2$ is an isometry-invariant measure on $\mathcal{A}_{3,2}\times\mathcal{A}_{3,2}$. This can be done in terms of the coefficients of the expansion of the support function of K in spherical harmonics and the coefficients of the Legendre series of the measurable even function $f:[-1,1]\to\mathbb{R}$ such that $\tilde{f}(E_1,E_2)=f(\langle u_1,u_2\rangle)$ (see Proposition 3.1).

The following result is a special case, with a different notation, of [7, Theorem 5], whose proof is based on the Funk–Hecke theorem [6, p. 98].

Theorem 4.1. *Let K be a compact convex set with support function p given in terms of spherical harmonics by (2.1). Let $\tilde{f}(E_1,E_2)dE_1\,dE_2$ be an isometry-invariant measure on $\mathcal{A}_{3,2}\times\mathcal{A}_{3,2}$ and $f:[-1,1]\to\mathbb{R}$ an even measurable function such that $\tilde{f}(E_1,E_2)=f(\langle u_1,u_2\rangle)$ where $\pi(E_i)^\perp=\operatorname{span}\{u_i\}$, $i=1,2$. Then*

$$\int_{E_i\cap K\neq\emptyset} \tilde{f}(E_1,E_2)dE_1\,dE_2=\frac{\lambda_0}{4\pi}M^2+\sum_{n=1}^{\infty}\lambda_{2n}\|\pi_{2n}(p)\|^2, \tag{4.1}$$

where $\lambda_{2n}=2\pi\int_{-1}^{1}f(t)P_{2n}(t)\,dt$ with P_{2n} the Legendre polynomial of degree $2n$.

Example 1. If $f(t)=\sqrt{1-t^2}$, then $f(\langle u_1,u_2\rangle)=\sin(\theta_{12})$ where $0\le\theta_{12}\le\pi$ is the angle between the planes E_1 and E_2 (that is, $\cos\theta_{12}=\pm\langle u_1,u_2\rangle$ where $\pi(E_i)^\perp=\operatorname{span}\{u_i\}$, $i=1,2$). Applying Theorem 4.1 with the corresponding coefficients

$$\lambda_{2n}=2\pi\int_{-1}^{1}f(t)P_{2n}(t)=-\frac{\Gamma(n+\frac{1}{2})\Gamma(n-\frac{1}{2})}{n!(n+1)!}\frac{\pi}{2},\ \lambda_0=\pi^2,\ \lambda_{2n+1}=0$$

(cf. [5, 7.132]), one gets

$$\int_{E_i\cap K\neq\emptyset}\sin(\theta_{12})dE_1\,dE_2=\frac{\pi}{4}M^2-\frac{\pi}{2}\left(\sum_{n=1}^{\infty}\frac{\Gamma(n+\frac{1}{2})\Gamma(n-\frac{1}{2})}{n!(n+1)!}\|\pi_{2n}(p)\|^2\right). \tag{4.2}$$

In the particular case that f is a Legendre polynomial, one obtains from Theorem 4.1 the following

Corollary 4.2. *Let K be a compact convex set with support function p given in terms of spherical harmonics by (2.1). Then if P_{2n} is the Legendre polynomial of even degree $2n$, one has*

$$\int_{E_i\cap K\neq\emptyset}P_{2n}(\langle u_1,u_2\rangle)\,dE_1\,dE_2=\frac{4\pi}{4n+1}\|\pi_{2n}(p)\|^2.$$

Proof. In this case, $\lambda_m = 0$ for $m \neq 2n$ and $\lambda_{2n} = \dfrac{4\pi}{4n+1}$. □

To end this section, we analyze equality (4.1) when K is a convex set of constant width. As said, this means that $\pi_n(p) = 0$ for $n \neq 0$ even.

Proposition 4.3. *Let K be a compact convex set of constant width $\mathcal{W}$ and let $f : [-1, 1] \longrightarrow \mathbb{R}$ be an even bounded measurable function. Then*

$$\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = \lambda_0 \pi \mathcal{W}^2, \tag{4.3}$$

where u_i are normal unit vectors to the planes E_i and $\lambda_0 = 2\pi \int_{-1}^{1} f(t)dt$. Moreover, if the above equality holds when $f(t) = P_{2n}(t)$ where P_{2n} is any Legendre polynomial of degree $2n$, $n \neq 0$, then K is of constant width.

Proof. Since K is of constant width, by (4.1), one gets

$$\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = \frac{\lambda_0}{4\pi} M^2$$

and remembering that $M = 2\pi\mathcal{W}$, the equality follows. If equality (4.3) holds for $f(t) = P_{2n}(t)$ with $n \neq 0$, and since the corresponding λ_0 vanishes, one has

$$\int_{E_i \cap K \neq \emptyset} P_{2n}(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = 0.$$

Therefore, by Corollary 4.2, it follows that $\|\pi_{2n}(p)\| = 0$ for every nonzero n and K is of constant width. □

5. Integrals of invariant measures in terms of the visual angle. The aim of this section is to write the integral of an isometry-invariant measure over the pairs of planes meeting a convex set K, given in Theorem 4.1, as an integral of an appropriate function of the visual angle.

Let us make precise what we mean by the angle of a plane about a straight line G and the visual angle of a convex set K from a line G not meeting K.

Definition 5.1.

1. Given a straight line G, let $(q; e_1, e_2)$ be a fixed affine orthonormal frame in $G^{\perp}$ with $q \in G$. For each plane E through G, let u be the unit normal vector to E pointing from the origin to it. Then the angle α associated to E is the one given by $u = \cos(\alpha)e_1 + \sin(\alpha)e_2$.
2. The visual angle of a convex set K from a line G not meeting K is the angle $\omega = \omega(G)$, $0 \leq \omega \leq \pi$, between the half-planes E_1, E_2 through G tangents to K.

If α_i are the angles associated to E_i, $i = 1, 2$, then

$$\cos(\pi - \omega) = \cos(\alpha_2 - \alpha_1) = \langle u_1, u_2 \rangle$$

where u_1, u_2 are the normal unit vectors to E_1, E_2 pointing from the origin, assuming the origin is inside of K.

The measure $dE_1\,dE_2$ in the space $\mathcal{A}_{3,2}\times\mathcal{A}_{3,2}$ of pairs of planes in $\mathbb{E}^3$ can be written according to Santaló (cf. [11], Section II.12.6) as

$$dE_1\,dE_2=\sin^2(\alpha_2-\alpha_1)\,d\alpha_1\,d\alpha_2\,dG. \tag{5.1}$$

Then we can prove the following

Theorem 5.2. *Let K be a compact convex set and let $f:[-1,1]\longrightarrow\mathbb{R}$ be an even continuous function. Let H be the $\mathcal{C}^2$ function on $[-\pi,\pi]$ satisfying*

$$H''(x)=f(\cos(x))\sin^2(x),\quad -\pi\le x\le\pi,\quad H(0)=H'(0)=0.$$

Then

$$\int\limits_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\,dE_1\,dE_2=\pi H(\pi)F+2\int\limits_{G\cap K=\emptyset} H(\omega)\,dG, \tag{5.2}$$

where u_i are normal unit vectors to the planes E_i, $\omega=\omega(G)$ is the visual angle from the line G, and F is the area of the boundary of K.

Proof. Let $G=q+\mathrm{span}\{u\}$ with u a unit director vector such that $K\cap G=\emptyset$. Let $E_i, i=1,2$, be the supporting planes of K through G. Take now an affine orthonormal frame $\{q;e_1,e_2,u\}$ in $\mathbb{E}^3$ such that $E_1=q+\mathrm{span}\{e_1,u\}$. Every plane E through G can be written as $E=q+\mathrm{span}\{v_\alpha,u\}$ where $v_\alpha=\cos\alpha e_1+\sin\alpha e_2$ with $\alpha\in[0,\pi)$ and the planes E intersecting K correspond to angles $\alpha\in[0,\omega(G)]$. Then, using (5.1), one has

$$\begin{aligned}\int\limits_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\,dE_1\,dE_2&=\int\limits_{G\cap K=\emptyset}\int_0^{\omega}\int_0^{\omega} H''(\alpha_2-\alpha_1)\,d\alpha_1\,d\alpha_2\,dG\\&\quad+\int\limits_{G\cap K\neq\emptyset}\int_0^{\pi}\int_0^{\pi} H''(\alpha_2-\alpha_1)\,d\alpha_1\,d\alpha_2\,dG.\end{aligned}$$

Evaluating the inner integrals and taking into account that $\int_{G\cap K\neq\emptyset} dG=\frac{\pi}{2}F$, it follows that

$$\begin{aligned}\int\limits_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\,dE_1\,dE_2&=\frac{1}{2}\pi(H(\pi)+H(-\pi))F+\int\limits_{G\cap K=\emptyset}(H(\omega)+H(-\omega))\,dG.\end{aligned}$$

Since $H(0)=H'(0)=0$ and $H''(x)=H''(-x)$, it is easy to see that $H(x)=H(-x)$ and the result follows. □

Example 2. Let $f(t)=\sqrt{1-t^2}$ be the function considered in Example 1. In this case, the corresponding function H such that

$$H''(x)=f(\cos(x))\sin^2(x)=|\sin^3(x)|$$

is given by

$$H(x) = \frac{2}{3}\left(|x| - |\sin x|\right) - \frac{1}{9}|\sin^3 x|.$$

Now, since $\omega \in [0, \pi]$, Theorem 5.2 and equality (4.2) lead to

$$\int\limits_{G\cap K=\emptyset} \left(\omega - \sin\omega - \frac{1}{3!}\sin^3\omega\right) dG$$
$$= \pi\left(\frac{3M^2}{16} - \frac{1}{2}\pi F - \frac{3}{8}\sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(n-\frac{1}{2})}{n!(n+1)!}\|\pi_{2n}(p)\|^2\right).$$

5.1. Crofton's formula in the space. In Blaschke's work [1, p. 75], the following Crofton–Herglotz formula is given

$$\int\limits_{G\cap K=\emptyset} (\omega^2 - \sin^2\omega)\, dG = 2M^2 - \frac{\pi^3 F}{2}. \tag{5.3}$$

We can easily recover (5.3) from Theorem 5.2. In fact, considering $f(t) = 1$, one gets $H(x) = (x^2 - \sin^2 x)/4$ and equality (5.2) gives

$$M^2 = \int\limits_{E_i\cap K\neq\emptyset} dE_1\, dE_2 = \frac{1}{4}\pi^3 F + \frac{1}{2}\int\limits_{G\cap K=\emptyset} (\omega^2 - \sin^2\omega)\, dG.$$

Blaschke's formula reveals the significance of the function of the visual angle $\omega^2 - \sin^2\omega$. One can ask what role the function $\omega - \sin\omega$ plays; this function, interpreting ω as the visual angle in the plane, is significant thanks to Crofton's formula $\int_{P\not\in K}(\omega - \sin\omega)dP = L^2/2 - \pi F$, where K is a compact convex set in the plane with area F and length of its boundary L (see [11]).

In [1, p. 85], Blaschke shows that

$$\int\limits_{G\cap K=\emptyset} (\omega - \sin\omega)dG = \frac{1}{4}\int\limits_{u\in S^2} L_u^2 du - \frac{\pi^2}{2}F, \tag{5.4}$$

where L_u is the length of the boundary of the projection of K on $\mathrm{span}\{u\}^{\perp}$.

It can be easily seen that $\int_{u\in S^2} L_u du = 2\pi M$ and from this equality, applying Schwarz's inequality, one gets

$$\int\limits_{u\in S^2} L_u^2 du \geq \pi M^2. \tag{5.5}$$

Introducing (5.5) into (5.4), one obtains

$$\int\limits_{G\cap K=\emptyset} (\omega - \sin\omega)dG \geq \frac{\pi}{4}(M^2 - 2\pi F). \tag{5.6}$$

As a consequence of Theorem 5.2, we can now evaluate the deficit in both inequalities (5.5) and (5.6).

Theorem 5.3. *Let K be a compact convex set with support function p, area of its boundary F, and mean curvature M. Let L_u be the length of the boundary of the projection of K on $\operatorname{span}\{u\}^\perp$ and let $\omega = \omega(G)$ be the visual angle of K from the line G. Then*

$$\text{(i)} \quad \int_{u\in S^2} L_u^2 du = \pi M^2 + 4\pi \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2} \|\pi_{2n}(p)\|^2,$$

$$\text{(ii)} \quad \int_{G\cap K=\emptyset} (\omega - \sin\omega) dG = \frac{\pi}{4}(M^2 - 2\pi F) + \pi \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2} \|\pi_{2n}(p)\|^2,$$

with $\pi_{2n}(p)$ the projection of the support function p of K on the vector space of spherical harmonics of degree $2n$.

Moreover, equality holds both in (5.5) and (5.6) if and only if K is of constant width.

Proof. We consider $f(t) = 1/\sqrt{1-t^2}$. For this function, the corresponding H in Theorem 5.2 is $H(x) = |x| - |\sin x|$. Applying equality (5.2) and Theorem 4.1 with the corresponding λ_{2n}'s given by

$$\lambda_{2n} = 2\pi \int_{-1}^{1} f(t) P_{2n}(t) dt = 2\pi \frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2}$$

(cf. [5, 7.226]), item (ii) follows. Equality (i) is a consequence of (ii) and (5.4).

The statement about equality in (5.5) and (5.6) is a consequence of the fact that K is of constant width if and only if $\pi_{2n}(p) = 0$ for $n \neq 0$. □

6. A formulation with the Fourier series. In this section, we give an alternative formulation of Theorem 5.2 in terms of the Fourier coefficients of the function $H''(x)$. Since f is even, one has that $H''(x) = f(\cos(x))\sin^2(x)$ is an even π-periodic function. Let

$$H''(x) = \frac{1}{2}a_0 + \sum_{n\geq 1} a_{2n}\cos(2nx) \tag{6.1}$$

be the Fourier expansion of $H''(x)$. Integrating twice and taking into account that $H(0) = H'(0) = 0$ one obtains

$$H(x) = \frac{a_0}{4}x^2 + \sum_{n\geq 1} \frac{a_{2n}}{4n^2}(1 - \cos(2nx)). \tag{6.2}$$

Using this expression of the function H, Theorem 5.2 can be written as

Proposition 6.1. *Let K be a compact convex set and let $f : [-1,1] \longrightarrow \mathbb{R}$ be an even continuous function. Let H be the $\mathcal{C}^2$ function on $[-\pi,\pi]$ satisfying*

$$H''(x) = f(\cos(x))\sin^2(x), \quad -\pi \leq x \leq \pi, \quad H(0) = H'(0) = 0. \tag{6.3}$$

If $H(x)$ is given by (6.2), then

$$\int\limits_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\,dE_1\,dE_2 = \frac{a_0}{4}\pi^3 F + \frac{1}{2}\int\limits_{G\cap K=\emptyset}\left(a_0\omega^2+\sum_{n\geq 1}\frac{a_{2n}}{n^2}(1-\cos(2n\omega))\right)dG, \quad (6.4)$$

where u_i are normal vectors to the planes E_i, the visual angle from the line G is ω, and F denotes the area of the boundary of K.

The right hand side of (6.4) can be written as a linear combination of integrals of even powers of $\sin\omega$. For this purpose, we will use Blaschke's formula (5.3) and the known equality

$$\cos 2nx = \sum_{m=0}^{n}\alpha_{n,m}\sin^{2m}x \quad \text{with} \quad \alpha_{n,m} = \frac{(-1)^m\, n\, 2^{2m}(n+m-1)!}{(2m)!(n-m)!}, \quad (6.5)$$

which follows easily from the equality $\cos(2nx) = (-1)^n T_{2n}(\sin(x))$ where T_{2n} is Chebyshev's polynomial of degree $2n$. We can state

Proposition 6.2. *Let K be a compact convex set and let $f:[-1,1]\longrightarrow\mathbb{R}$ be an even continuous function. Let H be the $\mathcal{C}^2$ function on $[-\pi,\pi]$ satisfying*

$$H''(x) = f(\cos(x))\sin^2(x), \quad -\pi\leq x\leq\pi, \quad H(0)=H'(0)=0.$$

If $H(x)$ is given by (6.2), then

$$\int\limits_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\,dE_1\,dE_2 = a_0M^2 - \frac{1}{2}\sum_{m=2}^{\infty}\left(\sum_{n=m}^{\infty}\frac{a_{2n}}{n^2}\alpha_{n,m}\int\limits_{G\cap K=\emptyset}\sin^{2m}\omega\,dG\right), \quad (6.6)$$

where u_i are normal vectors to the planes E_i, the visual angle from the line G is ω, F denotes the area of the boundary of K, the coefficients $\alpha_{n,m}$ are given by (6.5), and the coefficients a_{2n} by (6.1).

Proof. Using (6.5), the right hand side of (6.4) is written as

$$\frac{a_0\pi^3}{4}F + \frac{1}{2}\int\limits_{G\cap K=\emptyset}\left(a_0\omega^2 - \sum_{n=1}^{\infty}\frac{a_{2n}}{n^2}\alpha_{n,1}\sin^2\omega - \sum_{n=2}^{\infty}\frac{a_{2n}}{n^2}\sum_{m=2}^{n}\alpha_{n,m}\sin^{2m}\omega\right)dG$$
$$= \frac{a_0\pi^3}{4}F + \frac{1}{2}\int\limits_{G\cap K=\emptyset}\left(a_0(\omega^2-\sin^2\omega) - \sum_{n=2}^{\infty}\frac{a_{2n}}{n^2}\sum_{m=2}^{n}\alpha_{n,m}\sin^{2m}\omega\right)dG$$

where we have used that $\alpha_{n,0}=1, \alpha_{n,1}=-2n^2$, and $a_0 = -2\sum_{n=1}^{\infty}a_{2n}$ which is a consequence of the fact that $H''(0)=0$. Using Blaschke's formula (5.3) and reordering the double sum, the result follows. □

6.1. A basis for the integrals of invariant measures. As a consequence of Proposition 6.2, we can exhibit a simple family of polynomial functions that are in some sense a basis for the integrals in Theorem 4.1. Consider the polymomials

$$h_m(t) = m(2mt^2 - 1)(1 - t^2)^{m-2}, \quad m > 1. \tag{6.7}$$

Then for $H''(x) = h_m(\cos(x)) \sin^2(x)$, one easily checks that $H(\omega) = \frac{1}{2} \sin^{2m} \omega$ and Theorem 5.2 applied to $h_m(t)$ gives

$$\int_{E_i \cap K \neq \emptyset} h_m(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = \int_{G \cap K = \emptyset} \sin^{2m} \omega\, dG,$$

which together with Equation (6.6) leads to the following

Proposition 6.3. *Under the same hypotheses and notation as in Proposition 6.2, one has*

$$\begin{aligned} &\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 \\ &= a_0 M^2 - \frac{1}{2} \sum_{m=2}^{\infty} \left(\sum_{n=m}^{\infty} \frac{a_{2n}}{n^2} \alpha_{n,m} \int_{E_i \cap K \neq \emptyset} h_m(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 \right), \end{aligned}$$

where the polynomials h_m are given in (6.7).

So every invariant integral can be written as an infinite linear combination of the integrals of the invariant measures given by the polynomials h_m.

References

[1] Blaschke, W.: Vorlesungen über Integralgeometrie, 3rd edition. Deutscher Verlag der Wissenschaften, Berlin (1955)

[2] Crofton, M.W.: On the theory of local probability. Philos. Trans. R. Soc. Lond. **158**, 181–199 (1868)

[3] Cufí, J., Gallego, E., Reventós, A.: On the integral formulas of Crofton and Hurwitz relative to the visual angle of a convex set. Mathematika **65**(4), 874–896 (2019)

[4] Cufí, J., Gallego, E., Reventós, A.: Integral geometry about the visual angle of a convex set. Rend. Circ. Mat. Palermo (2) **69**(3), 1115–1130 (2020)

[5] Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 7th edn. Elsevier/Academic Press, Amsterdam (2007)

[6] Groemer, H.: Geometric Applications of Fourier Series and Spherical Harmonics. Encyclopedia of Mathematics and its Applications, vol. 61. Cambridge University Press, Cambridge (1996)

[7] Hug, D., Schneider, R.: Integral geometry of pairs of hyperplanes or lines. Arch. Math. (Basel) **115**(3), 339–351 (2020)

[8] Hurwitz, A.: Sur quelques applications géométriques des séries de Fourier. Ann. Sci. École Norm. Sup. (3) **19**, 357–408 (1902)

[9] Klain, D.A., Rota, G.C.: Introduction to Geometric Probability. Cambridge University Press, Lezioni Lincee (1997)

[10] Masotti, G.: La geometria integrale. Rend. Sem. Mat. Fis. Milano, **25**(1954–55), 164–231 (1955)

[11] Santaló, L.A.: Integral Geometry and Geometric Probability, 2nd edn. With a foreword by Mark Kac. Cambridge Mathematical Library. Cambridge University Press, Cambridge (2004)

[12] Schneider, R., Schuster, F.E.: Rotation invariant Minkowski classes of convex bodies. Mathematika **54**(1–2), 1–13 (2007)

Julià Cufí, Eduardo Gallego, and Agustí Reventós
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona Catalonia
Spain
e-mail: jcufi@mat.uab.cat

Eduardo Gallego
e-mail: egallego@mat.uab.cat

Agustí Reventós
e-mail: agusti@mat.uab.cat

Received: 26 March 2021

Revised: 3 July 2021

Accepted: 27 July 2021.